Iranian Journal of Mathematical Sciences and Informatics Vol. 7, No. 2 (2012), pp 83-91 DOI: 10.7508/ijmsi.2012.02.008

Weakly g(x)-Clean Rings

Nahid Ashrafi* and Zahra Ahmadi

Department of Mathematics, Faculty of Mathematics, Statistics and Computer Sciences, Semnan University, Semnan, Iran

> E-mail: nashrafi@semnan.ac.ir E-mail: zahmadiv@yahoo.com

Abstract. A ring R with identity is called "clean" if for every element $a \in R$, there exist an idempotent e and a unit u in R such that a = u + e. Let C(R) denote the center of a ring R and g(x) be a polynomial in C(R)[x]. An element $r \in R$ is called "g(x)-clean" if r = u + s where g(s) = 0 and u is a unit of R and R is g(x)-clean if every element is g(x)-clean. In this paper we define a ring to be weakly g(x)-clean if each element of R can be written as either the sum or difference of a unit and a root of g(x).

Keywords: Clean ring, g(x)-clean ring, Weakly g(x)-clean ring.

2000 Mathematics subject classification: 16U60, 16U99, 13A99

1. INTRODUCTION

Throughout this note, R is an associative ring with identity. A ring R is called clean if for every element $a \in R$, there exist an idempotent e and a unit u in R such that a = e + u [9] and R is called strongly clean if, in addition, eu = ue [10].

Let C(R) denote the center of a ring R and g(x) be a polynomial in C(R)[x]. Following Camillo and Simon [2], an element $r \in R$ is called g(x)-clean if r = u + s where g(s) = 0 and u is a unit of R, and R is g(x)-clean if every element in R is g(x)-clean. It is clear that the $(x^2 - x)$ - clean rings are precisely

^{*}Corresponding Author

Received 15 November 2011; Accepted 10 March 2012

 $[\]textcircled{C}2012$ Academic Center for Education, Culture and Research TMU

⁸³

the clean rings.

Camillo and Simon [2] proved that if V is a countable dimensional vector space over a division ring D and g(x) is any polynomial with coefficients in K = C(D)and two distinct roots in K, then $End(V_D)$ is g(x)-clean. Nicholson and Zhou [11] generalized Camillo and Simon's result by proving that $End(_RM)$ is g(x)clean where $_RM$ is a semisimple left R-module and $g(x) \in (x-a)(x-b)C(R)[x]$ with $a, b \in C(R)$ and $b, b - a \in U(R)$. g(x)-clean rings have also been studied in [3], [7] and [6].

It is easy to see that a ring R is g(x)-clean if and only if each $x \in R$ can be written in the form x = u - s where $u \in U(R)$ and g(s) = 0. This raises the question of whether a ring with the property that, for each $x \in R$, either x = u + s or x = u - s for some $u \in U(R)$ and g(s) = 0 must be cleaned. Let us call rings with this property weakly g(x)-clean. Here we study weakly g(x)-clean rings and also investigate the general properties of weakly g(x)-clean rings which are similar to those of g(x)-clean rings. For example we prove the following results:

Proposition 1.1. Let $g(x) \in \mathbb{Z}[x]$ and $\{R_i\}_{i \in I}$ be a family of rings. Then $\prod_{i \in I} R_i$ is weakly g(x)-clean if and only if for all $i \in I$, R_i is weakly g(x)-clean.

Theorem 1.2. Let R be a ring, $g(x) \in C(R)[x]$, and $n \in \mathbb{N}$. Then R is weakly g(x)-clean if and only if the upper triangular matrix ring $\mathbb{T}_n(R)$ is weakly g(x)-clean.

Theorem 1.3. Let R be a commutative ring and M an R-module. Let $g(x) \in C(R)[x]$. If R is weakly g(x)-clean, then the idealization R(M) of R and M is also weakly g(x)-clean.

In section 3 we consider the weakly $(x^n - x)$ -clean rings and weakly 2-clean rings.

An usual, $\mathbb{T}_n(R)$ denotes the upper triangular matrix ring of order n over R; $GL_n(R)$ denotes the general linear group over R; and gcd(m, n) means the greatest common divisor of the integers m and n. All polynomials are in the polynomial ring C(R)[x] and U(R) denotes the multiplicative unit group of R.

