

Some Results on TVS-cone Normed Spaces and Algebraic Cone Metric Spaces

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ABSTRACT. In this paper we introduce the cone bounded linear mapping and show that the cone norm is continuous. Among other things, we prove the open mapping theorem and the closed graph theorem in TVS-cone normed spaces. We also show that under some restrictions on the cone, two TVS-cone norms are equivalent if and only if they induce equivalent topologies. In the sequel, the notion of algebraic cone metric is introduced and it is shown that every algebraic cone metric space has a topology and the Banach fixed point theorem for contraction mappings on algebraic cone metric spaces is proved.

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1. INTRODUCTION

Ordered normed spaces and cones have applications in applied mathematics and optimization theory [6]. Replacing the real numbers, as the codomain of metrics, by ordered Banach spaces we obtain a generalization of metric spaces. Such generalized spaces called cone metric spaces, were introduced by Rzepecki

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[12]. In that paper, the author introduced not only a generalized metric but also a generalized norm. Several authors attempt to adjust the theory of cone metric to ordinary metric space, by proving the most important standard results on fixed point theory and functional analysis, such as fixed point theorem for contraction mappings. Recently, Erdal Karapinar [9] studied fixed point theorems in cone Banach spaces, and Abdeljawad et al. [14] studied some properties of cone Banach spaces. In [13], Sonmez and Cakalli studied the main properties of cone normed spaces and proved some results in cone normed spaces and complete cone normed spaces. In this paper, we also try to prove some results on these spaces.

Let E be a topological vector space (TVS, for short) with its zero vector θ . A nonempty subset P of E is called a convex cone if $P + P \subseteq E$ and $\lambda P \subseteq P$ for all $\lambda \geq 0$. A convex cone P is said to be pointed if $P \cap (-P) = \{\theta\}$. For a given convex cone P in E , a partial ordering \preceq on E with respect to P is defined by $x \preceq y$ if and only if $y - x \in P$. We shall write $x \prec y$ if $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int}P$, where $\text{int}P$ denotes the topological interior of P .

In the following, unless otherwise specified, we always suppose that E is a locally convex Hausdorff TVS with its zero vector θ , P a proper closed and convex pointed cone in E with $\text{int}P \neq \emptyset$, $e \in \text{int}P$ and \preceq the partial ordering with respect to P .

Recently Wie-Shih Du [15] introduced the notion of TVS-cone metric space and replaced the set of ordered Banach space by locally convex Hausdorff TVS. Now we first recall the concept of topological vector valued cone metric space.

Definition 1.1. Let X be a nonempty set and $d : X \times X \rightarrow E$ be a mapping that satisfies:

- (CM1) For all $x, y \in X$, $d(x, y) \succeq \theta$ and $d(x, y) = \theta$ if and only if $x = y$,
- (CM2) $d(x, y) = d(y, x)$ for all $x, y \in X$,
- (CM3) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a topological vector space valued cone metric (TVS cone metric, for short) on X and (X, d) is said to be a topological vector space valued cone metric space.

Let (X, d) be a TVS-cone metric space, $x \in X$ and $\{x_n\}$ a sequence in X . Then

- (1) $\{x_n\}$ is said to be convergent to x if for every $c \gg \theta$ there exists a positive integer N such that for all $n > N$, $d(x_n, x) \ll c$. We denote this by $\lim_{n \rightarrow \infty} x_n = x$;
- (2) $\{x_n\}$ is said to be a TVS-cone Cauchy sequence if for every $c \gg \theta$ there

exists a positive integer N such that for all $m, n > N$, $d(x_n, x_m) \ll c$;
 (3) (X, d) is called a complete TVS-cone metric space if every Cauchy sequence is convergent.

Remark 1. Cone metric spaces in the Huang-Zhang sense [7] are included in the above definition. Indeed, every Banach space is a locally convex Hausdorff topological vector space.

Let (X, d) be a TVS-cone metric space, $c \in \text{int}P$ and $x \in X$. Define $B(x, c) := \{y \in X : d(x, y) \ll c\}$. It is well-known that the collection

$$\tau_c = \{U \subseteq X : \forall x \in U, \exists c \in \text{int}P, B(x, c) \subseteq U\}$$

is a topology for X under which the above definitions of convergent and Cauchy sequences and it is called TVS-cone metric topology.

