# Weakly $g(x)$-Clean Rings 

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#### Abstract

A ring $R$ with identity is called "clean" if for every element $a \in R$, there exist an idempotent $e$ and a unit $u$ in $R$ such that $a=u+e$. Let $C(R)$ denote the center of a ring $R$ and $g(x)$ be a polynomial in $C(R)[x]$. An element $r \in R$ is called " $\mathrm{g}(\mathrm{x})$-clean" if $r=u+s$ where $g(s)=0$ and $u$ is a unit of $R$ and $R$ is $g(x)$-clean if every element is $g(x)$-clean. In this paper we define a ring to be weakly $g(x)$-clean if each element of $R$ can be written as either the sum or difference of a unit and a root of $g(x)$.


Keywords: Clean ring, $g(x)$-clean ring, Weakly $g(x)$-clean ring.

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## 1. Introduction

Throughout this note, $R$ is an associative ring with identity. A ring $R$ is called clean if for every element $a \in R$, there exist an idempotent $e$ and a unit $u$ in $R$ such that $a=e+u[9]$ and $R$ is called strongly clean if, in addition, $e u=u e[10]$.
Let $C(R)$ denote the center of a ring $R$ and $g(x)$ be a polynomial in $C(R)[x]$. Following Camillo and Simon [2], an element $r \in R$ is called $g(x)$-clean if $r=u+s$ where $g(s)=0$ and $u$ is a unit of $R$, and $R$ is $g(x)$-clean if every element in $R$ is $g(x)$-clean. It is clear that the $\left(x^{2}-x\right)$ - clean rings are precisely

[^0]the clean rings.
Camillo and Simon [2] proved that if $V$ is a countable dimensional vector space over a division ring $D$ and $g(x)$ is any polynomial with coefficients in $K=C(D)$ and two distinct roots in $K$, then $\operatorname{End}\left(V_{D}\right)$ is $g(x)$-clean. Nicholson and Zhou [11] generalized Camillo and Simon's result by proving that $\operatorname{End}\left({ }_{R} M\right)$ is $g(x)$ clean where ${ }_{R} M$ is a semisimple left $R$-module and $g(x) \in(x-a)(x-b) C(R)[x]$ with $a, b \in C(R)$ and $b, b-a \in U(R) . g(x)$-clean rings have also been studied in [3], [7] and [6].
It is easy to see that a ring $R$ is $g(x)$-clean if and only if each $x \in R$ can be written in the form $x=u-s$ where $u \in U(R)$ and $g(s)=0$. This raises the question of whether a ring with the property that, for each $x \in R$, either $x=u+s$ or $x=u-s$ for some $u \in U(R)$ and $g(s)=0$ must be cleaned. Let us call rings with this property weakly $g(x)$-clean. Here we study weakly $g(x)$-clean rings and also investigate the general properties of weakly $g(x)$-clean rings which are similar to those of $g(x)$-clean rings. For example we prove the following results:

Proposition 1.1. Let $g(x) \in \mathbb{Z}[x]$ and $\left\{R_{i}\right\}_{i \in I}$ be a family of rings. Then $\prod_{i \in I} R_{i}$ is weakly $g(x)$-clean if and only if for all $i \in I, R_{i}$ is weakly $g(x)$-clean.

Theorem 1.2. Let $R$ be a ring, $g(x) \in C(R)[x]$, and $n \in \mathbb{N}$. Then $R$ is weakly $g(x)$-clean if and only if the upper triangular matrix ring $\mathbb{T}_{n}(R)$ is weakly $g(x)$ clean.

Theorem 1.3. Let $R$ be a commutative ring and $M$ an $R$-module. Let $g(x) \in$ $C(R)[x]$. If $R$ is weakly $g(x)$-clean, then the idealization $R(M)$ of $R$ and $M$ is also weakly $g(x)$-clean.

In section 3 we consider the weakly $\left(x^{n}-x\right)$-clean rings and weakly 2 -clean rings.