2. Weakly g(x)-clean rings

In this section first we define the weakly g(x)-clean rings, then we explain the relation between weakly g(x)-clean and g(x)-clean rings.

Definition 2.1. Let g(x) be a fixed polynomial in C(R)[x]. An element $r \in R$ is called weakly g(x)-clean if r = u + s or r = u - s where g(s) = 0 and

84

 $u \in U(R)$. We say that R is weakly g(x)-clean if every element is weakly g(x)-clean.

Obviously, g(x)-clean rings are weakly g(x)-clean and also if g(x) is an odd or an even polynomial (i.e g(-x) = -g(x) or g(-x) = g(x)), then the concepts g(x)-clean and weakly g(x)-clean coincide, that is, if R is a weakly g(x)-clean ring then R is also g(x)-clean. So the interesting case is when g(x) is neither an even nor an odd polynomial. In [1, Proposition 16] it was shown that if Rhas exactly two maximal ideals and $2 \in U(R)$, then each $x \in R$ has the form x = u + e or x = u - e where $u \in U(R)$ and $e \in \{0, 1\}$. Thus $\mathbb{Z}_{(3)} \cap \mathbb{Z}_{(5)}$ is weakly clean but is not clean since an indecomposable clean ring is quasilocal [1, Thedrem 3]. But since weakly $(x^2 - x)$ -clean rings are precisely the weakly clean rings, we can say that for $g(x) = x^2 - x$, the ring $\mathbb{Z}_{(3)} \cap \mathbb{Z}_{(5)}$ is weakly g(x)-clean, but it is not g(x)-clean.

The following two examples explain the relations between weakly g(x)-clean rings and weakly clean rings.

Example 2.2. Let $R = \mathbb{Z}_{(p)} = \{\frac{m}{n} ; gcd(p,n) = 1 \text{ and } p \text{ prime}\}$ be the localization of \mathbb{Z} at the prime ideal $p\mathbb{Z}$ and $g(x) = (x-a)(x^2+1) \in C(R)[x]$. Then R is a weakly clean ring, because local rings are strongly clean, thus R is clean (it is of course weakly clean). But as a is the single root of g(x), R is not a weakly g(x)-clean ring.

Example 2.3. Let R be a Boolean ring with the number of elements |R| > 2 and $c \in R$ with $0 \neq c \neq 1$. Define g(x) = (x+1)(x+c). Then R is not weakly g(x)-clean.

Because if $c = u \pm s$ where $u \in U(R)$ and g(s)=0, then it must be that u = 1 and $s = \pm (c \pm u)$. But, clearly, $g(c+1) \neq 0$. However, R is certainly weakly clean.

Let R and S be rings and $\theta: C(R) \longrightarrow C(S)$ be a ring homomorphism with $\theta(1) = 1$. Then θ induces a map θ' from C(R)[x] to C(S)[x] such that For $g(x) = \sum_{i=0}^{n} a_i x^i \in C(R)[x], \, \theta'(g(x)) := \sum_{i=0}^{n} \theta(a_i) x^i \in C(S)[x]$. Clearly, if g(x) is a polynomial with coefficients in \mathbb{Z} then $\theta'(g(x)) = g(x)$. We give some prop

a polynomial with coefficients in \mathbb{Z} , then $\theta'(g(x)) = g(x)$. We give some properties of weakly g(x)-clean rings which are similar to those of weakly clean rings.

Proposition 2.4. Let $\theta : R \longrightarrow S$ be a ring epimorphism. If R is weakly g(x)-clean, then S is weakly $\theta'(g(x))$ -clean.

Proof. Let $g(x) = a_0 + a_1x + ... + a_nx^n \in C(R)[x]$. Then $\theta'(g(x)) = \theta(a_0) + \theta(a_1)x + ... + \theta(a_n)x^n \in C(S)[x]$. As θ is a ring epimorphism so for any $s \in S$, there exists $r \in R$ such that $\theta(r) = s$. Since R is weakly g(x)-clean, there

exist $u \in U(R)$ and $s_0 \in R$ such that $r = u \pm s_0$ and $g(s_0) = 0$. Then $s = \theta(r) = \theta(u \pm s_0) = \theta(u) \pm \theta(s_0)$ with $\theta(u) \in U(S)$. But $\theta'(g(\theta(s_0))) = \theta(a_0) + \theta(a_1)\theta(s_0) + \ldots + \theta(a_n)\theta(s_0^n) = \theta(a_0 + a_1s_0 + \ldots + a_ns_0^n) = \theta(g(s_0)) = \theta(0) = 0$, we have s is weakly $\theta'(g(x))$ -clean. Therefore S is weakly $\theta'(g(x))$ -clean.

