Iranian Journal of Mathematical Sciences and Informatics
Vol. 9, No. 2 (2014), pp 1-6
DOI: 10.7508/ijmsi.2014.02.001

On Generalized Coprime Graphs

S. Mutharasu[a,*], N. Mohamed Rilwan[b], M. K. Angel Jebitha[c], T. Tamizh Chelvam[b]

[a]Department of Mathematics, CBM college, Coimbatore-641 042, Tamil nadu, India
[b]Department of Mathematics, Manonmaniam Sundaranar University, Tirunelveli-627 012, Tamil nadu, India
[c]Department of Mathematics, Loyola Institute of Technology and Science, Thovalai-629 302, Tamil nadu, India

E-mail: skannanmunna@yahoo.com
E-mail: rilwan2020@gmail.com
E-mail: jebidom@gmail.com
E-mail: tamche59@gmail.com

Abstract. Paul Erdos defined the concept of coprime graph and studied about cycles in coprime graphs. In this paper this concept is generalized and a new graph called Generalized coprime graph is introduced. Having observed certain basic properties of the new graph it is proved that the chromatic number and the clique number of some generalized coprime graphs are equal.

Keywords: Coprime graph, Semi-perfect, Clique number, Chromatic number.

2000 Mathematics subject classification: 05C 17.

1. Introduction

In 1996, Paul Erdos and Gabor N. Sarkozy [4] have introduced the coprime graph of integers and studied about cycles in coprime graph of integers.

*Corresponding Author

Received 10 June 2012; Accepted 02 March 2013

Further Gabor N. Sarkozy [3] has studied about the complete tripartite subgraphs in the coprime graph of integers. The *coprime graph* on the integer set $X = \{1, 2, \ldots, n\}$(n is a positive integer) is $G = (V, E)$ where $V = X$ and $E = \{(x, y) : x, y \in X$ and $gcd(x, y) = 1\}$. Note that coprime graphs are different from prime graphs [2]. Here, we generalize this definition of coprime graph and define generalized coprime graph on a positive integer n and $A \subseteq X$ as follows: Let $n \geq 2, X = \{1, 2, \ldots, n\}$ and $A \subseteq X$. Then the *generalized coprime graph* on n and A, denoted by $CP(n, A) = (V, E)$, where $V = X$ and $E = \{(x, y) : x, y \in X$ and $gcd(x, y) \in A\}$. Note that coprime graph need not be a subgraph of a generalized coprime graph. Let G be a graph. The *girth* of G, denoted by $g(G)$, is the length of a shortest cycle in G. The *circumference* $c(G)$ of G is the length of a cycle of maximum length in G. The *chromatic number* $\chi(G)$ of G is defined to be the minimum number of colours requires to colour the vertices of G in such a way that no two adjacent vertices have the same colour. The *clique number* $\omega(G)$ of G is the order of the maximum complete subgraph of G. A graph G is said to be *perfect* if the chromatic number and the clique number are same for every induced subgraph of G. In generalization of this, a graph G is said to be *semi-perfect* if the chromatic number and the clique number of G are same. For basic definitions in graph theory, we follow [1].

Throughout this paper we have follow the following notations:
Let $X = \{1, 2, \ldots, n\}$. For any k with $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, let $CP(n, A^{(k)})$ be the generalized coprime graph corresponding to $A^{(k)} = \{1, 2, \ldots, k\}$ and $CP(n, B^{(k)})$ be the generalized coprime graph corresponding to $B^{(k)} = \{xk \in X : x \in \mathbb{N}$ and $xk \leq \lfloor \frac{n}{2} \rfloor\}$. Let S be the set of all primes in X and $S_1 = \{p \in S : p^2 \leq n\}$. Without loss of generality we can assume that $S_1 = \{p_1, p_2, \ldots, p_g\}$ with $1 < p_1 < p_2 < \ldots < p_g$. For $1 \leq k \leq g$, let $C^{(k)} = \{p_1, p_2, \ldots, p_k\} \cup \{1\}$ and $CP(n, C^{(k)})$ be the generalized coprime graph corresponding to $C^{(k)}$ for $1 \leq k \leq g$. Note that coprime graph is a subgraph of $CP(n, A^{(k)}), CP(n, B^{(k)})$ and $CP(n, C^{(k)})$, special classes of generalized coprime graphs. We use the following result in sequel.