An usual, $\mathbb{T}_{n}(R)$ denotes the upper triangular matrix ring of order $n$ over $R$; $G L_{n}(R)$ denotes the general linear group over $R$; and $\operatorname{gcd}(m, n)$ means the greatest common divisor of the integers $m$ and $n$. All polynomials are in the polynomial ring $C(R)[x]$ and $U(R)$ denotes the multiplicative unit group of $R$.

## 2. Weakly $g(x)$-CLEAN RINGS

In this section first we define the weakly $g(x)$-clean rings, then we explain the relation between weakly $g(x)$-clean and $g(x)$-clean rings.

Definition 2.1. Let $g(x)$ be a fixed polynomial in $C(R)[x]$. An element $r \in R$ is called weakly $g(x)$-clean if $r=u+s$ or $r=u-s$ where $g(s)=0$ and
$u \in U(R)$. We say that $R$ is weakly $g(x)$-clean if every element is weakly $g(x)$-clean.

Obviously, $g(x)$-clean rings are weakly $g(x)$-clean and also if $g(x)$ is an odd or an even polynomial (i.e $g(-x)=-g(x)$ or $g(-x)=g(x))$, then the concepts $g(x)$-clean and weakly $g(x)$-clean coincide, that is, if $R$ is a weakly $g(x)$-clean ring then $R$ is also $g(x)$-clean. So the interesting case is when $g(x)$ is neither an even nor an odd polynomial. In [1, Proposition 16] it was shown that if $R$ has exactly two maximal ideals and $2 \in U(R)$, then each $x \in R$ has the form $x=u+e$ or $x=u-e$ where $u \in U(R)$ and $e \in\{0,1\}$. Thus $\mathbb{Z}_{(3)} \cap \mathbb{Z}_{(5)}$ is weakly clean but is not clean since an indecomposable clean ring is quasilocal [1, Thedrem 3]. But since weakly $\left(x^{2}-x\right)$-clean rings are precisely the weakly clean rings, we can say that for $g(x)=x^{2}-x$, the ring $\mathbb{Z}_{(3)} \bigcap \mathbb{Z}_{(5)}$ is weakly $g(x)$-clean, but it is not $g(x)$-clean.

The following two examples explain the relations between weakly $g(x)$-clean rings and weakly clean rings.

Example 2.2. Let $R=\mathbb{Z}_{(p)}=\left\{\frac{m}{n} ; \operatorname{gcd}(p, n)=1\right.$ and p prime $\}$ be the localization of $\mathbb{Z}$ at the prime ideal $p \mathbb{Z}$ and $g(x)=(x-a)\left(x^{2}+1\right) \in C(R)[x]$. Then $R$ is a weakly clean ring, because local rings are strongly clean, thus $R$ is clean (it is of course weakly clean). But as $a$ is the single root of $g(x), R$ is not a weakly $g(x)$-clean ring.

Example 2.3. Let $R$ be a Boolean ring with the number of elements $|R|>2$ and $c \in R$ with $0 \neq c \neq 1$. Define $g(x)=(x+1)(x+c)$. Then $R$ is not weakly $g(x)$-clean.
Because if $c=u \pm s$ where $u \in U(R)$ and $\mathrm{g}(\mathrm{s})=0$, then it must be that $u=1$ and $s= \pm(c \pm u)$. But, clearly, $g(c+1) \neq 0$. However, $R$ is certainly weakly clean.

Let $R$ and $S$ be rings and $\theta: C(R) \longrightarrow C(S)$ be a ring homomorphism with $\theta(1)=1$. Then $\theta$ induces a map $\theta^{\prime}$ from $\mathrm{C}(\mathrm{R})[\mathrm{x}]$ to $C(S)[x]$ such that For $g(x)=\sum_{i=0}^{n} a_{i} x^{i} \in C(R)[x], \theta^{\prime}(g(x)):=\sum_{i=0}^{n} \theta\left(a_{i}\right) x^{i} \in C(S)[x]$. Clearly, if $g(x)$ is a polynomial with coefficients in $\mathbb{Z}$, then $\theta^{\prime}(g(x))=g(x)$. We give some properties of weakly $g(x)$-clean rings which are similar to those of weakly clean rings.

