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4−PLACEMENT OF ROOTED TREES

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Abstract. A tree T of order n is called $k-$placeable if there are k edge-disjoint copies of T into K_n. In this paper we prove some results about $4-$placement of rooted trees.

Keywords:Embedding, packing, k-placement, star-path, star-path-star.

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1. Introduction

We use standard graph theory notation. We consider in our study only simple undirected graphs. Suppose $G_1, G_2, \cdots, G_k$ are graphs of order n. We say that there is a packing of $G_1, G_2, \cdots, G_k$ (into the complete graph K_n) if there is injections $\alpha_i : V(G_i) \to V(K_n), i = 1, 2, \cdots, k$ such that $\alpha_i^*(E(G_i)) \bigcap \alpha_j^*(E(G_j)) = \emptyset$ for $i \neq j$ where the map $\alpha_i^* : E(G_i) \to E(k_n)$ is the one induced by α_i.

A packing of k copies of a graph G is called a $k-placement$ of G. Then we say G is $k-placeable$ (into K_n). The main references of the paper are the last chapter of Bollobás book [1] and the survey paper [10], (cf. also [4,7,8]).

Let P_n and S_n be the path and star on n vertices respectively. A tree of order n obtained from one path and one star by joining two end-vertices of theirs is called path-star and denoted by $(PS)_n$. A $(PS)_n$ in which the order of path is r denoted by $(P_rS)_n$. Similarly a tree of order n is obtained from two stars by joining their centers by a path is called star-path-star and denoted by $(SPS)_n$.

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A rooted tree T of order n with root $v_0 \in V(T)$, by three branches T_1, T_2, T_3 of T (the components of $T - v_0$) is denoted by $RT_n(T_1, T_2, T_3)$. In this paper we prove any $RT_{n-2}(T_1, T_2, T_3)$, $T_i = (PS)_{n_i}$ $i = 1, 2, 3$, $n_1 + n_2 + n_3 = n - 3$ is $4-$placeable.

2. Preliminaries

The following conjecture stated by Bollobás and Eldridge [2].

Conjecture 1. Let $G_1, G_2, \ldots, G_k$ be k graphs of order n. If $|E(G_i)| \leq n - k$, $i = 1, \ldots, k$ then $G_1, G_2, \ldots, G_k$ are packable into K_n.

The cases $k = 2, 3$ of the above-mentioned conjecture, was proved in [6] and [5], respectively.

Let us mention that some related problems have already been considered. For instance in the case of trees the hypothesis on the size may be improved. The first theorem concerning the packing of three trees was probably proved in connection with the following conjecture stated by Gyárfás in [3].

Conjecture 2. Let T_i denote a tree of order i. The sequence of trees $T_2, T_3, \ldots, T_n$ can be packed into K_n.

This conjecture is sometimes called tree packing conjecture (TPC). The TPC has been proved valid for quite a few cases. We list some of the results below.

Theorem 2.1. *(Gyárfás and Lehel[3]) Any sequence of trees $T_2, T_3, \ldots, T_n$ in which all but two are stars, can be packed into K_n.*

Theorem 2.2. *(Gyárfás and Lehel[3]) Any sequence of trees $T_2, T_3, \ldots, T_n$ where $T_i \in \{S_i, P_i\}$ $i = 2, 3, \ldots, n$, can be packed into K_n.*

Hobbs in [4] proved that :

Theorem 2.3. *Any three trees of order $n_1 < n_2 < n_3 \leq n$, respectively, can be packed into K_n.*

Inspired by the above theorem, a similar result was obtained in [9] :

Theorem 2.4. *Any three trees of order $n - 1$ can be packed into K_n.*

The authors in [11] proved the following Theorem :

Theorem 2.5. *Suppose T is a tree of order $n-2$ where $T \in \{P_{n-2}, S_{n-2}, (PS)_{n-2}$ $, (SPS)_{n-2}\}$. Then four copies of T can be packed into K_n.*

3. Packing four copies of rooted trees

In this paper we suppose c_i $i = 1, 2, 3, 4$ are four different colors. The notation $(T)_{c_i}$ means that the edges of tree T are colored by c_i. In the Figureures we used four different lines instead of colors. Suppose $N = \{1, 2, \cdots, n\}$ is the