Theorem 1.1. [1] *For every graph G of order $n, \chi(G) \geq \omega(G)$ and $\chi(G) \geq \frac{n}{\beta(G)}$.*

2. Properties of Generalized Coprime graphs

In this section, certain basic properties of generalized coprime graphs are obtained. Since K_3 is a subgraph of $CP(n, A^{(k)})$ for all $k \geq 3$, we have the following.

Lemma 2.1. *Let $n \geq 3$ be an integer. Then $g(CP(n, A^{(k)}))$ is 3 for all $k \geq 3$.*

Since $gcd(x, x+1) = 1$ for all $x \in X - \{n\}$ and $gcd(1, n) = 1$, one can prove the following.

Lemma 2.2. *Let $n \geq 3$ be an integer. For all k, $CP(n, A^{(k)})$ is Hamiltonian.*

Lemma 2.3. *Let n and k be integers such that $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$. Then $CP(n, A^{(k)})$ is bipartite if and only if $n = 2$.*

Proof. Suppose $CP(n, A^{(k)})$ is bipartite graph and $n \geq 3$, then by Lemma 2.1, $CP(n, A^{(k)})$ contains an odd cycle C_3 which is a contradiction to $CP(n, A^{(k)})$ is bipartite. Hence $n = 2$. Converse is trivial. □

From the definition of generalized coprime graph, one can observe the following:

Lemma 2.4. *If $A \subseteq B$, then $CP(n, A)$ is a subgraph of $CP(n, B)$.*

Lemma 2.5. *Let $n \geq 2$ be an integer. Then $CP(n, A^{(k)})$ is complete if and only if $k = \lfloor \frac{n}{2} \rfloor$.*

Proof. Suppose $CP(n, A^{(k)})$ is complete and $k < \lfloor \frac{n}{2} \rfloor$. Take $x = \lfloor \frac{n}{2} \rfloor$. Then $x, 2x \in X$ and $gcd(x, 2x) = x = \lfloor \frac{n}{2} \rfloor \notin A^{(k)}$. Therefore x and $2x$ are non-adjacent in $CP(n, A^{(k)})$, a contradiction to $CP(n, A^{(k)})$ is complete. Conversely, assume that $k = \lfloor \frac{n}{2} \rfloor$. Since $gcd(a, b) \leq \lfloor \frac{n}{2} \rfloor$ for all $a, b \in X$, one can conclude that $CP(n, A^{(k)})$ is complete. □

In the following theorem we prove that K_n, the complete graph on n vertices is the union of generalized coprime graphs.

Lemma 2.6. *Let $n \geq 3$ and $S = \{k : k$ is prime and $k \leq \lfloor \frac{n}{2} \rfloor\}$. Then $K_n = H \cup CP(n, A^{(1)})$ where $H = \bigcup_{k \in S} CP(n, B^{(k)})$.*

Proof. Obviously $K_n \supseteq H \cup G^{(1)}$. Let $x, y \in X$ and $gcd(x, y) = d$.
Case(i): Suppose $d = 1$. Then x and y are adjacent in $CP(n, A^{(1)})$.
Case(ii): Suppose d is a prime. Then $d \in S$ and hence x and y are adjacent in $CP(n, B^{(d)}) \subseteq H$.
Case (iii): Suppose $d = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$ where $p_i's$ are primes and $\alpha_i \geq 1$. Then $d = p_1.s$ where $s = p_1^{\alpha_1 - 1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$. Hence $d \in B^{(p_1)}, p_1 \in S$ and so x and y are adjacent in $CP(n, B^{(p_1)}) \subseteq H$. Hence $K_n \subseteq H \cup CP(n, A^{(1)})$. □

3. Semi-Perfect Graphs

In this section, we find the clique number and the chromatic number for $CP(n, A^{(1)}), CP(n, A^{(2)})$ and $CP(n, C^{(k)})$. We also prove that $CP(n, A^{(1)})$, $CP(n, A^{(2)})$ and $CP(n, C^{(k)})$ are semi-perfect.