The nonlinear scalarization function $\xi_e : E \rightarrow \mathbb{R}$ is defined as follows:

$$\xi_e(y) = \inf\{r \in \mathbb{R} : y \in re - P\}$$

for all $y \in E$.

Lemma 1.2. [5] *Let $r \in \mathbb{R}$ and $y \in E$, then*

- (i) $\xi_e(y) \leq r \Leftrightarrow re - y \in P$;
- (ii) $\xi_e(y) < r \Leftrightarrow re - y \in \text{int}P$;
- (iii) $\xi_e(\cdot)$ is positively homogeneous and continuous on E ;
- (iv) If $y_1 \in y_2 + P$, then $\xi_e(y_2) \leq \xi_e(y_1)$;
- (v) $\xi_e(y_1 + y_2) \leq \xi_e(y_1) + \xi_e(y_2)$.

In [2], authors proved that $d_e = \xi_e \circ d$ is a metric on X which induces the same topology on X as the TVS-cone metric topology induced by d .

2. TVS-CONE NORMED SPACES

In this section we first recall the concept of topological vector space valued cone normed space.

Definition 2.1. Let X be a vector space over F (\mathbb{R} or \mathbb{C}) and $\|\cdot\|_c : X \rightarrow E$ be a mapping that satisfies:

- (CN1) $\|x\|_c \succeq \theta$ for all $x \in X$ and $\|x\|_c = \theta$ if and only if $x = \theta_X$, where θ_X is the zero vector in X ,
- (CN2) $\|\alpha x\|_c = |\alpha| \|x\|_c$ for all $x \in X$ and $\alpha \in F$,
- (CN3) $\|x + y\|_c \preceq \|x\|_c + \|y\|_c$.

Then $\|\cdot\|_c$ is called a cone norm on X and $(X, \|\cdot\|_c)$ is called a cone normed

space.

It is clear that each cone normed space is a cone metric space. In fact, the cone metric is given by $d(x, y) = \|x - y\|_c$.

The authors in [2] proved that $\|\cdot\| = \xi_e \circ \|\cdot\|_c$ is a real valued norm on X . In fact, they proved that every TVS-cone normed space $(X, \|\cdot\|_c)$ is normable in the usual sense, i.e. the topology induced by topological vector space valued cone norm coincides with the topology induced by the norm obtained via a nonlinear scalarization function.

Definition 2.2. Let X be a vector space, $\|\cdot\|_{c_1} : X \rightarrow E$ and $\|\cdot\|_{c_2} : X \rightarrow E$ be two TVS-cone norms on X . $\|\cdot\|_{c_1}$ is said to be equivalent to $\|\cdot\|_{c_2}$ if there exist $\alpha, \beta > 0$ such that

$$\alpha\|x\|_{c_1} \preceq \|x\|_{c_2} \preceq \beta\|x\|_{c_1},$$

for each $x \in X$.

Theorem 2.3. Let X be a vector space. If $\|\cdot\|_{c_1}$ and $\|\cdot\|_{c_2}$ are two equivalent TVS-cone norms on X , then $\tau_{c_1} = \tau_{c_2}$. Moreover, the converse is valid if all elements of P are comparable. (i.e for all $c_1, c_2 \in P$, $c_1 \preceq c_2$ or $c_2 \preceq c_1$).

Proof. Fix $e \in \text{int}P$ and suppose that $\|\cdot\|_i = \xi_e(\|\cdot\|_{c_i})$, $i = 1, 2$. We know there exist $\alpha, \beta > 0$ such that $\alpha\|x\|_{c_1} \preceq \|x\|_{c_2} \preceq \beta\|x\|_{c_1}$, for each $x \in X$, also by Lemma 1.2(iv), ξ_e is an increasing function on E , thus

$$\alpha\|x\|_1 \leq \|x\|_2 \leq \beta\|x\|_1$$

for each $x \in X$. Hence, $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent norms on X , so they induce same topology on X . On the other hand, $\|\cdot\|_i$ induces τ_{c_i} , $i = 1, 2$. Therefore, $\tau_{c_1} = \tau_{c_2}$.