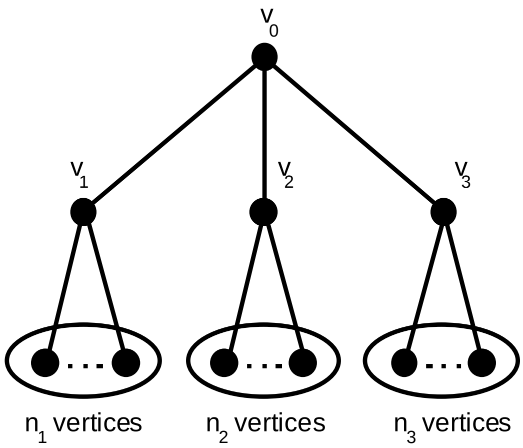

FIGURE 1

vertex set of K_n. The star S_n with center v is denoted by $S_n(V)$, in particular if $A = V(S_n) - \{v\}$ then we use the notation $S_n(v, A)$.

The main result of this paper is as follows :

Theorem 3.1. *Every* $RT_{n-2}(T_1, T_2, T_3)$, $T_i = (PS)_{n_i}$, $i = 1, 2, 3$, $n_1 + n_2 + n_3 = n - 3$ *(Figure 16) is* $4-$*placeable.*

We prove this theorem in the end of this section.

Lemma 3.2. *Every* $RT_{n-2}(T_1, T_2, T_3)$ *with children* v_i , $i = 1, 2, 3$, $T_i = S_{n_i+1}(v_i)$, $n_1 + n_2 + n_3 = n - 6$ *(Figure 1) is* $4-$*placeable.*

Proof. Suppose $A_1 = \{3, 4, 5\}$, $A_2 = \{2, 3, 5\}$, $A_3 = \{1, 3, 6\}$, $A_4 = \{2, 5, 6\}$ be subsets of $V(K_6)$. Then the edge-disjoint stars $(T)_{c_1} = S_4(2, A_1)$, $(T)_{c_1} = S_4(6, A_6)$, $(T)_{c_3} = S_4(4, A_3)$, $(T)_{c_4} = S_4(1, A_4)$ are a $4-$placement of S_4 into K_6 (Figure 2). Now we partition the $n - 6$ other vertices of k_n (except the above six vertices) to three sets B_i such that $|B_i| = n_i$, $i = 1, 2, 3$. Then color the edges of $T_1 = S_{n_1+1}(4, B_1)$, $T_2 = S_{n_2+1}(3, B_2)$, $T_3 = S_{n_3+1}(5, B_3)$ with color c_1. So the $RT_{n-2}(T_1, T_2, T_3)$ with children $v_1 = 3$, $v_2 = 4$, $v_3 = 5$ and root $v_0 = 2$ is embedded into K_n. By the same way we embed three another copies of $RT_{n-2}(T_1, T_2, T_3)$ into K_n as follows :
1. Let $T_1 = S_{n_1+1}(2, B_1)$, $T_2 = S_{n_1+1}(5, B_2)$, $T_3 = S_{n_3+1}(3, B_3)$ colored by c_2.
2. Let $T_1 = S_{n_1+1}(5, B_1)$, $T_2 = S_{n_1+1}(2, B_2)$, $T_3 = S_{n_3+1}(6, B_3)$ colored by c_3.
3. Let $T_1 = S_{n_1+1}(3, B_1)$, $T_2 = S_{n_1+1}(6, B_2)$, $T_3 = S_{n_3+1}(1, B_3)$ colored by c_4.

For example we show packing of two copies (colored by c_2, c_4) of these trees into K_n in Figure 3. Clearly these four copies of $RT_{n-2}(T_1, T_2, T_3)$ are edge-disjoint and so it is 4-placeable.