Proposition 2.4. Let $\theta: R \longrightarrow S$ be a ring epimorphism. If $R$ is weakly $g(x)$-clean, then $S$ is weakly $\theta^{\prime}(g(x))$-clean.

Proof. Let $g(x)=a_{0}+a_{1} x+\ldots+a_{n} x^{n} \in C(R)[x]$. Then $\theta^{\prime}(g(x))=\theta\left(a_{0}\right)+$ $\theta\left(a_{1}\right) x+\ldots+\theta\left(a_{n}\right) x^{n} \in C(S)[x]$. As $\theta$ is a ring epimorphism so for any $s \in S$, there exists $r \in R$ such that $\theta(r)=s$. Since $R$ is weakly $g(x)$-clean, there
exist $u \in U(R)$ and $s_{0} \in R$ such that $r=u \pm s_{0}$ and $g\left(s_{0}\right)=0$. Then $s=\theta(r)=\theta\left(u \pm s_{0}\right)=\theta(u) \pm \theta\left(s_{0}\right)$ with $\theta(u) \in U(S)$. But $\theta^{\prime}\left(g\left(\theta\left(s_{0}\right)\right)\right)=$ $\theta\left(a_{0}\right)+\theta\left(a_{1}\right) \theta\left(s_{0}\right)+\ldots+\theta\left(a_{n}\right) \theta\left(s_{0}^{n}\right)=\theta\left(a_{0}+a_{1} s_{0}+\ldots+a_{n} s_{0}^{n}\right)=\theta\left(g\left(s_{0}\right)\right)=$ $\theta(0)=0$, we have $s$ is weakly $\theta^{\prime}(g(x))$-clean. Therefore $S$ is weakly $\theta^{\prime}(g(x))$ clean.

Corollary 2.5. If $R$ is weakly $g(x)$-clean, then for any ideal $I$ of $R, R / I$ is weakly $\bar{g}(x)$-clean where $\bar{g}(x) \in C(R / I)[x]$.

Proposition 2.6. Let $g(x) \in \mathbb{Z}[x]$ and $\left\{R_{i}\right\}_{i \in I}$ be a family of rings. Then $\prod_{i \in I} R_{i}$ is weakly $g(x)$-clean if and only if for all $i \in I, R_{i}$ is weakly $g(x)$-clean.

Proof. Let $\prod_{i \in I} R_{i}$ be a weakly $g(x)$-clean. Define $\pi_{j}: \prod_{i \in I} R_{i} \longrightarrow R_{j}$ by $\pi_{j}\left(\left\{a_{i}\right\}_{i \in I}\right)=$ $a_{j}$. Since for all $j \in I, \pi_{j}$ is a ring epimorphism, so by Proposition 2, for every $i \in I$, each $R_{i}$ is a weakly $g(x)$-clean ring.
For the converse, let $x=\left\{x_{i}\right\}_{i \in I} \in R=\prod_{i \in I} R_{i}$. In $R_{i_{0}}$, we can write $x_{i}=u_{1_{0}}+s_{i_{0}}$ or $x_{i}=u_{i_{0}}-s_{i_{0}}$ where $u_{i_{0}} \in U\left(R_{i_{0}}\right)$ and $g\left(s_{i_{0}}\right)=0$. If $x_{i_{0}}=u_{i_{0}}+s_{i_{0}}$, for $i \neq i_{0}$, let $x_{i}=u_{i}+s_{i}$ where $u_{i} \in U\left(R_{i}\right), g\left(s_{i}\right)=0$; while if $x_{i_{0}}=u_{i_{0}}-s_{i_{0}}$, for $i \neq i_{0}$, let $x_{i}=u_{i}-s_{i}$ where $u_{i} \in U\left(R_{i}\right), g\left(s_{i}\right)=0$. Then $u=\left\{u_{i}\right\}_{i \in I} \in U(R)$ and