Conversely, let $\tau_{c_1} = \tau_{c_2}$, then $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent norms on X . Therefore, there exist scalars $\alpha, \beta > 0$ such that $\alpha\|x\|_1 \leq \|x\|_2 \leq \beta\|x\|_1$, for each $x \in X$. So we have

$$\alpha\xi_e(\|x\|_{c_1}) \leq \xi_e(\|x\|_{c_2}) \leq \beta\xi_e(\|x\|_{c_1})$$

for each $x \in X$. On the other hand, the elements of P are comparable with each other and ξ_e is increasing on E , hence

$$\alpha\|x\|_{c_1} \preceq \|x\|_{c_2} \preceq \beta\|x\|_{c_1}$$

for each $x \in X$. □

Definition 2.4. Let $(X, \|\cdot\|_c)$ and $(Y, \|\cdot\|_c)$ be two TVS-cone normed spaces and T be a linear map from X into Y . T is called a cone bounded linear map if there exists $M > 0$ such that $\|Tx\|_c \preceq M\|x\|_c$ for all $x \in X$. We denote by

$\|T\|$ the infimum of such M , i.e. $\|T\| = \inf\{M > 0 : \|Tx\|_c \preceq M\|x\|_c\}$.

Example 2.5. Let $E = \ell^1$ and $P = \{\{x_n\} \in \ell^1 : x_n \geq 0, \forall n\}$. Then P is a cone in E . Put $X = C^1[0, 1]$ and $Y = C[0, 1]$. Moreover, define $\|\cdot\|_{c_1} : X \rightarrow E$ and $\|\cdot\|_{c_2} : Y \rightarrow E$ as follows:

$$\|f\|_{c_1} = \left\{ \frac{\|f\|_1}{2^n} \right\}_{n=1}^{\infty}, \quad \|g\|_{c_2} = \left\{ \frac{\|g\|_2}{2^n} \right\}_{n=1}^{\infty}$$

where $\|f\|_1 = \|f\|_{\infty} + \|f'\|_{\infty}$ and $\|g\|_2 = \|g\|_{\infty}$ for $f \in X = C^1([0, 1])$ and $g \in Y = C([0, 1])$. Obviously, $\|\cdot\|_{c_1}$ and $\|\cdot\|_{c_2}$ are two cone norms on X and Y respectively. Now define $T : (X, \|\cdot\|_{c_1}) \rightarrow (Y, \|\cdot\|_{c_2})$ by $Tf = f'$. Therefore, $\|Tf\|_{c_2} \preceq \|f\|_{c_1}$ implies that T is a cone bounded linear map.

Note that every cone bounded linear map $T : X \rightarrow Y$ is continuous, since any TVS-cone normed space is a normable space with respect to $\|\cdot\| = \xi_e \circ \|\cdot\|_c$, so by lemma 1.2(iv), $\|Tx\|_c \preceq M\|x\|_c$ implies that $\|Tx\| \leq M\|x\|$. Hence, T is a bounded linear map. Therefore it is continuous and so we have the following cone normed version of open mapping theorem.

Theorem 2.6. (Open mapping theorem). *Let $(X, \|\cdot\|_c)$, $(Y, \|\cdot\|_c)$ be two complete TVS-cone normed spaces and $T : X \rightarrow Y$ be a surjective cone-bounded linear map, then T is an open mapping (i.e $T(G)$ is an open set in (Y, τ_c) whenever G is an open set in (X, τ_c)).*

Theorem 2.7. (The inverse mapping theorem). *If X and Y are two cone Banach spaces and $T : X \rightarrow Y$ is a bijective cone-bounded linear map, then $T^{-1} : Y \rightarrow X$ is continuous.*

Remark 2. Let $(X, \|\cdot\|_{c_1})$ and $(Y, \|\cdot\|_{c_2})$ be two complete TVS-cone normed spaces, then the vector space $X \times Y$ is a TVS-cone normed space by the following cone norm:

$$\|(x, y)\|_c = \|x\|_{c_1} + \|y\|_{c_2}$$

it is easy to check that the TVS-cone normed space $(X \times Y, \|\cdot\|_c)$ is complete.