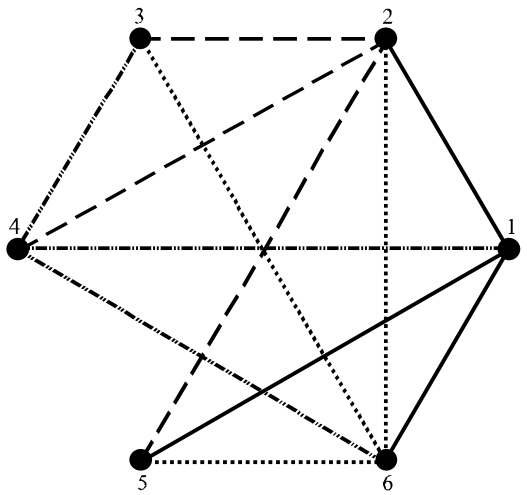

FIGURE 2

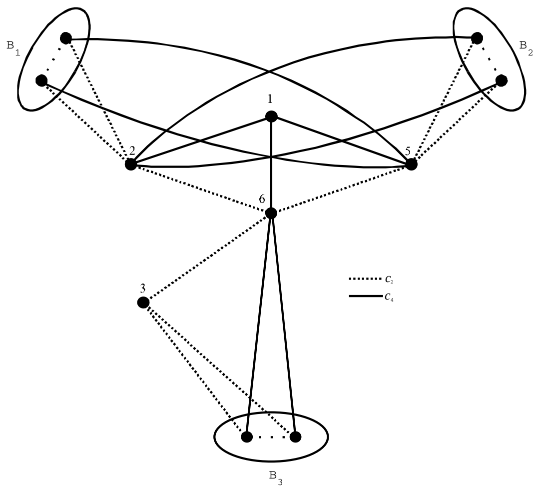

FIGURE 3

Lemma 3.3. *Every* $RT_{n-2}(T_1, T_2, T_3)$ *with children* v, v_i , $i = 1, 2$, $T_i = S_{n_i+1}(v_i)$, $i = 1, 2$, $T_3 = S_{n_3+1}(v)$, $v \neq v_3$, $n_1 + n_2 + n_3 = n - 7$ *(Figure 4) is 4-placeable.*

Proof. We consider the rooted tree of Figure 5. There is a packing of four copies of it into K_7 (Figure 6). Now we partition the $n - 7$ other vertices of K_n (except the above seven vertices) to three sets B_i such that $|B_i| = n_i$, $i = 1, 2, 3$. Then we use the method of Lemma 1 as follows:

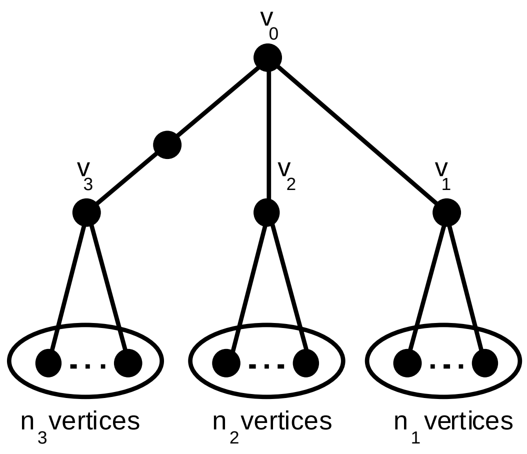

FIGURE 4

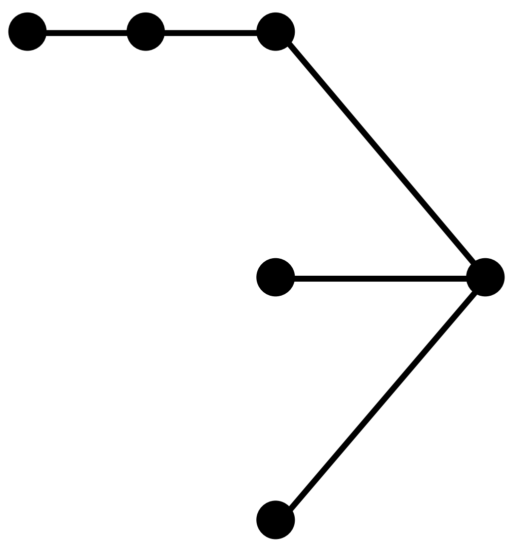

FIGURE 5

1. Let $T_1 = S_{n_1+1}(4, B_1)$, $T_2 = S_{n_2+1}(3, B_2)$, $T_3 = S_{n_3+1}(5, B_3)$ colored by c_1.
2. Let $T_1 = S_{n_1+1}(2, B_1)$, $T_2 = S_{n_2+1}(5, B_2)$, $T_3 = S_{n_3+1}(3, B_3)$ colored by c_2.
3. Let $T_1 = S_{n_1+1}(5, B_1)$, $T_2 = S_{n_2+1}(2, B_2)$, $T_3 = S_{n_3+1}(6, B_3)$ colored by