$$
\begin{aligned}
g\left(s=\left\{s_{i}\right\}_{i \in I}\right) & =a_{0}\left\{1_{R_{i}}\right\}_{i \in I}+a_{1}\left\{s_{i}\right\}_{i \in I}+\ldots+a_{n}\left\{s_{i}^{n}\right\}_{i \in I} \\
& =\left\{a_{0}\right\}_{i \in I}+\left\{a_{1} s_{i}\right\}_{i \in I}+\ldots+\left\{a_{n} s_{i}^{n}\right\}_{i \in I} \\
& =\left\{a_{0}+a_{1} s_{i}+\ldots+a_{n} s_{i}^{n}\right\}_{i \in I} \\
& =\left\{g\left(s_{i}\right)\right\}_{i \in I}=0
\end{aligned}
$$

That is, $\prod_{i \in I} R_{i}$ is weakly $\mathrm{g}(\mathrm{x})$-clean.
Define $\pi_{n}: C(R) \longrightarrow M_{n}(R)$ by $a \longmapsto a I_{n}$ with $I_{n}$ being the identity matrix of $M_{n}(R)$ and $a \in C(R)$. Then $M_{n}(R)$ is a $C(R)$-algebra.

Theorem 2.7. Let $R$ be a ring, $g(x) \in C(R)[x]$, and $n \in \mathbb{N}$. Then $R$ is weakly $g(x)$-clean if and only if the upper triangular matrix ring $\mathbb{T}_{n}(R)$ is weakly $g(x)$ clean.

Proof. Let $R$ be weakly $g(x)$-clean and $A=\left(a_{i j}\right) \in \mathbb{T}_{n}(R)$ with $a_{i j}=0$ for $1 \leq j<i \leq n$. Since $R$ is weakly $g(x)$-clean, for any $1 \leq i \leq n$, there exist $s_{i i} \in R$ and $u_{i i} \in U(R)$ such that $a_{i i}=u_{i i} \pm s_{i i}$ with $g\left(s_{i i}\right)=0$. So we have

$$
A=\left[\begin{array}{cccc}
a_{11} & a_{12} & \ldots & a_{1 n} \\
0 & a_{22} & \ldots & a_{2 n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & a_{n n}
\end{array}\right]=\left[\begin{array}{cccc}
u_{11} \pm s_{11} & a_{12} & \ldots & a_{1 n} \\
0 & u_{22} \pm s_{22} & \ldots & a_{2 n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & u_{n n} \pm s_{n n}
\end{array}\right]
$$

In $R$ for any $0 \leq i \leq n$, we can write $a_{i i}=u_{i i}+s_{i i}$ or $a_{i i}=u_{i i}-s_{i i}$ where $u_{i i} \in U(R)$ and $g\left(s_{i i}\right)=0$. If $a_{i i}=u_{i i}+s_{i i}$ for $j \neq i$, let $a_{j j}=u_{j j}+s_{j j}$ where $\left(u_{j j} \in U(R), g\left(s_{j j}\right)=0\right)$; while if $a_{i i}=u_{i i}-s_{i i}$, for $j \neq i$, let $a_{j j}=u_{j j}-s_{j j}$ such that $\left(u_{j j} \in U(R), g\left(s_{j j}\right)=0\right)$. Then by elementary row and column operations we can see that,
$U=\left[\begin{array}{ccccc}u_{11} & a_{12} & a_{13} & \ldots & a_{1 n} \\ 0 & u_{22} & a_{23} & \ldots & a_{2 n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & u_{n n}\end{array}\right] \in G L_{n}(R)$.
Suppose $g(x)=\sum_{i=0}^{m} a_{i} x^{i} \in C(R)[x]$, then

