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Generalized Symmetric Berwald Spaces

Parastoo Habibi [a*] and Asadollah Razavi [b]

[a] Department of Mathematics
Islamic Azad University, Astara branch, Astara, Iran
p.habibi@iau-astara.ac.ir

[b] Faculty of Mathematics and Computer Science
Amirkabir University of Technology, 424 Hafez Ave., 15914 Tehran, Iran
arazavi@aut.ac.ir

Abstract. In this paper we study generalized symmetric Berwald spaces. We show that if a Berwald space (M, F) admits a parallel $s-$structure then it is locally symmetric. For a complete Berwald space which admits a parallel s-structure we show that if the flag curvature of (M, F) is everywhere nonzero, then F is Riemannian.

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1. Introduction

Let (M, g) be a Riemannian symmetric space. Then for any $x \in M$, there exists an isometry $s_x : M \longrightarrow M$ such that x is an isolated fixed point of s_x and $s_x^2 = Id$. Then we have $(s_x)_{*x} = (-Id)_x$, $v_x \longrightarrow -v_x$. Now we consider a generalization of the notion of Riemannian symmetric spaces. Let (M, g) be a connected Riemannian manifold. An isometry of (M, g) with an isolated fixed point $x \in M$ is called a symmetry of (M, g) at x. A family $\{s_x | x \in M\}$ of symmetries of a connected Riemannian manifold (M, g) is called an $s-$structure on

*Corresponding Author

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(M, g). Clearly if each s_x satisfies the additional property s_x^2 =identity, then (M, g) is nothing but a Riemannian symmetric space.

Let (M, F) be a Finsler space, where F is positively homogeneous of degree one. Then we have two ways to define the notion of an isometry of (M, F). On the one hand, we call a diffeomorphism σ of M onto itself an isometry if $F(d\sigma_x(y)) = F(y)$, for any $x \in M$ and $y \in T_xM$. On the other hand, we can also define an isometry of (M, F) to be a one-to-one mapping of M onto itself which preserves the distance of each pair of points of M. It is well known that the two definitions are equivalent if the metric F is Riemannian. The equivalence of these two definitions in the general Finsler case is a result of S. Deng and Z. Hou [3]. Using these result, they proved that the group of isometries $I(M, F)$ of a Finsler space (M, F) is a Lie transformation group of M and for any point $x \in M$, the isotropic subgroup $I_x(M, F)$ is a compact subgroup of $I(M, F)$. These results are important to study homogeneous Finsler spaces. In this paper we study Berwald spaces admitting an $s-$structuer.

2. Preliminaries

We first review the basics of Finsler geometry. Standard references are [1] and [2]. We will follow the notations in [2].

2.1. **Finsler Spaces.**

Let M be an n-dimensional C^∞ manifold and $TM = \bigcup_{x\in M} T_xM$ the tangent bundle.

A Finsler structure is a function $F : TM \longrightarrow [0, \infty)$ satisfying the following conditions:

(i): F is C^∞ on $TM \setminus \{0\}$;
(ii): $F(cv) = cF(v)$ for all $v \in TM$ and $c \geq 0$;
(iii): The matrix

$$g_{ij}(v) = \frac{1}{2}\frac{\partial^2 F^2}{\partial v^i \partial v^j}(v)$$

is positive definite for all $v \in TM \setminus \{0\}$.

The positive definite matrix $(g_{ij}(v))$ defines a Riemannian structure g_v of T_xM through

$$g_v(\sum_i a^i \frac{\partial}{\partial x^i}, \sum_j b^j \frac{\partial}{\partial x^j}) = \sum_{i,j} g_{ij}(v) a^i b^j.$$

Note that $g_v(v, v) = F(v)^2$. If (M, F) is Riemannian, then g_v always coincide with the original Riemannian metric.

Let $\gamma : [0, r] \longrightarrow M$ be a piecewise C^∞ curve. Its length is defined as

$$L(\gamma) = \int_0^r F(\gamma(t), \dot{\gamma}(t))dt.$$

For $x_0, x_1 \in M$ denote by $\Gamma(x_0, x_1)$ the set of all piecewise C^∞ curve $\gamma : [0, r] \longrightarrow M$ such that $\gamma(0) = x_0$ and $\gamma(r) = x_1$. Define a map $d_F : M \times M \longrightarrow [0, \infty)$ by

$$d_F(x_0, x_1) = \inf_{\gamma \in \Gamma(x_0, x_1)} L(\gamma).$$

Of course we have $d_F(x_0, x_1) \geq 0$, where the equality holds if and only if $x_0 = x_1$; $d_F(x_0, x_2) \leq d_F(x_0, x_1) + d_F(x_1, x_2)$. In general, since F is only a positive homogeneous function, $d_F(x_0, x_1) \neq d_F(x_1, x_0)$, therefore (M, d_F) is only a non-reversible metric space.