Theorem 3.1. *Let $n \geq 2$ be a positive integer. Then $\omega(CP(n, A^{(1)})) = |S| + 1$ where $S = \{x \in X : x$ is prime$\}$.*

Proof. Let $S_1 = S \cup \{1\}$. Since $gcd(p, q) = 1$ for all $p, q \in S_1$, $< S_1 >$ is a complete subgraph of $G^{(1)}$ with $|S| + 1$ vertices.
Suppose there exists a maximal complete subgraph $< S_2 >$ of $CP(n, A^{(1)})$ such

that $|S_2| > |S_1|$. Then S_2 must contains at least one composite number v such that $v = v_1^{a_1} v_2^{a_2} \dots v_r^{a_r}$, $v_i's$ are prime and $v_i \geq 1$. Let Y be the set of all proper divisors of v. Suppose $x \in Y \cap S_2$. Then $gcd(x, v) = x > 1$ and so x and v are not adjacent in $< S_2 >$, a contradiction to the fact that $< S_2 >$ is complete. Hence $Y \cap S_2 = \emptyset$. Therefore $gcd(v_i, y) = 1$ for all $y \in S_2$. In particular, $gcd(v_1, y) = 1$ for all $y \in S_2$. Thus $< S_2 \cup \{v_1\} >$ is a complete subgraph of $CP(n, A^{(1)})$, which properly contains S_2, a contradiction to the maximality of S_2. Hence $\omega(CP(n, A^{(1)})) = |S| + 1$. □

Theorem 3.2. *Let $n \geq 2$ be a positive integer. Then $\chi(CP(n, A^{(1)})) = |S| + 1$ where $S = \{x \in X : x \text{ is prime}\}$ and hence $CP(n, A^{(1)})$ is semi perfcet.*

Proof. By Theorem 1.1 and Theorem 3.1, $\chi(CP(n, A^{(1)})) \geq \omega(CP(n, A^{(1)})) = |S| + 1$. Let $S_1 = S \cup \{1\}$. Colour each vertex of S_1 by a different colour. Let $m \in X - S_1$ and p be the least prime divisor of m. Now colour the vertex m by $col(p)$.

Let $a, b \in X$ be two adjacent vertices in $CP(n, A^{(1)})$. Since $gcd(a, b) = 1$, the prime factorization for a and b will contain disjoint set of primes and so $col(a) \neq col(b)$. Hence $\chi(CP(n, A^{(1)})) \leq |S_1| = |S| + 1$ and so $\chi(CP(n, A^{(1)})) = |S| + 1$. □

Theorem 3.3. *Let $n \geq 2$ be a positive integer. Then $\omega(CP(n, A^{(2)})) = |S| + 2$ where $S = \{x \in X : x \text{ is prime}\}$.*

Proof. Let $S_1 = S \cup \{1, 4\}$. Since $gcd(p, q) \leq 2$ for all $p, q \in S_1$, $< S_1 >$ is a complete subgraph of $CP(n, A^{(2)})$. Suppose there exists a maximal complete subgraph $< S_2 >$ such that $|S_2| > |S_1|$. Then there exists a composite number $v \in S_2$.

Case(i): Suppose S_2 contains composite numbers only of the form $x = 2^a, a \geq 2$. Then $v = 2^\alpha$ for some $\alpha \geq 2$. Since $|S_2| > |S_1|$ and by the definition of S_1, S_2 contains another composite number w such that $w = 2^\beta$ such that $\beta \geq 2$ and $\alpha \neq \beta$. Now $gcd(v, w) \geq 4$, a contradiction to S_2 is complete.

Case(ii): Suppose $v = v_1^{a_1} v_2^{a_2} \dots v_r^{a_r}$, $v_i's$ are prime such that $v_1 \neq 2$ and $a_i \geq 1$. Let $Y = \{v_1^{b_1} v_2^{b_2} \dots v_r^{b_r} : v_i \neq 2,\ 1 \leq b_i \leq a_i \text{ and } 1 \leq i \leq r\} - \{v\}$. Suppose $x \in Y \cap S_2$. Then $gcd(x, v) = x > 2$, a contradiction to the fact that $< S_2 >$ is complete. Hence $Y \cap S_2 = \emptyset$. Therefore $gcd(v_i, y) = 1$ for all $y \in S_2$. In particular, $gcd(v_1, y) = 1$ for all $y \in S_2$. Thus $< S_2 \cup \{v_1\} >$ is a complete subgraph of $CP(n, A^{(2)})$, which properly contains S_2, a contradiction to the maximality of S_2. Hence $\omega(CP(n, A^{(2)})) = |S| + 2$. □