$$
\begin{aligned}
g(S & \left.=\left[\begin{array}{cccc}
s_{11} & 0 & \ldots & 0 \\
0 & s_{22} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & s_{n n}
\end{array}\right]\right)=a_{0} I_{n}+a_{1} S+\ldots+a_{n} S^{n} \\
& =\left[\begin{array}{cccc}
a_{0} & 0 & \ldots & 0 \\
0 & a_{0} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & a_{0}
\end{array}\right]+\left[\begin{array}{cccc}
a_{1} s_{11} & 0 & \ldots & 0 \\
0 & a_{1} s_{22} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & a_{1} s_{n n}
\end{array}\right]+\ldots \\
& +\left[\begin{array}{cccc}
a_{m} s_{11}^{m} & 0 & \ldots & 0 \\
0 & a_{m} s_{22}^{m} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & a_{m} s_{n n}^{m}
\end{array}\right] \\
& =\left[\begin{array}{cccc}
g\left(s_{11}\right) & 0 & \ldots & 0 \\
0 & g\left(s_{22}\right) & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & g\left(s_{n n}\right)
\end{array}\right]=0
\end{aligned}
$$

So $\mathbb{T}_{n}(R)$ is weakly $g(x)$-clean.
Now let $\mathbb{T}_{n}(R)$ be weakly $g(x)$-clean. Define $\theta: \mathbb{T}_{n}(R) \longrightarrow R$ by $\theta(A)=a_{11}$
where $A=\left[\begin{array}{cccc}a_{11} & a_{12} & \ldots & a_{1 n} \\ 0 & a_{22} & \ldots & a_{2 n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_{n n}\end{array}\right]$.
Then $\theta$ is a ring epimorphism. For any
$a \in R$, let $B$ be the diagonal matrix $\operatorname{diag}(a, \ldots a)$. Then $a=\theta(B)=\theta(U \pm S)=$ $\theta(U) \pm \theta(S)$ where $U \in G L_{n}(R)$ and

$$
\begin{aligned}
g(\theta(S)) & =a_{0}+a_{1} \theta(S)+\ldots+a_{n} \theta\left(S^{n}\right) \\
& =\theta\left(B_{0}\right)+\theta\left(B_{1}\right) \theta(S)+\ldots+\theta\left(B_{n}\right) \theta\left(S^{n}\right) \\
& =\theta\left(B_{0}+B_{1} S+\ldots+B_{n} S^{n}\right) \\
& =\theta\left(a_{0} I_{n}+\left(a_{1} I_{n}\right) S+\ldots+\left(a_{n} I_{n}\right) S^{n}\right) \\
& =\theta(g(S))=0
\end{aligned}
$$

Thus $a$ is weakly $g(x)$-clean, i.e., $R$ is a weakly $g(x)$-clean ring.

Remark 2.8. Let $R$ be a ring with identity, then the following hold:
(1) $f=\sum_{i=0}^{\infty} a_{i} x^{i} \in R[[x]]$ is a unit if and only if $a_{0}$ is a unit of $R$.
(2) $U(R[t])=\left\{r_{0}+r_{1} t+\ldots+r_{n} t^{n} \mid r_{0} \in U(R), r_{i} \in \sqrt{(0)}\right.$ for $\left.i=0,1, \ldots, n\right\}$

Proposition 2.9. Let $R$ be a ring and $g(x) \in C(R)[x]$. Then the formal power series ring $R[[t]]$ is weakly $g(x)$-clean if and only if $R$ is weakly $g(x)$-clean.

Proof. Let $R$ be weakly $g(x)$-clean and $f=\sum_{i \geq 0} a_{i} t^{i} \in R[[t]]$. Since $R$ is weakly $g(x)$-clean, $a_{0}=u \pm s$ for some $s \in R$ and $u \in U(R)$ and $g(s)=0$. Then $f=\left(u+\sum_{i \geq 1} a_{i} t^{i}\right) \pm s$. By Remark $6, u+\sum_{i \geq 1} a_{i} t^{i} \in U(R[[t]])$. So $f$ is weakly $g(x)$-clean, i.e., $R[[t]]$ is weakly $g(x)$-clean.
For the converse, let $R[[t]]$ be weakly $g(x)$-clean. Since $\theta: R[[t]] \longrightarrow R$ with $\theta(f)=a_{0}$ is a ring epimorphism where $f=\sum_{i \geq 0} a_{i} t^{i} \in R[[t]]$, by Proposition $2, R$ is weakly $g(x)$-clean.