Theorem 2.8. (The closed graph theorem). *If X and Y are two complete TVS-cone normed spaces and $T : X \rightarrow Y$ is a linear map such that the graph of T*

$$Gr(T) = \{(x, Tx) \in X \times Y : x \in X\}$$

is closed, then T is continuous.

Let E be a Banach space and P be a cone in E . The cone P is called normal if there exists a constant $K > 0$ such that for all $a, b \in P$, $a \preceq b$ implies that

$\|a\| \leq K\|b\|$. The least positive number satisfying the above inequality is called the normal constant of P .

In the sequel of this section we suppose that P is a normal cone in a Banach space E .

Let $\{a_n\}, \{b_n\}$ be arbitrary sequences in E such that $\theta \preceq a_n \preceq b_n$, for each $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} b_n = \theta$, then by normality of P we have $\lim_{n \rightarrow \infty} a_n = \theta$. Consequently, sandwich theorem holds in ordered Banach space E when P is a normal cone. The following example shows that in general sandwich theorem does not hold.

Let $E = \ell^2(\mathbb{R})$ and $P = \{x \in \ell^2(\mathbb{R}) : \sum_{k=1}^n x_k \geq 0, \forall n\}$. Then P is a cone in E (see [8]). Put

$$u_n = \frac{1}{\sqrt{2n}} \underbrace{(1, -1, 1, -1, \dots, 1, -1, 0, 0, \dots)}_{2n \text{ terms}}$$

and

$$v_n = \frac{1}{\sqrt{2n}} \underbrace{(0, 1, -1, 1, -1, \dots, 1, -1, 0, 0, \dots)}_{2n \text{ terms}}.$$

Clearly $\{u_n\}$ and $\{v_n\}$ are two sequences in P . Then $0 \preceq u_n \preceq u_n + v_n$, $\|u_n\|_2 = 1$ and $\|u_n + v_n\|_2 = \frac{1}{\sqrt{n}} \rightarrow 0$ as $n \rightarrow \infty$.

Theorem 2.9. *Suppose that $(X, \|\cdot\|_c)$ is a cone normed space and τ_c is the cone topology on X . Define $f : X \rightarrow E$ by $f(x) = \|x\|_c$, then f is $(\tau_c, \|\cdot\|)$ -continuous.*

Proof. Let $\{x_n\} \subseteq X$, $x \in X$ and $\|x_n - x\|_c \rightarrow \theta$ as $n \rightarrow \infty$. Then, by (CN3), we have

$$-\|x_n - x\|_c \preceq \|x_n\|_c - \|x\|_c \preceq \|x_n - x\|_c.$$

It follows from the sandwich theorem that $\lim_{n \rightarrow \infty} \|x_n\|_c = \|x\|_c$ in E . \square

3. ALGEBRAIC CONE METRIC SPACES

In this section we introduce the concept of algebraic cone metric space and some of its elementary properties is studied and also a fixed point theorem in this space is proved.

Let E be a real vector space and $L \subseteq E$ be a subspace of E . The translation $A = u + L$ where $u \in E$ is called an affine subspace of E . The dimension of A is defined as the dimension of L . If $\dim A = 1$, then the set A is called a straight line. A straight line can be written in the parametric form $A = \{u + rv : r \in \mathbb{R}\}$,

where $u, v \in E$ and $v \neq \theta$.

Definition 3.1. [11] Let E be a real vector space and P be a convex subset of E . A point $x \in P$ is said to be an algebraic interior point of P if for each $v \in E$ there exists $\epsilon > 0$ such that $x + tv \in P$, for all $t \in [0, \epsilon]$.

Note that the above definition is equivalent to the following statement:

A point x is an algebraic interior point of the convex set $P \subseteq E$ if $x \in P$ and for each $v \in E$ there exists $\epsilon > 0$ such that $[x, x + \epsilon v] \subseteq P$, where $[x, x + \epsilon v] = \{\lambda x + (1 - \lambda)(x + \epsilon v) : \forall \lambda \in [0, 1]\}$.