Define the Cartan tensor

$$C_{ijk}(x, y) = \frac{1}{4} \frac{\partial^3 F^2(x, y)}{\partial y^i \partial y^j \partial y^k},$$

we also define the formal Christoffel symbol

$$\gamma_{ij}^k = \frac{1}{2} g^{km} \left(\frac{\partial g_{mj}}{\partial x^i} + \frac{\partial g_{im}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^m} \right).$$

Using these, we further define the nonlinear connection

$$N_j^i = \gamma_{jk}^i y^k - C_{jk}^i \gamma_{rs}^k y^r y^s.$$

According to [2], the pulled-back bundle $\pi^* TM$ admits a unique linear connection, called the Chern connection. Its connection forms are characterized by the structure equation:

- Torsion freeness

$$dx^j \wedge \omega_j^i = 0;$$

- Almost g−compatibility:

$$dg_{ij} - g_{kj}\omega_i^k - g_{ik}\omega_j^k = 2C_{ijk}(dy^k + N_l^k dx^l).$$

It is easy to know that torsion freeness is equivalent to the $\omega_j^i = \Gamma_{jk}^i dx^k$ and $\Gamma_{jk}^i = \Gamma_{kj}^i$.

Definition 2.1. *A Finsler metric F on a manifold M is called a Berwald metric if in any standard local coordinate system (x^i, y^i) in TM, the Christoffel symbols Γ_{jk}^i are the functions of $x \in M$ only, i.e., $\Gamma_{jk}^i = \Gamma_{jk}^i(x)$.*

2.2. **Symmetric Fisler spaces.**

Let G be a Lie group, H a closed subgroup of G. The coset space G/H has a unique smooth structure such that G is a Lie transformation group of G/H. It is called reductive if there exists a subspace $\mathfrak{m}$ of the Lie algebra $\mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$$

where $\mathfrak{h}$ is the Lie algebra of H and $Ad(h)\mathfrak{m} \subset \mathfrak{m}$, $\forall h \in H$. The study of invariant structures on coset spaces is an important problem in differential geometry.

Definition 2.2. *A Finsler space (M, F) is called homogeneous Finsler space if the group of isometries of (M, F), $I(M, F)$, acts transitively on M.*

Every homogeneous Finsler space is forward complete [7]. Let G be a Lie group, H be a closed subgroup of G. Suppose there exists an invariant Finsler metric on G/H. Then there exists an invariant Riemannian metric on G/H.

The definition of globally symmetric Finsler space is a natural generalization of É. Cartan's definition of Riemannian globally symmetric spaces.

Definition 2.3. *A connected Finsler space (M, F) is said to be symmetric if to each $p \in M$ there is associated an isometry $\sigma_p : M \longrightarrow M$ which is*

- **(i):** *involutive (σ_p^2 is the identity).*
- **(ii):** *has p as an isolated fixed point, that is, there is a neighborhood U of p in which p is the only fixed point of σ_p.*

σ_p is called the *symmetry* at point p.

As p is an isolated fixed point of σ_p it follows that $(d\sigma_p)_p = -id$, and therefore symmetric Finsler spaces have reversible metrics and geodesics.

Let (M, F) be a connected symmetric Finsler space, Then (M, F) is (forward-backward) complete and homogeneous that is the group of isometries of (M, F) acts transitively on M [7], [5].

Theorem 2.4 ([5])**.** *Let (M, F) be a symmetric Finsler space. Then (M, F) is a Berwald space. Furthermore, the connection of F coincides with the Levi-Civita connection of a Riemannian metric g such that (M, g) is a Riemannian symmetric space.*

3. Generalized symmetric Berwald spaces

Let (M, F) be a connected Berwald space. An isometry s_x of (M, F) for which $x \in M$ is an isolated fixed point will be called a symmetry of M at x.

Clearly, if s_x is a symmetry of (M, g) at x, then the tangent map $S_x = (s_{x*})_x$ has no invariant vector.

An $s-$structure on (M, F) is a family $\{s_x | x \in M\}$ of symmetries of (M, F). The corresponding tensor field S of type (1,1) defined by $S_x = (s_{x*})_x$ for each $x \in M$ is called the symmetry tensor field of $s-$structure [6], [8].

An $s-$structure $\{s_x | x \in M\}$ is called of order k $(k \geq 2)$ if $(s_x)^k = id$ for all $x \in M$ and k is the least integer of this property. Obviously a Berwald space is symmetric if and only if it admits an $s-$structure of order 2.