Remark 2.10. Generally, the polynomial ring $R[t]$ is not weakly $g(x)$-clean for non-zero polynomial $g(x) \in C(R)[x]$. For example let $R$ be a commutative ring and also let $g(x)=x$, we show that $t$ is not weakly $g(x)$-clean. If $t=u \pm s$ then it must be that $s=0$ and so $t=u$. But, by Remark 6 , clearly $t \notin U(R[t])$, i.e., $R[t]$ is not weakly $g(x)$-clean.

For more examples of weakly $g(x)$-clean rings, we consider the method of idealization. Let $R$ be a commutative ring and $M$ an $R$-module. The idealization of $R$ and $M$ is the ring $R(M)=R \bigoplus M$ with product $(r, m)\left(r^{\prime}, m^{\prime}\right)=$ $\left(r r^{\prime}, r m^{\prime}+r^{\prime} m\right)$ and addition $(r, m)\left(r^{\prime}, m^{\prime}\right)=\left(r+r^{\prime}, m+m^{\prime}\right)$.

Theorem 2.11. Let $R$ be a commutative ring, $M$ an $R$-module and $g(x)=$ $\sum_{i=0}^{n} a_{i} x^{i} \in R[x]$. If $R$ is a weakly $g(x)$-clean ring, then the idealization $R(M)$ of $R$ and $M$ is weakly $g(x)$-clean.

Proof. Let $(r, m) \in R(M)$. Since $R$ is a weakly $g(x)$-clean ring, we have $r=u \pm s$ where $u \in U(R)$ and $g(s)=0$. So $(r, m)=(u \pm s, m)=(u, m) \pm(s, 0)$. We have $(u, m)\left(u^{-1},-u^{-1} m u^{-1}\right)=\left(u u^{-1}, u\left(-u^{-1} m u^{-1}\right)+m u^{-1}\right)=(1,0)$. Therefore $(u, m) \in U(R(M))$. Also we have

$$
\begin{aligned}
g((s, 0)) & =a_{0}(1,0)+a_{1}(s, 0)+\ldots+a_{n}(s, 0)^{n} \\
& =a_{0}(1,0)+a_{1}(s, 0)+\ldots+a_{n}\left(s^{n}, 0\right) \\
& =\left(a_{0}, 0\right)+\left(a_{1} s, 0\right)+\ldots+\left(a_{n} s^{n}, 0\right) \\
& =\left(a_{0}+a_{1} s+\ldots+a_{n} s^{n}, 0\right)=(g(s), 0)=(0,0)
\end{aligned}
$$

Thus $(r, m)$ is weakly $g(x)$-clean and so $R(M)$ is a weakly $g(x)$-clean ring.

## 3. WEAKLY $\left(x^{n}-x\right)$-CLEAN RINGS

In this section we consider the weakly $\left(x^{n}-x\right)$-clean rings and weakly 2 -clean rings.

Theorem 3.1. Let $R$ be a ring, $n \in \mathbb{N}$ and $a, b \in R$. Then $R$ is weakly $\left(a x^{2 n}-b x\right)$-clean if and only if $R$ is weakly $\left(a x^{2 n}+b x\right)$-clean.

Proof. Suppose $R$ is weakly $\left(a x^{2 n}-b x\right)$-clean. Then for any $r \in R,-r=u \pm s$ where $\left(a s^{2 n}-b s\right)=0$ and $u \in U(R)$. So $r=(-u) \pm(-s)$ where $(-u) \in U(R)$ and $a(-s)^{2 n}+b(-s)=0$. Hence, $r$ is weakly $\left(a x^{2 n}+b x\right)$-clean. Therefore, $R$ is weakly c $\left(a x^{2 n}+b x\right)$-clean. Now suppose $R$ is weakly $\left(a x^{2 n}+b x\right)$-clean. Let $r \in R$. Then there exist $s$ and $u$ such that $-r=u \pm s, a s^{2 n}+b s=0$ and $u \in U(R)$. So $r=(-u) \pm(-s)$ satisfies $\left(a s^{2 n}-b s\right)=0$. Hence, $R$ is weakly $\left(a x^{2 n}-b x\right)$-clean.