Corollary 2.5. If R is weakly g(x)-clean, then for any ideal I of R, R/I is weakly $\bar{g}(x)$ -clean where $\bar{g}(x) \in C(R/I)[x]$.

Proposition 2.6. Let $g(x) \in \mathbb{Z}[x]$ and $\{R_i\}_{i \in I}$ be a family of rings. Then $\prod_{i \in I} R_i$ is weakly g(x)-clean if and only if for all $i \in I$, R_i is weakly g(x)-clean.

Proof. Let $\prod_{i \in I} R_i$ be a weakly g(x)-clean. Define $\pi_j : \prod_{i \in I} R_i \longrightarrow R_j$ by $\pi_j(\{a_i\}_{i \in I}) = a_j$. Since for all $j \in I$, π_j is a ring epimorphism, so by Proposition 2, for every

 $i \in I$, each R_i is a weakly g(x)-clean ring. For the converse, let $x = \{x_i\}_{i \in I} \in R = \prod_{i \in I} R_i$. In R_{i_0} , we can write $x_i = u_{i_0} + s_{i_0}$ or $x_i = u_{i_0} - s_{i_0}$ where $u_{i_0} \in U(R_{i_0})$ and $g(s_{i_0}) = 0$. If $x_{i_0} = u_{i_0} + s_{i_0}$, for $i \neq i_0$, let $x_i = u_i + s_i$ where $u_i \in U(R_i)$, $g(s_i) = 0$; while if $x_{i_0} = u_{i_0} - s_{i_0}$, for $i \neq i_0$, let $x_i = u_i - s_i$ where $u_i \in U(R_i)$, $g(s_i) = 0$. Then $u = \{u_i\}_{i \in I} \in U(R)$ and

$$g(s = \{s_i\}_{i \in I}) = a_0\{1_{R_i}\}_{i \in I} + a_1\{s_i\}_{i \in I} + \dots + a_n\{s_i^n\}_{i \in I}$$
$$= \{a_0\}_{i \in I} + \{a_1s_i\}_{i \in I} + \dots + \{a_ns_i^n\}_{i \in I}$$
$$= \{a_0 + a_1s_i + \dots + a_ns_i^n\}_{i \in I}$$
$$= \{g(s_i)\}_{i \in I} = 0$$

That is, $\prod_{i \in I} R_i$ is weakly g(x)-clean.

Define $\pi_n : C(R) \longrightarrow M_n(R)$ by $a \longmapsto aI_n$ with I_n being the identity matrix of $M_n(R)$ and $a \in C(R)$. Then $M_n(R)$ is a C(R)-algebra.

Theorem 2.7. Let R be a ring, $g(x) \in C(R)[x]$, and $n \in \mathbb{N}$. Then R is weakly g(x)-clean if and only if the upper triangular matrix ring $\mathbb{T}_n(R)$ is weakly g(x)-clean.

Proof. Let R be weakly g(x)-clean and $A = (a_{ij}) \in \mathbb{T}_n(R)$ with $a_{ij} = 0$ for $1 \leq j < i \leq n$. Since R is weakly g(x)-clean, for any $1 \leq i \leq n$, there exist $s_{ii} \in R$ and $u_{ii} \in U(R)$ such that $a_{ii} = u_{ii} \pm s_{ii}$ with $g(s_{ii}) = 0$. So we have

Weakly g(x)-clean rings

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix} = \begin{bmatrix} u_{11} \pm s_{11} & a_{12} & \dots & a_{1n} \\ 0 & u_{22} \pm s_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & u_{nn} \pm s_{nn} \end{bmatrix}$$