The set of all algebraic interior points of P is called its algebraic interior and is denoted by $\text{aint}P$. Moreover, P is called algebraically open if $P = \text{aint}P$, equivalently, P is called algebraically open if its intersection with every straight line in E is an open interval (possibly empty). For example every convex open set in \mathbb{R}^d is algebraically open.

Suppose that E is a real vector space and $P \subseteq E$ is a convex non-empty set such that $P \cap (-P) = \{\theta\}$, $\lambda P \subseteq P$ ($\lambda \geq 0$), $P + P \subseteq P$ and $P \neq \{\theta\}$. In this case we will say that P is an algebraic cone in E and the partial ordering on E with respect to P is denoted by \preceq_a . Moreover, we will write $x \ll_a y$ if and only if $y - x \in \text{aint}P$ and we say that P has the Archimedean property if for each $x, y \in P$ there exists $n \in \mathbb{N}$ such that $x \preceq_a ny$.

One can easily see that $P = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_i \geq 0, i = 1, 2, \dots, n\}$ is an algebraic cone in \mathbb{R}^n with the Archimedean property. Also $\{f \in C_{\mathbb{R}}[a, b] : f(x) \geq 0, \forall x \in [a, b]\}$ is an algebraic cone with the Archimedean property in the real vector space $C_{\mathbb{R}}[a, b]$. But there exists a real vector space with an algebraic cone which does not have the Archimedean property. For example, in the real vector space $C_{\mathbb{R}}(0, \infty)$ if we consider the algebraic cone $P = \{f \in C_{\mathbb{R}}(0, \infty) : f(x) \geq 0, x \in (0, \infty)\}$ then it is easy to see that P does not have the Archimedean property. Indeed, if $f(x) = x^2$ and $g(x) = \frac{1}{x}$ then there is not any $n \in \mathbb{N}$ such that $g(x) \leq nf(x)$, for all $x \in (0, \infty)$.

Lemma 3.2. Let E be a real vector space and P be an algebraic cone in E with non-empty algebraic interior. Then

- (i) $P + \text{aint}P \subset \text{aint}P$;
- (ii) $\alpha \text{aint}P \subset \text{aint}P$, for each scalar $\alpha > 0$;
- (iii) For any $x, y, z \in X$, $x \preceq_a y$ any $y \ll_a z$ implies that $x \ll_a z$.

Proof. (i). Let $x \in \text{aint}P$, $y \in P$ and v be an arbitrary element in E . By definition of algebraic point, there exists $\epsilon > 0$ such that $[x, x + \epsilon v] \subseteq P$, so for each λ , $0 \leq \lambda \leq 1$, $\lambda x + (1 - \lambda)(x + \epsilon v) \in P$. Thus we have $\lambda x + (1 - \lambda)(x +$

$\epsilon v) + y \in P$, since $P + P \subset P$ and $y \in P$. Hence for each $\lambda \in [0, 1]$,

$$\begin{aligned} \lambda(x + y) + (1 - \lambda)((x + y) + \epsilon v) &= \lambda x + (1 - \lambda)(x + \epsilon v) + \lambda y + (1 - \lambda)y \\ &= \lambda x + (1 - \lambda)(x + \epsilon v) + y \in P. \end{aligned}$$

Therefore the proof of (i) is complete.

(ii). Let $x \in \text{aint}P$, $\alpha > 0$ and v be an arbitrary element in E . By definition of algebraic point, there exists $\epsilon > 0$ such that $x + t\frac{v}{\alpha} \in P$, for all $t \in [0, \epsilon]$, hence $\alpha x + tv \in P$, for all $t \in [0, \epsilon]$.

(iii) is trivial by using (i). \square

Definition 3.3. Let X be a nonempty set, P be an algebraic cone in E with non-empty algebraic interior and $d_a : X \times X \rightarrow E$ be a vector-valued function that satisfies:

(ACM1) For all $x, y \in X$, such that $x \neq y$, $\theta \ll_a d_a(x, y)$ and $d_a(x, y) = \theta$ if and only if $x = y$,

(ACM2) $d_a(x, y) = d_a(y, x)$ for all $x, y \in X$,

(ACM3) $d_a(x, y) \preceq_a d_a(x, z) + d_a(z, y)$ for all $x, y, z \in X$.