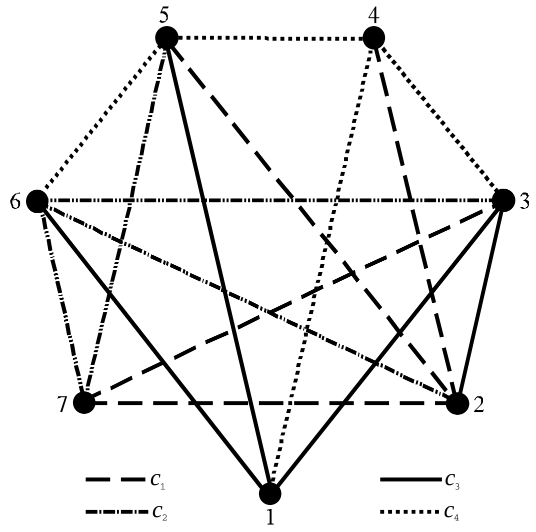

FIGURE 6

c_3.
4. Let $T_1 = S_{n_1+1}(3, B_1)$, $T_2 = S_{n_2+1}(6, B_2)$, $T_3 = S_{n_3+1}(1, B_3)$ colored by c_4.

Then four copies of $RT_{n-2}(T_1 2, T_2, T_3)$ are edge-disjoint and so they form a packing (into K_n).

Lemma 3.4. *Every $RT_{n-2}(T_1, T_2, T_3)$ with children v_i , $i = 1,2,3$, $T_i = S_{n_i+1}(v_i)$, $i = 1,2$, $T_3 = (P_4 S)_{n_3+1}$, $n_1 + n_2 + n_3 = n - 9$ (Figure 7) is 4-placeable.*

Proof. By Figure 9, the tree pictured in Figure 8 is 4-placeable into K_9. We partition $n - 9$ vertices of K_n (except the above 9 vertices) into three sets B_i such that $|B_i| = n_i$ $i = 1, 2, 3$. Set
1. $T_1 = S_{n_1+1}(5, B_1)$, $T_2 = S_{n_2+1}(1, B_2)$, $T_3 = S_{n_3+1}(9, B_3)$ colored by c_1.
2. $T_1 = S_{n_1+1}(9, B_1)$, $T_2 = S_{n_2+1}(4, B_2)$, $T_3 = S_{n_3+1}(5, B_3)$ colored by c_2.
3. $T_1 = S_{n_1+1}(4, B_1)$, $T_2 = S_{n_2+1}(2, B_2)$, $T_3 = S_{n_3+1}(1, B_3)$ colored by c_3.
4. $T_1 = S_{n_1+1}(2, B_1)$, $T_2 = S_{n_2+1}(5, B_2)$, $T_3 = S_{n_3+1}(3, B_3)$ colored by c_4.

Clearly these four copies of $RT_{n-2}(T_1, T_2, T_3)$ are edge-disjoint, so they form a 4-placement.

Lemma 3.5. *Every $RT_{n-2}(T_1, T_2, T_3)$ with children v_i , $i = 1,2,3$, $T_i = S_{n_i+1}(v_i)$, $i = 1,2$, $T_3 = (P_m S)_{n_3+1}$, $m > 8$, $n_1 + n_2 + n_3 = n - m - 6$ (Figure 10) is 4-placeable.*

Proof. It is well known that every K_n contains $\lfloor \frac{n-1}{2} \rfloor$ edge-disjoint Hamiltonian paths. So there is a 4-placement of P_3 into K_m for $m > 8$. There are