In R for any $0 \le i \le n$, we can write $a_{ii} = u_{ii} + s_{ii}$ or $a_{ii} = u_{ii} - s_{ii}$ where $u_{ii} \in U(R)$ and $g(s_{ii}) = 0$. If $a_{ii} = u_{ii} + s_{ii}$ for $j \ne i$, let $a_{jj} = u_{jj} + s_{jj}$ where $(u_{jj} \in U(R), g(s_{jj}) = 0)$; while if $a_{ii} = u_{ii} - s_{ii}$, for $j \ne i$, let $a_{jj} = u_{jj} - s_{jj}$ such that $(u_{jj} \in U(R), g(s_{jj}) = 0)$. Then by elementary row and column operations we can see that,

$$\begin{split} U &= \begin{bmatrix} u_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & u_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & u_{nn} \end{bmatrix} \in GL_n(R). \\ \text{Suppose } g(x) &= \sum_{i=0}^m a_i x^i \in C(R)[x], \text{ then} \\ g(S &= \begin{bmatrix} s_{11} & 0 & \dots & 0 \\ 0 & s_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & s_{nn} \end{bmatrix}) = a_0 I_n + a_1 S + \dots + a_n S^n \\ &= \begin{bmatrix} a_0 & 0 & \dots & 0 \\ 0 & a_0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_0 \end{bmatrix} + \begin{bmatrix} a_1 s_{11} & 0 & \dots & 0 \\ 0 & a_1 s_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_1 s_{nn} \end{bmatrix} + \dots \\ &+ \begin{bmatrix} a_m s_{11}^m & 0 & \dots & 0 \\ 0 & a_m s_{22}^m & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_m s_{nn}^m \end{bmatrix} \\ &= \begin{bmatrix} g(s_{11}) & 0 & \dots & 0 \\ 0 & g(s_{22}) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & g(s_{nn}) \end{bmatrix} = 0. \end{split}$$

So $\mathbb{T}_n(R)$ is weakly g(x)-clean.

Now let $\mathbb{T}_n(R)$ be weakly g(x)-clean. Define $\theta : \mathbb{T}_n(R) \longrightarrow R$ by $\theta(A) = a_{11}$

where $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}$. Then θ is a ring epimorphism. For any

 $a \in R$, let B be the diagonal matrix diag(a, ...a). Then $a = \theta(B) = \theta(U \pm S) = \theta(U) \pm \theta(S)$ where $U \in GL_n(R)$ and

$$g(\theta(S)) = a_0 + a_1\theta(S) + \dots + a_n\theta(S^n)$$

= $\theta(B_0) + \theta(B_1)\theta(S) + \dots + \theta(B_n)\theta(S^n)$
= $\theta(B_0 + B_1S + \dots + B_nS^n)$
= $\theta(a_0I_n + (a_1I_n)S + \dots + (a_nI_n)S^n)$
= $\theta(g(S)) = 0.$

Thus a is weakly g(x)-clean, i.e., R is a weakly g(x)-clean ring.

Remark 2.8. Let R be a ring with identity, then the following hold:

(1) $f = \sum_{i=0}^{\infty} a_i x^i \in R[[x]]$ is a unit if and only if a_0 is a unit of R. (2) $U(R[t]) = \{r_0 + r_1 t + ... + r_n t^n \mid r_0 \in U(R), r_i \in \sqrt{(0)} \text{ for } i = 0, 1, ..., n\}$

Proposition 2.9. Let R be a ring and $g(x) \in C(R)[x]$. Then the formal power series ring R[[t]] is weakly g(x)-clean if and only if R is weakly g(x)-clean.

Proof. Let R be weakly g(x)-clean and $f = \sum_{i\geq 0} a_i t^i \in R[[t]]$. Since R is weakly g(x)-clean, $a_0 = u \pm s$ for some $s \in R$ and $u \in U(R)$ and g(s) = 0. Then $f = (u + \sum_{i\geq 1} a_i t^i) \pm s$. By Remark 6, $u + \sum_{i\geq 1} a_i t^i \in U(R[[t]])$. So f is weakly g(x)-clean, i.e., R[[t]] is weakly g(x)-clean.

For the converse, let R[[t]] be weakly g(x)-clean. Since $\theta : R[[t]] \longrightarrow R$ with $\theta(f) = a_0$ is a ring epimorphism where $f = \sum_{i \ge 0} a_i t^i \in R[[t]]$, by Proposition 2, R is weakly g(x)-clean.