Definition 3.1. *An $s-$structure $\{s_x | x \in M\}$ on a Berwald space (M, F) is said to be regular if it satisfies the rule*

$$s_x \circ s_y = s_z \circ s_x, \qquad z = s_x(y) \tag{3.1}$$

for every two points $x, y \in M$.

Lemma 3.2. *An $s-$structure $\{s_x\}$ on a connected Berwald space (M, F) is regular if and only if the tensor field S is invariant with respect to all symmetries s_x, i.e.*

$$s_{x*}(S) = S, \qquad x \in M \tag{3.2}$$

Proof: The proof is similar to the Riemmanian case.□

Definition 3.3. *An $s-$structure $\{s_x\}$ on a Berwald space (M, F) is said to be parallel if the tensor field S is parallel with respect to the Chern connection i.e. $\nabla S = 0$.*

Theorem 3.4. *Each parallel $s-$structure on a Berwald space is regular.*

Proof: Suppose $\{s_x\}$ to be a parallel $s-$structure on (M, F). Let $p \in M$ be a fixed point and put $S' = s_{p*}(S)$. Because $\nabla S = 0$ and s_p is connection preserving, we have $\nabla S' = 0$. Now $S'_p = (s_{p*})_p(S_p) = S_p$, from the uniqueness of a parallel extension we have $S' = S$. Thus for all points $p \in M$ we get $(s_{p*})(S) = S$ and hence $\{s_x\}$ is regular by Lemma 3.2.□ .

Theorem 3.5. *If a Berwald space (M, F) admits a parallel $s-$structure then it is locally symmetric.*

Proof: Let (M, F) be a Berwald space and let $\{s_x\}$ be a parallel $s-$structure on (M, F). Let $X, Y, Z \in T_pM$ be tangent vectors and $\omega \in T_p^*M$ a covector at $p \in M$. By parallel translation along each geodesic through p, X, Y, Z, ω can be extended to local vector fields $\widetilde{X}, \widetilde{Y}, \widetilde{Z}, \widetilde{\omega}$ with vanishing covariant derivatives at p. Because S is parallel, the local vector fields $S\widetilde{X}, S\widetilde{Y}, S\widetilde{Z}, S^{*-1}\widetilde{\omega}$ have also vanishing covariant derivative at p. Now, because R is invariant with respect to the affine transformation $s_x, x \in M$ [5], we have

$$R(\omega, \widetilde{X}, \widetilde{Y}, \widetilde{Z}) = R(S^{*-1}\widetilde{\omega}, S\widetilde{X}, S\widetilde{Y}, S\widetilde{Z}) \tag{3.3}$$

$$\nabla R(\omega, X, Y, Z, U) = \nabla R(S^{*-1}\widetilde{\omega}, X, Y, Z, U) \tag{3.4}$$

Differentiating covariantly (3) in the direction of SU at p and using (4) we get $\nabla R(\omega, X, Y, Z, SU) = \nabla R(S^{*-1}\widetilde{\omega}, SX, SY, SZ, SU) = \nabla R(\omega, X, Y, Z, U)$. Thus $(\nabla R)_p(\omega, X, Y, Z, (I - S)U) = 0$ for all $\omega \in T_p^*M$, $X, Y, Z, U \in T_pM$ and because $(I - S)_p$ is non-singular transformation, we obtain $(\nabla R)_p = 0$. This holds for all $p \in M$ and hence $\nabla R = 0$.□

Let (M, F) be a Berwald space, $p \in M$. Then there exists a neighborhood N_0 of the origin of the tangent space T_pM such that the exponential mapping exp_p is C^∞ diffeomorphism of N_0 on to a neighborhood N_p of p in M [4]. We can also assume that $N_0 = -N_0$. Now we define a mapping of N_p onto itself by

$$s_p : exp(y) \longrightarrow exp(-y) \qquad y \in N_0$$

Then s_p is called the geodesic symmetry with respect to p. M is called locally geodesic symmetric if for any $p \in M$, there exists N_p such that s_p is an isometry of N_p.

Since any isometry of (M, F) is an affine transformation with respect to the connection of F, we see that a locally geodesic symmetric Berwald space (M, F) must be locally symmetric. If F is absolutely homogeneous and (M, F) is locally symmetric, then (M, F) is locally geodesic symmetric.

Corollary 3.6. *If a Berwald space (M, F) admits a parallel $s-$structure and F is absolutely homogeneous then it is locally geodesic symmetric.*

Corollary 3.7. *If a Berwald space (M, F) admits a parallel $s-$structure then its flag curvature is invariant under all parallel displacements.*

Proof: It is a consequence of Theorem 3.5.□

Corollary 3.8. *Let (M, F) be a complete Berwald space which admits a parallel $s-$structure. If the flag curvature of (M, F) is everywhere nonzero, then F is Riemannian.*

Proof: It is a consequence of Theorem 3.5.□

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