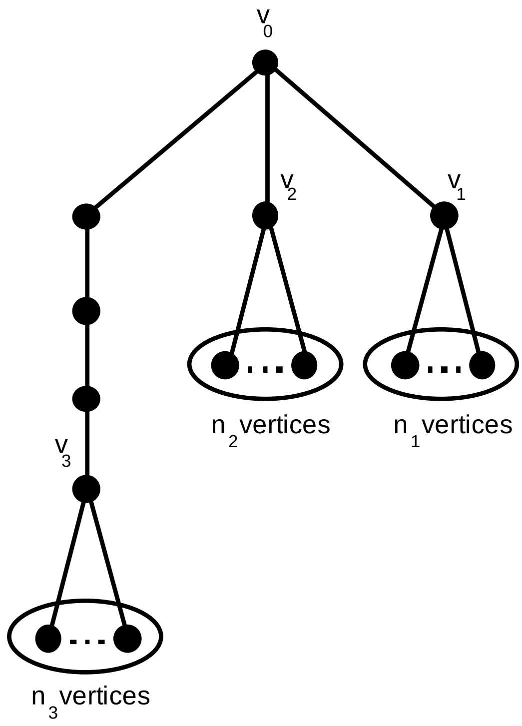

Figure 7

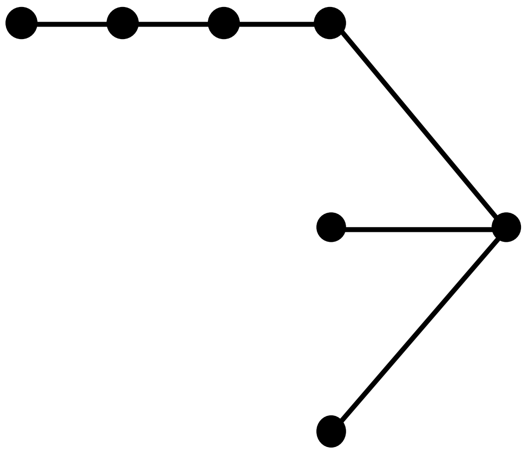

Figure 8

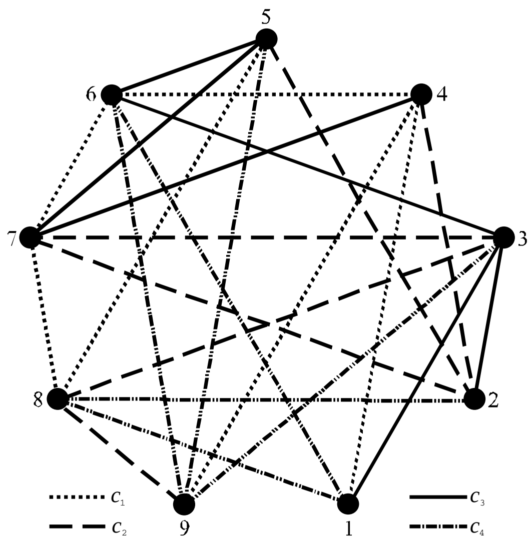

FIGURE 9

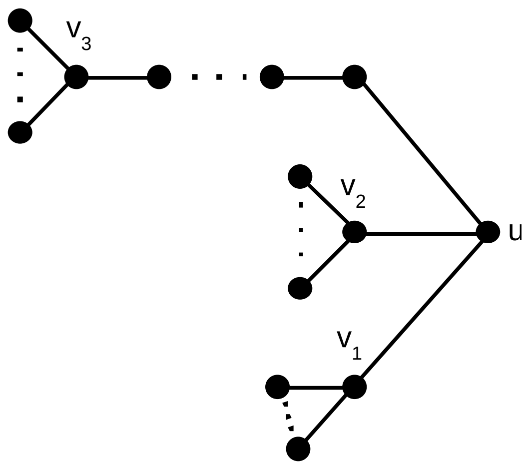

FIGURE 10

also 4-placement of S_2 into K_6. Suppose K_6, K_m be above complete graphs such that $V(K_6) = \{1, 2, \ldots, 6\}$, $V(K_m) = \{7, 8, \ldots, m + 6\}$. If we join the center of every above S_2 to end-vertex of one of the four paths then we get a 4-placement of tree in Figure 11 into K_{m+6} (Figure 12). Now we partition $n - m - 6$ other vertices of k_n (except $m + 6$ above vertices) to three sets B_i $i = 1, 2, 3$ such that $|B_i| = n_i$, $n_1 + n_2 + n_3 = n - m - 6$. Finally let:

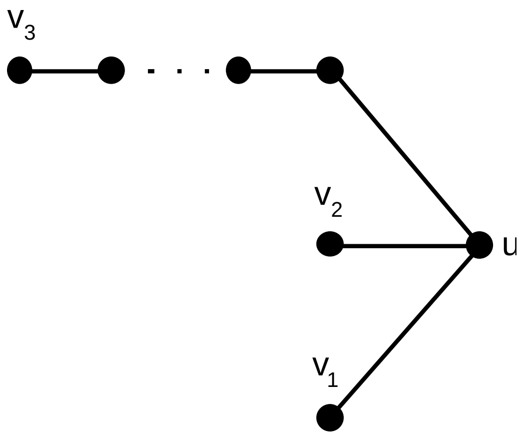

FIGURE 11

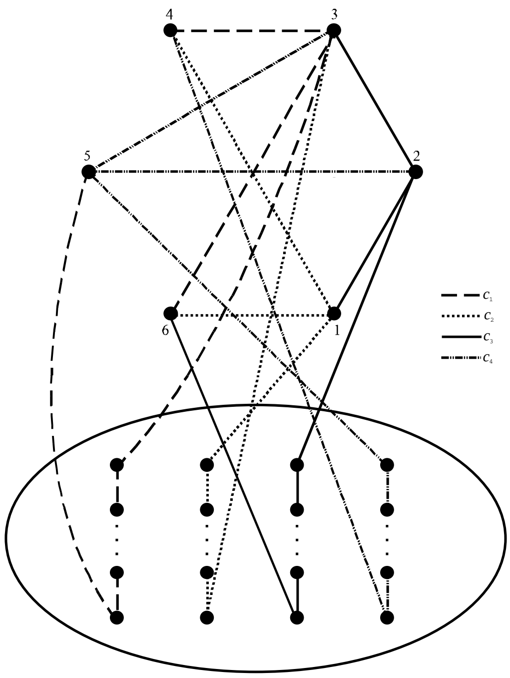

FIGURE 12

1. $(T)_{c_1} = S_{n_1+1}(5, B_1)$ $(T)_{c_1} = S_{n_1+1}(4, B_2)$ $(T)_{c_1} = S_{n_3+1}(6, B_3)$
2. $(T)_{c_2} = S_{n_1+1}(3, B_1)$ $(T)_{c_2} = S_{n_1+1}(6, B_2)$ $(T)_{c_2} = S_{n_3+1}(4, B_3)$
3. $(T)_{c_3} = S_{n_1+1}(6, B_1)$ $(T)_{c_3} = S_{n_1+1}(3, B_2)$ $(T)_{c_3} = S_{n_3+1}(1, B_3)$

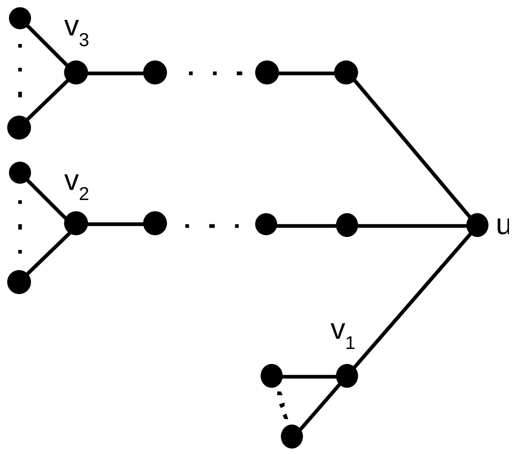

FIGURE 13

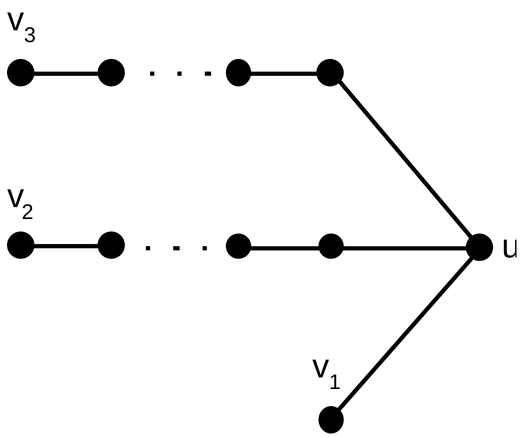

FIGURE 14

4. $(T)_{c_4} = S_{n_1+1}(4, B_1)$ $(T)_{c_4} = S_{n_1+1}(2, B_2)$ $(T)_{c_4} = S_{n_3+1}(3, B_3)$

So we have embedded four edge-disjoint copies of tree in Figure 10 into K_n.