Remark 2.10. Generally, the polynomial ring R[t] is not weakly g(x)-clean for non-zero polynomial $g(x) \in C(R)[x]$. For example let R be a commutative ring and also let g(x) = x, we show that t is not weakly g(x)-clean. If $t = u \pm s$ then it must be that s = 0 and so t = u. But, by Remark 6, clearly $t \notin U(R[t])$, i.e., R[t] is not weakly g(x)-clean. For more examples of weakly g(x)-clean rings, we consider the method of idealization. Let R be a commutative ring and M an R-module. The idealization of R and M is the ring $R(M) = R \bigoplus M$ with product (r, m)(r', m') = (rr', rm' + r'm) and addition (r, m)(r', m') = (r + r', m + m').

Theorem 2.11. Let R be a commutative ring, M an R-module and $g(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$. If R is a weakly g(x)-clean ring, then the idealization R(M) of R and M is weakly g(x)-clean.

Proof. Let $(r,m) \in R(M)$. Since *R* is a weakly g(x)-clean ring, we have $r = u \pm s$ where $u \in U(R)$ and g(s) = 0. So $(r,m) = (u \pm s,m) = (u,m) \pm (s,0)$. We have $(u,m)(u^{-1}, -u^{-1}mu^{-1}) = (uu^{-1}, u(-u^{-1}mu^{-1}) + mu^{-1}) = (1,0)$. Therefore $(u,m) \in U(R(M))$. Also we have

$$g((s,0)) = a_0(1,0) + a_1(s,0) + \dots + a_n(s,0)^n$$

= $a_0(1,0) + a_1(s,0) + \dots + a_n(s^n,0)$
= $(a_0,0) + (a_1s,0) + \dots + (a_ns^n,0)$
= $(a_0 + a_1s + \dots + a_ns^n,0) = (g(s),0) = (0,0).$

Thus (r, m) is weakly g(x)-clean and so R(M) is a weakly g(x)-clean ring. \Box

3. WEAKLY $(x^n - x)$ -CLEAN RINGS

In this section we consider the weakly $(x^n - x)$ -clean rings and weakly 2-clean rings.

Theorem 3.1. Let R be a ring, $n \in \mathbb{N}$ and $a, b \in R$. Then R is weakly $(ax^{2n} - bx)$ -clean if and only if R is weakly $(ax^{2n} + bx)$ -clean.

Proof. Suppose R is weakly $(ax^{2n} - bx)$ -clean. Then for any $r \in R$, $-r = u \pm s$ where $(as^{2n} - bs) = 0$ and $u \in U(R)$. So $r = (-u) \pm (-s)$ where $(-u) \in U(R)$ and $a(-s)^{2n} + b(-s) = 0$. Hence, r is weakly $(ax^{2n} + bx)$ -clean. Therefore, R is weakly $(ax^{2n} + bx)$ -clean. Now suppose R is weakly $(ax^{2n} + bx)$ -clean. Let $r \in R$. Then there exist s and u such that $-r = u \pm s$, $as^{2n} + bs = 0$ and $u \in U(R)$. So $r = (-u) \pm (-s)$ satisfies $(as^{2n} - bs) = 0$. Hence, R is weakly $(ax^{2n} - bx)$ -clean.

Proposition 3.2. Let $2 \leq n \in \mathbb{N}$. If for every $a \in R$, $a = u \pm v$ where $u \in U(R)$ and $v^{n-1} = 1$, then R is weakly $(x^n - x)$ -clean.

The following Lemma is well-known.

Lemma 3.3. Let $a \in R$. The following statements are equivalent for $n \ge 1$:

- (1) $a = a(ua)^n$ for some $u \in U(R)$;
- (2) $a = ve \text{ for some } e^{n+1} = e \text{ and some } v \in U(R);$
- (3) a = fw for some $f^{n+1} = f$ and some $w \in U(R)$.

Proof. See Lemma 4.3 of [3].

Proposition 3.4. Let R be a weakly $(x^n - x)$ -clean ring where $n \ge 2$ and $a \in R$. Then either (i) $a = u \pm v$ where $u \in U(R)$ and $v^{n-1} = 1$ or (ii) both aR and Ra contain nontrivial idempotents.