Then d_a is called an algebraic cone metric on X and (X, d_a) is said to be an algebraic cone metric space.

Theorem 3.4. Let (X, d_a) be an algebraic cone metric space. Then the collection $\{B_a(x, c) : c \in \text{aint}P, x \in X\}$ forms a subbasis for a Hausdorff topology on X , where $B_a(x, c) := \{y \in X : d_a(x, y) \ll_a c\}$.

Proof. Trivially $\bigcup_{x \in X, c \in \text{aint}P} B_a(x, c) = X$, so the collection $\{B_a(x, c) : c \in \text{aint}P, x \in X\}$ forms a subbasis for a topology on X . Now we show that this topology is Hausdorff. Let $x, y \in X$ and $x \neq y$, take $\theta \ll_a c = d_a(x, y)$. The facts that $P \cap (-P) = \{\theta\}$ and d_a has the property (ACM3) imply that $B_a(x, \frac{c}{3}) \cap B_a(y, \frac{c}{3}) = \emptyset$. Therefore the topology induced by the above collection is Hausdorff. \square

Let (X, d_a) be an algebraic cone metric space, $x \in X$ and $\{x_n\}$ be a sequence in X . Then one can prove that if $\{x_n\}$ converges to x with respect to the topology of X then for every $\theta \ll_a c$ there exists a positive integer N such that for all $n > N$, $d_a(x_n, x) \ll_a c$.

Definition 3.5. A sequence $\{x_n\}$ is called a Cauchy sequence if for every $\theta \ll_a c$ there exists a positive integer N such that for all $m, n > N$, $d_a(x_n, x_m) \ll_a c$. Moreover, (X, d_a) is said to be a complete algebraic cone metric space if every Cauchy sequence is convergent.

In the sequel of this section we suppose that P has the Archimedean property.

The Banach fixed point theorem is an important tool in the theory of metric spaces, for more information one can see ([1], [3], [4],[10]). Now we prove the Banach fixed point theorem in this framework.

Theorem 3.6. *Let (X, d_a) be a complete algebraic cone metric space. Suppose that a mapping $T : X \rightarrow X$ satisfies the contractive condition*

$$d_a(Tx, Ty) \preceq_a \lambda d_a(x, y),$$

for all $x, y \in X$, where $\lambda \in (0, 1)$ is a constant. Then T has a unique fixed point in X . Moreover, for each $x \in X$, the iterative sequence $T^n x$ converges to the fixed point.

Proof. Choose $x_0 \in X$. Set $x_1 = Tx_0, x_2 = Tx_1 = T^2x_0, \dots, x_{n+1} = Tx_n = T^{n+1}x_0$. We have:

$$\begin{aligned} d_a(x_{n+1}, x_n) &= d_a(Tx_n, Tx_{n-1}) \preceq_a \lambda d_a(x_n, x_{n-1}) \\ &\preceq_a \lambda^2 d_a(x_{n-1}, x_{n-2}) \preceq_a \dots \preceq_a d_a(x_1, x_0). \end{aligned}$$

So for $n > m$,

$$\begin{aligned} d_a(x_n, x_m) &\preceq_a d_a(x_n, x_{n-1}) + d_a(x_{n-1}, x_{n-2}) + \dots + d_a(x_{m+1}, x_m) \\ &\preceq_a (\lambda^{n-1} + \lambda^{n-2} + \dots + \lambda^m) d_a(x_1, x_0) \preceq_a \frac{\lambda^m}{1-\lambda} d_a(x_1, x_0). \end{aligned}$$