Lemma 3.6. *Every $RT_{n-2}(T_1, T_2, T_3)$ with children v_i , $i = 1, 2, 3$, $T_1 = S_{n_1+1}(v_1)$, $T_i = (P_{m_i}S)_{n_i+1}$, $m_i > 8$, $i = 1, 2$, $n_1+n_2+n_3 = n-(m_1+m_2)-6$ (Figure 13) is 4-placeable.*

Proof. The Figure 15 shows there are 4-placement of tree in Figure 14 into $K_{m_1+m_2+6}$ with the following vertices and colors:
Copy 1 : $u = 3$ $v_1 = 6$ $v_2 = 4$ $v_3 = 5$ colored by c_1.

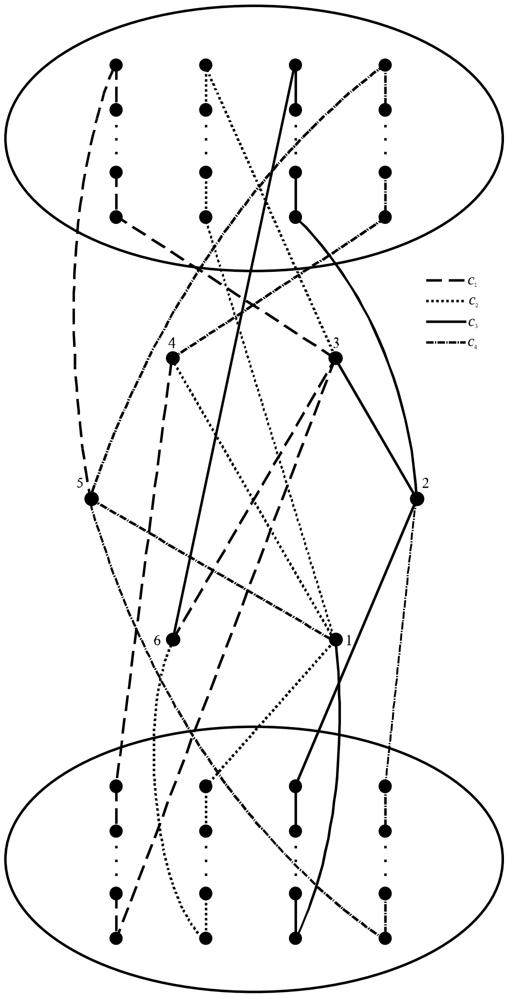

FIGURE 15

Copy 2 : $u = 1$ $v_1 = 4$ $v_2 = 6$ $v_3 = 3$ colored by c_2.
Copy 3 : $u = 2$ $v_1 = 3$ $v_2 = 1$ $v_3 = 6$ colored by c_3.
Copy 4 : $u = 5$ $v_1 = 1$ $v_2 = 2$ $v_3 = 4$ colored by c_4.

Now we partition $n-(m_1+m_2)-6$ other vertices of k_n (except m_1+m_2+6 above vertices) to three sets B_i $i = 1, 2, 3$ such that $|B_i| = n_i$, $n_1 + n_2 + n_3 = n - (m_1 + m_2) - 6$. Then we consider four edge-disjoint trees.
1. $S_{n_1+1}(5, B_1)$, $S_{n_2+1}(4, B_2)$, $S_{n_3+1}(6, B_3)$ colored by c_1.
2. $S_{n_1+1}(3, B_1)$, $S_{n_2+1}(6, B_2)$, $S_{n_3+1}(4, B_3)$ colored by c_2.