Proof. Since R is weakly $(x^n - x)$ -clean, write $a = u \pm e$ where $u \in U(R)$ and $e^n = e$. Then $ae^{n-1} = ue^{n-1} \pm e$. So $a(1 - e^{n-1}) = u(1 - e^{n-1})$. Since $1 - e^{n-1}$ is an idempotent, by Lemma 12, $u(1 - e^{n-1}) = fw$ where $w \in U(R)$ and $f^2 = f \in R$. So $f = a(1 - e^{n-1})w^{-1} \in aR$. Suppose (i) does not hold. Then $1 - e^{n-1} \neq 0$, hence $f \neq 0$. Thus, aR contains a nontrivial idempotent. Similarly, Ra contains a nontrivial idempotent.

Definition 3.5. An element $r \in R$ is called weakly *n*-clean if $r = u_1 + u_2 + \dots + u_n \pm e$ with $e^2 = e \in R$ and $u_i \in U(R)$ for $1 \le i \le n$ and R is called weakly *n*-clean if every element of R is weakly *n*-clean.

Definition 3.6. An element $a \in R$ is called right π -regular if it satisfies the following equivalent conditions,

- (1) $a^n \in a^{n+1}R$ for some integer $n \ge 1$;
- (2) $a^n R = a^{n+1} R$ for some integer $n \ge 1$;
- (3) The chain $aR \supseteq a^2R \supseteq$...terminates.

The left π -regular elements are defined analogously. An element $a \in R$ is called strongly π -regular if it is both left and right π -regular, and R is called strongly π -regular if every element is strongly π -regular [10].

Proposition 3.7. Let $n \in \mathbb{N}$, if the ring R is weakly $(x^n - x)$ -clean, then R is weakly 2-clean.

Proof. Let $r \in R$. Then $r = u \pm t$ for some $t^n = t \in R$ and $u \in U(R)$. Since t is a strongly π -regular element and strongly π -regular elements are strongly clean [10] (it is of course clean and weakly clean), $t = v \pm e$ for some $e^2 = e$ and

 $v \in U(R)$. Then $r = u \pm v \pm e$ is weakly 2-clean. Hence, R is weakly 2-clean.

In fact, all weakly $(x^2 - x)$ -clean rings and weakly $(x^2 + cx + d)$ -clean rings with $d \in U(R)$ discussed above, are weakly 2-clean rings.

Acknowledgments. The authors are grateful to the referee for useful suggestions and careful reading.

References

- D. D. Anderson and V. P. Camillo, Cammutative rings whose elements are a sum of a unit and an idempotent, *Comm. Algebra*, 30, (2002), 3327-3336.
- [2] V. P. Camillo and J. J. Simon, The Nicholson-Varadarajan theorem on clean linear transformations, *Glasg. Math. J.*, 44, (2002), 365-369.
- [3] L. Fan and X. Yang, On rings whose elements are the sum of a unit and a root of a fixed polynomial, *Comm. Algebra*, 36(1), (2008), 269-278.
- [4] J. Han and W. K. Nicholson, Extensions of clean rings, Comm. Algebra, 29, (2001), 2589-2595.
- [5] M. Henriksen, Two classes of rings generated by their units, J. Algebra, 31, (1974), 182-193.
- [6] R. Jahani-Nezhad, The dual of a strongly prime ideal, Iranian Journal of Mathematical Sciences and Informatics, 5(1), (2010), 19-26.
- [7] G. R. Mohtashei Borzabaran, A not via diagonality of 2 by 2 bhattacharyya matrices, Iranian Journal of Mathematical Sciences and Informatics, 1(2), (2006), 79-84.
- [8] A. Myung-Sook, and D. D. Anderson, Weakly clean rings and almost clean rings, Rocky Mountain J. Math., 36(3), (2006), 783-798.
- [9] W. K. Nicholson, Lifting idempotents and exchange rings, Trans. Amer. Math. Soc., 229, (1977), 269-278.
- [10] W. K. Nicholson, Strongly clean rings and Fitting's lemma, Comm. Algebra, 27(3), (1999), 3583-3592.
- [11] W. K. Nicholson and Y. Zhou, Endomorphism that are the sum of a unit and a root of a fixed polynomial, *Canad. Math. Bull.*, 49, (2006), 265-269.