But P has the Archimedean property, so for each $\theta \ll_a c$ there exists $\epsilon > 0$ such that $\epsilon d_a(x_1, x_0) \ll_a c$. Hence, there exists $N \in \mathbb{N}$ such that for each $m > N$, we have

$$\frac{\lambda^m}{1-\lambda} d_a(x_1, x_0) \ll_a c.$$

Therefore, Lemma 3.2(iii) and $d_a(x_n, x_m) \preceq_a \frac{\lambda^m}{1-\lambda} d_a(x_1, x_0)$, for $n > m$, imply that $\{x_n\}$ is a Cauchy sequence. By the completeness of X , there exists $x^* \in X$ such that $x_n \rightarrow x^*$ ($n \rightarrow \infty$). We know

$$d_a(Tx^*, x^*) \preceq_a d_a(Tx_n, Tx^*) + d_a(Tx_n, x^*) \preceq_a \lambda d_a(x_n, x^*) + d_a(x_{n+1}, x^*),$$

so convergence of $\{x_n\}$ to x^* and the fact that the topology of X is Hausdorff imply that $Tx^* = x^*$. The uniqueness of fixed point of T is clearly obtained by the facts $P \cap (-P) = \{\theta\}$ and $\lambda \in (0, 1)$. □

Corollary 3.7. *Let $\alpha \in \mathbb{R}$ with $\alpha > 1$ and let (X, d_a) be a complete algebraic cone metric space, $T : X \rightarrow X$ be an onto mapping which satisfies the condition*

$$\alpha d_a(x, y) \preceq_a d_a(Tx, Ty).$$

Then T has a unique fixed point.

Proof. Let $x \neq y$ and $Tx = Ty$, then by assumption, one can observe $\alpha d_a(x, y) \preceq_a \theta$ which is a contradiction, since $P \cap (-P) = \{\theta\}$. Thus, T is one-to-one and it has an inverse, say S . Hence,

$$\alpha d_a(x, y) \preceq_a d_a(Tx, Ty) \Leftrightarrow d_a(Sx, Sy) \preceq_a \frac{1}{\alpha} d_a(x, y).$$

Therefore, by the above theorem, S has a unique fixed point and so T has a unique fixed point. \square

The above corollary has been already proved in the case that X is a cone metric space and P is a normal cone (see [9]).

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REFERENCES

1. S. Banach, Sur les operations dans les ensembles abstraits et leurs applications aux equations integrals, *Fund. Math.*, **3**, (1922), 133-181.
2. H. Cakalli, A. Sonmez, C. Genc, On an equivalence of topological vector space valued cone metric spaces and metric spaces, *Applied Mathematics Letters*, **25**, (2012), 429-433.
3. S. Chandok and T. D. Narang, Common fixed points and invariant approximations for C_q -commuting Generalized nonexpansive mappings, *Iranian Journal of Mathematical Sciences and Informatics*, **7** (1), (2012), 21-34.
4. S. K. Chatterjea, Fixed-point theorems, *C. R. Acad. Bulgare Sci.*, **25**, (1972), 727-730.
5. G. Y. Chen, X. X. Huang, X. Q. Yang, *Vector Optimization*, Springer-Verlag, Berlin, Heidelberg, 2005.
6. K. Deimling, *Nonlinear Functional Analysis*, Springer, Berlin, 1985.
7. L. G. Huang and X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, *J. Math. Anal. Appl.*, **323**, (2007), 1468-1476.
8. A. Iusem, A. Seeger, Normality and modulability indices. Part II: Convex cones in Hilbert spaces, *J. Math. Anal. Appl.*, **338**, (2008), 392-406.
9. E. Karapinar, Fixed point theorems in cone Banach spaces, *Fixed Point Theory and Applications*, **2009**, (2009), 1-9.
10. S. Moradi and A. Beiranvand, Fixed point of T_F -contractive single valued mappings, *Iranian Journal of Mathematical Sciences and Informatics*, **5**(2), (2010), 25-32.
11. N. K. Nikolskij, *Functional Analysis I*, Springer-Verlag, Berlin, Heidelberg, 1992.
12. B. Rzepecki, On fixed point theorems of Maia type, *Publications de l'Institut Mathematique*, **28**(42), (1980), 179-186.
13. A. Sonmez, H. Cakalli, Cone normed spaces and weighted means, *Mathematical and Computer Modelling*, **52**, (2010), 1660-1666.
14. D. Turkoglu, M. Abuloha, T. Abdeljawad, Some theorems and examples of cone Banach spaces, *Journal of Computational Analysis and Applications*, **12**(4), (2010), 739-753.
15. Wei-Shih Du, A note on cone metric fixed point theory and it's equivalence, *Nonlinear Analysis*, **72**, (2010), 2259-2261.