Theorem 3.4. *Let $n \geq 2$ be a positive integer. Then $\chi(CP(n, A^{(2)})) = |S| + 2$ where $S = \{x \in X : x \text{ is prime}\}$ and hence $CP(n, A^{(2)})$ is semi perfect.*

Proof. By Theorem 1.1 and Theorem 3.3, $\chi(CP(n, A^{(2)})) \geq \omega(CP(n, A^{(2)})) = |S| + 2$. Let $S_1 = S \cup \{1, 4\}$. Colour each vertex of S_1 by a different colour. Let

$m \in X - S_1$. If $m = 2^\alpha, \alpha \geq 3$, then colour the vertex m by $col(4)$. Otherwise, let $p \neq 2$ be the least prime divisor of m and colour the vertex m by $col(p)$. Let $a, b \in X$ be two adjacent vertices in $CP(n, A^{(2)})$. Then $gcd(a, b) = 1$ or 2.
Case(i): If $gcd(a, b) = 1$, then $a = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}$ and $b = q_1^{b_1} q_2^{b_2} \dots q_t^{b_t}$ such that $p_i's$ and $q_j's$ are primes, $p_i \neq q_j$.
Subcase(i): Suppose $a = 2^\alpha,\ \alpha \geq 2$. Then 2 is not the least prime divisor of b and $col(a) = col(4)$. Hence $col(a) \neq col(b)$.
Subcase(ii): Suppose $b = 2^\alpha,\ \alpha \geq 2$. Then 2 is not the least prime divisor of a and $col(b) = col(4)$. Hence $col(a) \neq col(b)$.
Subcase(iii): Suppose $a \neq 2^\alpha$ and $b \neq 2^\beta$ where $\alpha, \beta \geq 2$ and $\alpha \neq \beta$. Then the least prime divisors of a and b are different and hence a and b have different colours.
Case(ii): If $gcd(a, b) = 2$, then $a = 2(p_1^{a_1} p_2^{a_2} \dots p_r^{a_r})$ and $b = 2(q_1^{b_1} q_2^{b_2} \dots q_t^{b_t})$ such that $p_i's$ and $q_j's$ are primes, $p_i \neq q_j$.
Subcase(i): Suppose $a = 2^\alpha,\ \alpha \geq 2$. Then $col(b) \neq col(4)$ and $col(a) = col(4)$. Hence $col(a) \neq col(b)$.
Subcase(ii): Suppose $b = 2^\alpha,\ \alpha \geq 2$. Then $col(a) \neq col(4)$ and $col(b) = col(4)$. Hence $col(a) \neq col(b)$.
Subcase(iii): Suppose $a \neq 2^\alpha$ and $b \neq 2^\beta,\ \alpha, \beta \geq 2$ and $\alpha \neq \beta$. Then the least prime divisor (greater than 2) of a and b are different and hence a and b have different colours.
Hence $\chi(CP(n, A^{(2)})) \leq |S_1| = |S| + 2$ and so $\chi(CP(n, A^{(2)})) = |S| + 2$. □

Now we obtain a class of graphs which are semi-perfect.

Theorem 3.5. *Let g be the number of primes p such that $1 \leq p \leq n$ and $p^2 \leq n$. For $1 \leq k \leq g, CP(n, C^{(k)})$ is semi-perfect.*