Proposition 3.2. Let $2 \leq n \in \mathbb{N}$. If for every $a \in R$, $a=u \pm v$ where $u \in U(R)$ and $v^{n-1}=1$, then $R$ is weakly $\left(x^{n}-x\right)$-clean.

The following Lemma is well-known.

Lemma 3.3. Let $a \in R$. The following statements are equivalent for $n \geq 1$ :
(1) $a=a(u a)^{n}$ for some $u \in U(R)$;
(2) $a=v e$ for some $e^{n+1}=e$ and some $v \in U(R)$;
(3) $a=f w$ for some $f^{n+1}=f$ and some $w \in U(R)$.

Proof. See Lemma 4.3 of [3].

Proposition 3.4. Let $R$ be a weakly $\left(x^{n}-x\right)$-clean ring where $n \geq 2$ and $a \in R$. Then either (i) $a=u \pm v$ where $u \in U(R)$ and $v^{n-1}=1$ or (ii) both $a R$ and $R a$ contain nontrivial idempotents.

Proof. Since $R$ is weakly $\left(x^{n}-x\right)$-clean, write $a=u \pm e$ where $u \in U(R)$ and $e^{n}=e$. Then $a e^{n-1}=u e^{n-1} \pm e$. So $a\left(1-e^{n-1}\right)=u\left(1-e^{n-1}\right)$. Since $1-e^{n-1}$ is an idempotent, by Lemma $12, u\left(1-e^{n-1}\right)=f w$ where $w \in U(R)$ and $f^{2}=f \in R$. So $f=a\left(1-e^{n-1}\right) w^{-1} \in a R$. Suppose (i) does not hold. Then $1-e^{n-1} \neq 0$, hence $f \neq 0$. Thus, $a R$ contains a nontrivial idempotent. Similarly, $R a$ contains a nontrivial idempotent.

Definition 3.5. An element $r \in R$ is called weakly $n$-clean if $r=u_{1}+u_{2}+$ $\ldots+u_{n} \pm e$ with $e^{2}=e \in R$ and $u_{i} \in U(R)$ for $1 \leq i \leq n$ and $R$ is called weakly $n$-clean if every element of $R$ is weakly $n$-clean.

Definition 3.6. An element $a \in R$ is called right $\pi$-regular if it satisfies the following equivalent conditions,
(1) $a^{n} \in a^{n+1} R$ for some integer $n \geq 1$;
(2) $a^{n} R=a^{n+1} R$ for some integer $n \geq 1$;
(3) The chain $a R \supseteq a^{2} R \supseteq \ldots$..terminates.

The left $\pi$-regular elements are defined analogously. An element $a \in R$ is called strongly $\pi$-regular if it is both left and right $\pi$-regular, and R is called strongly $\pi$-regular if every element is strongly $\pi$-regular [10].

Proposition 3.7. Let $n \in \mathbb{N}$, if the ring $R$ is weakly $\left(x^{n}-x\right)$-clean, then $R$ is weakly 2-clean.

Proof. Let $r \in R$. Then $r=u \pm t$ for some $t^{n}=t \in R$ and $u \in U(R)$. Since $t$ is a strongly $\pi$-regular element and strongly $\pi$-regular elements are strongly clean [10] (it is of course clean and weakly clean), $t=v \pm e$ for some $e^{2}=e$ and
$v \in U(R)$. Then $r=u \pm v \pm e$ is weakly 2-clean. Hence, $R$ is weakly 2-clean.

In fact, all weakly $\left(x^{2}-x\right)$-clean rings and weakly $\left(x^{2}+c x+d\right)$-clean rings with $d \in U(R)$ discussed above, are weakly 2-clean rings.

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