

## Stability and numerical solution of time variant linear systems with delay in both the state and control

Salameh Sedaghat and Yadollah Ordokhani\*

Department of Mathematics, Alzahra University, Tehran, Iran

E-mail: s.sedaghat@alzahra.ac.ir

E-mail: ordokhani@alzahra.ac.ir

**ABSTRACT.** In this paper, stability for uncertain time variant linear systems with time delay is studied. A new sufficient condition for delay-dependent systems is given in matrix inequality form which depends on the range of delay. Then, we introduce a new direct computational method to solve delay systems. This method consists of reducing the delay problem to a set of algebraic equations by first expanding the candidate function as hybrid functions with unknown coefficients. By using collocation method, the coefficients of the hybrid functions are obtained. Some numerical examples are given to illustrate and compare our results with other existing methods in the literature.

**Keywords:** Stability, Time delay, Lyapunov-Krasovskii functional, Uncertainty, Hybrid, Gauss Legendre nodes.

**2000 Mathematics subject classification:** 37B25, 37N30.

### 1. INTRODUCTION

Dynamical systems with time delays have been of considerable interest for decades. Delay systems represent a class of infinite-dimensional systems largely used to describe economic systems, biology [2,9,10,20,21,25], engineering [27], neural network [30], transport phenomena and population dynamics [11-18]. The main motivation for the stability analysis for delay systems is related to

---

\*Corresponding author

the difficulty in having good estimates of the delay values in systems. Different approaches to this problem have already been proposed. Two of the most popular approaches are characteristic equation and Lyapunov function. The characteristic equation can be defined as the equation obtained from the original system by looking for a nontrivial solution of the form  $e^{\lambda t} c_0$ , ( $t > 0$ ), where  $c_0$  is a constant vector of appropriate dimension. Due to its form, the characteristic equation associated with delay system is transcendental, which leads to its having an infinite number of roots in the complex plane. The main idea of Lyapunov function method is to transform the original stability issue into the evolution of some Lyapunov function in an Euclidian space [6, 22]. Also, delay systems are a very important class of systems whose control and optimization have been of interest to many investigators. Much progress has been made towards the solution of delay system using the orthogonal functions. Special attention has been given to applications of Walsh functions [5], block-pulse functions [25], Laguerre polynomials [10], Legendre polynomials [12], Chebyshev polynomials [7] and Fourier series [19].

We provide an overview of two application problems, the air-to-fuel ratio control in gasoline engines and inverted pendulum systems.

The air-to-fuel ratio control is imperative for conventional gasoline vehicles. The analysis of stability for the air-to-fuel ratio control system when the engine speed, and air flow into the engine, are constant is of interest to the researchers. In this case,  $t_d$  (measured output) and  $\tau_e$  (estimated output) are constant. The error between  $t_d$  and  $\tau_e$ ,  $e$ , can be shown to satisfy the following equations, [1]

$$\begin{aligned} \dot{z} &= e, \\ (1.1) \quad \dot{e} &= \frac{1}{\tau_e}(t) - \frac{1}{\tau_e}(k_p e(t - t_d) + k_i z(t - t_d)). \end{aligned}$$

System (1) is a linear time-invariant system with a single delay. To apply the proposed method in this paper it is convenient to transform system (1) into this form:

$$\begin{aligned} (1.2) \quad \frac{d}{d\theta} x &= A(\varepsilon)x(\theta) + B(\varepsilon)x(\theta - 1), \\ x &= [z, e]^T, \quad \varepsilon = [t_d, \tau_e, k_p, k_i]^T, \quad \theta = \frac{t}{t_d}, \end{aligned}$$

where  $\theta$  is the scaled time and  $\varepsilon$  is a parameter vector.