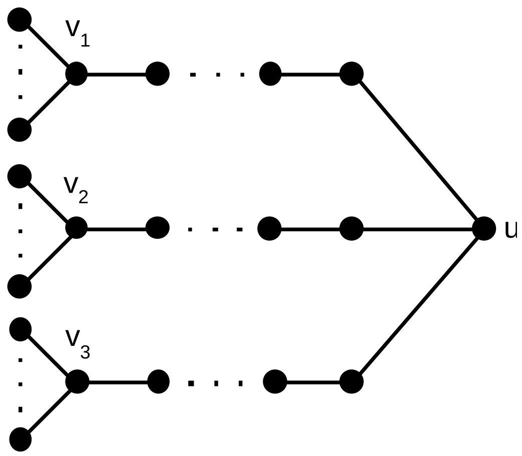

FIGURE 16

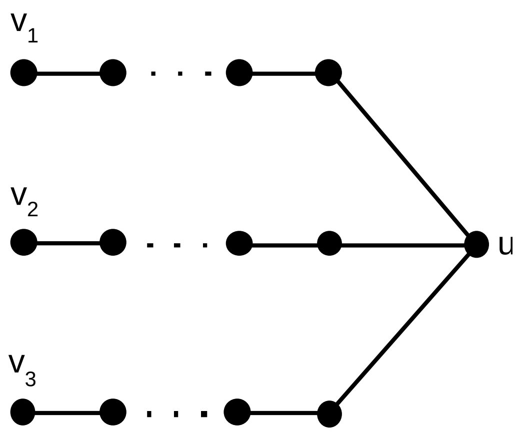

FIGURE 17

3. $S_{n_1+1}(6,B_1)$, $S_{n_2+1}(1,B_2)$, $S_{n_3+1}(3,B_3)$ colored by c_3.
4. $S_{n_1+1}(4,B_1)$, $S_{n_2+1}(2,B_2)$, $S_{n_3+1}(1,B_3)$ colored by c_4.

So four copies of tree in Figure 13 can be packed into K_n.

Proof of Theorem 6. As the proof of Lemma 5 there are a 4-placement of tree in Figure 17 into $K_{m_1+m_2+m_3+6}$ with the following vertices and colors :
Copy 1 : $u = 2$ $v_1 = 4$ $v_2 = 3$ $v_3 = 5$ colored by c_1.
Copy 2 : $u = 6$ $v_1 = 2$ $v_2 = 5$ $v_3 = 3$ colored by c_2.
Copy 3 : $u = 1$ $v_1 = 5$ $v_2 = 2$ $v_3 = 6$ colored by c_3.
Copy 4 : $u = 4$ $v_1 = 3$ $v_2 = 6$ $v_3 = 1$ colored by c_4. Now we partition $n-(m_1+m_2+m_3)-6$ other vertices of K_n (except $m_1+m_2+m_3+6$ above

vertices) to three sets B_i $i = 1, 2, 3$ such that $|B_i| = n_i$, $n_1 + n_2 + n_3 = n - (m_1 - m_2 - m_3) - 6$. Then let :

1. $S_{n_1+1}(4, B_1)$, $S_{n_2+1}(3, B_2)$, $S_{n_3+1}(5, B_3)$ colored by c_1.
2. $S_{n_1+1}(2, B_1)$, $S_{n_2+1}(5, B_2)$, $S_{n_3+1}(3, B_3)$ colored by c_2.
3. $S_{n_1+1}(5, B_1)$, $S_{n_2+1}(2, B_2)$, $S_{n_3+1}(6, B_3)$ colored by c_3.
4. $S_{n_1+1}(3, B_1)$, $S_{n_2+1}(6, B_2)$, $S_{n_3+1}(1, B_3)$ colored by c_4.

Therefore $RT_{n-2}(T_1, T_2, T_3)$ of Figure 15 is 4-placeable and we have proved the main Theorem.

References

[1] B. Bollobás, *Exteremal Graph Theory*, Academic Press, London, 1978.

[2] B. Bollobas and S. E. Eldridge, Packing of graphs and applications to computational complexity, *J. Combin. Theory Ser.* B 25 (1978) 105-124.

[3] A. Gyárfás and J. Lehel, Packing trees of different order into K_n, *Colloq. Math. Soc. J. Bolyai*, 18 (1976), 463-469.

[4] A. M. Hobbas, B. A. Bourgeois and J. Kasiraj, Packing trees in complete graphs, *Discrete Math.*, 67 (1987), 27-42.

[5] H. Kheddouci, S. Marshall, J. S. Sacle and M. Wozniak, On the packing of three graphs, *Discrete Math.*, 236 (2001), 197-225.

[6] N. Sauer and J. Spencer. Edge disjoint placement of graphs, *J. Combin. Theory Ser. B,* 25 (1978), 295-302.

[7] H. Wang and N. Sauer, Packing three copies of a tree into a complete graph, *European J. Combin.*, 14 (1993), 137-142.

[8] M. Wozniak, Embedding graphs of small size, *Discrete Appl. Math.*, 51 (1994), 233-241.

[9] M. Wozniak, Packing three trees, *Discrete Math.*, 150 (1996), 393-402.

[10] H. P. Yap, Packing of graphs-a survey, *Discrete Math.*, 72 (1988), 395-404.

[11] H.Yousefi-Azari and A.Goodarzi, *On the packing of four trees*, Global J. Math. Math. Sci. (to appear).