Proof. Define $S^{(1)} = S \cup \{1\}$ and $S^{(i)} = S^{(i-1)} \cup \{p_1.p_{i-1}, p_2.p_{i-1}, \dots, p_{i-1}.p_{i-1}\}$ for $2 \leq i \leq k$.
Claim 1: $\chi(CP(n, C^{(k)})) \leq |S^{(k)}|$. Initially color all the vertices of $S^{(k)}$ by $|S^{(k)}|$ different colors. Let $v \in X - S^{(k)}$.
Case A: If v has at least one divisor of the form p_i^2, for some $p_i \in C^{(k)}$ and v has no prime divisor outside $C^{(k)}$. Choose the least among such divisors and let it be p_i. Now assign for v the color $col(p_i^2)$.
Case B: If v has at least two distinct prime divisors in $C^{(k)}$ and v has no prime divisor outside $C^{(k)}$. Let p_i, p_j be the least prime divisors of v such that $p_i, p_j \in C^{(k)}$. Now assign the color $col(p_i.p_j)$ for v.
Case C: If v has one prime divisor which is not in $C^{(k)}$. Let q_i be the least prime divisor of v such that $q_i \notin C^{(k)}$. Take $col(v)$ as $col(q_i)$. Let $(a, b) \in E(G)$. Then $gcd(a, b) = 1$ or p_i for some $p_i \in C^{(k)}$.
Case(i): If a and b are of different types, then it is easy to verify that $col(a) \neq col(b)$.
Case(ii): If a and b are of Case A. Let p_i be the least divisor of a such that

$p_i \in C^{(k)}$ and p_i^2 divides a. Then $col(a) = col(p_i^2)$. Similarly for b also there exists a least divisor p_j, for some $1 \leq j \leq k$ such that $p_j \in C^{(k)}$ and p_j^2 divides b. Then $col(b) = col(p_j^2)$. Since $gcd(a,b) = 1$ or p_i for some $p_i \in C^{(k)}$, we have $p_i \neq p_j$. Hence $col(a) \neq col(b)$.

Case(iii): If a and b are of Case B. Let p_a, p_b be the least prime divisors of a such that $p_a, p_b \in C^{(k)}$ and let p_c, p_d be the least prime divisors of b such that $p_c, p_d \in C^{(k)}$. Then $col(a) = col(p_a.p_b)$ and $col(b) = col(p_c.p_d)$. Since $gcd(a,b) = 1$ or p_i for some $p_i \in C^{(k)}$, we have $p_a.p_b \neq p_c.p_d$. Hence $col(a) \neq col(b)$.

Case(iv): If a and b are of Case C. Let q_i, q_j be the least prime divisor of a and b respectively such that $q_i, q_j \notin C^{(k)}$. Then $col(a) = col(q_i)$ and $col(b) = col(q_j)$. Since $gcd(a,b) = 1$ or p_i for some $p_i \in C^{(k)}$, we have $q_i \neq q_j$. Hence $col(a) \neq col(b)$.

Claim 2: Now we claim that $\omega(CP(n, C^{(k)})) \geq |S^{(k)}|$. For this we prove that any two vertices of $S^{(k)}$ are adjacent. Let $a, b \in S^{(k)}$ with $a \neq b$. The set $S^{(k)}$ can be written as $S^{(k)} = S \cup \{1\} \cup B$ where $B = \{p_i.p_j : p_i, p_j \in C^{(k)} \cap S\}$.

Case(i): Suppose $a = 1$ or $b = 1$. Then $gcd(a,b) = 1$ and hence a and b are adjacent.

Case(ii): If $a, b \in S$, then $gcd(a,b) = 1$ and hence a and b are adjacent.

Case(iii): If $a, b \in B$. Then $a = q_1 \times q_2$ and $b = q_3 \times q_4$, where $q_1, q_2, q_3, q_4 \in C^{(k)} \cap S$. since $a \neq b$, we have $q_1 \times q_2 \neq q_3 \times q_4$ and so $gcd(a,b) = q_1$ or q_2 or q_3 or q_4 or 1. This means that $gcd(a,b) \in C^{(k)}$ and so a and b are adjacent.

Case(iv): If $a \in S$, $b \in B$. Then $a = p, p \in S$ and $b = q_1 \times q_2$ where $q_1, q_2 \in C^{(k)} \cap S$. Then $gcd(a,b) = 1$ or q_1 or q_2 where $q_1, q_2 \in C^{(k)}$. Hence a and b are adjacent.

Then by Theorem 1.1, we have $|S^{(k)}| \leq \omega(H^{(k)}) \leq \chi(H^{(k)}) \leq |S^{(k)}|$.
$\chi(H^{(k)}) = \omega(H^{(k)}) = |S^{(k)}|$. □

Acknowledgments

The second, third and fourth authors thank the University Grants Commission, Government of India for the financial assistance provided through UGC-SAP(DRS-I) programme of Department of Mathematics, Manonmaniam Sundaranar University.

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