**Pendulum systems:** Consider the inverted pendulum on a cart such as in [31]. The physical structure is shown in Fig. 1 (which is taken from [31]) where  $M$  is the mass of the cart,  $m$  is the mass of the pendulum rod,  $b$  is the friction coefficient of the cart.  $l$  is the length of the corresponding rod;  $F$  is the force acting on the cart,  $x$  is the horizontal displacement of the cart,  $\phi$  is the angle between the pendulum rod and the vertical. Let state variables be  $x$ ,  $\dot{x}$ ,  $\phi$  and  $\dot{\phi}$  which correspond to the horizontal position, horizontal velocity of the cart,

angle and angular velocity of the pole respectively. The equations of motion derived by using Newton's Second Law can be obtained as follows:

$$\begin{aligned}\dot{x} &= \dot{x}, \\ \ddot{x} &= \frac{-(I+ml^2)b}{I(M+m)+Mml^2}\dot{x} + \frac{m^2gl^2}{I(M+m)+Mml^2}\phi + \frac{(I+ml^2)b}{I(M+m)+Mml^2}u, \\ \dot{\phi} &= \dot{\phi}, \\ \ddot{\phi} &= \frac{-mlb}{I(M+m)+Mml^2}\dot{x} + \frac{mgl(M+m)}{I(M+m)+Mml^2}\phi + \frac{ml}{I(M+m)+Mml^2}u,\end{aligned}$$

where  $M, m, b, l$  and  $I$  are known constants. Let state variables be  $x = x_1(t)$ ,  $\dot{x} = x_2(t)$ ,  $\phi = x_3(t)$  and  $\dot{\phi} = x_4(t)$ , the two delays be  $\tau$  and  $d$ , then from [28] the system can be rewritten in the form of

$$(1.3) \quad \dot{x}(t) = Ex(t) + Fx(t - \tau) + Gu(t) + Hu(t - d).$$

To apply the proposed method in this paper, we consider the case of  $\tau = d$ .

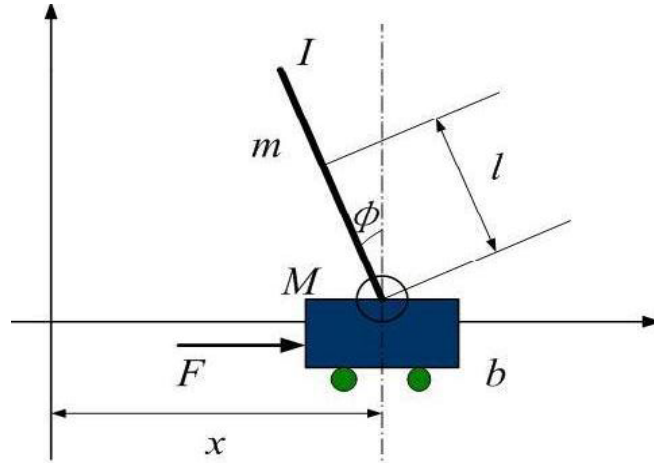


Fig. 1.

In the current work, first, we investigate the stability of time variant systems with time delay where the delay is bounded. In order to establish new delay-dependent sufficient conditions for stability of such a system, we use the Lyapunov-Krasovskii functional method with linear matrix inequality (LMI) approach together with the parameterized first-order model transformation. Also in the present paper we introduce a new direct computational method to solve the delay system. This method consists of reducing the delay problem to a set of algebraic equations by first expanding the candidate function as hybrid functions with unknown coefficients. These hybrid functions, which consist of block-pulse functions plus Legendre polynomials are provided. Then by using the Gauss (-Legendre) nodes, the coefficients of the hybrid function for the

solution of delay system are obtained. Finally we will demonstrate the results by considering four illustrative examples.

## 2. STABILITY

The analysis of the stability of time-varying systems with delay in both the state and control are important in theory and in practice [3]-[29]. Recently, Richard [24] summarized some current researches on time-delay systems and listed four problems, one of which is:

$$(2.1) \quad \dot{x}(t) = E(t)x(t) + F(t)x(t - \tau) + G(t)u(t) + H(t)u(t - \tau), \quad 0 \leq t \leq 1,$$

$$(2.2) \quad x(0) = x_0,$$

$$(2.3) \quad x(t) = \varphi(t), \quad -\tau \leq t < 0,$$

$$(2.4) \quad u(t) = \psi(t), \quad -\tau \leq t < 0,$$

where  $x(t) \in \mathbb{R}^l$  is state vector,  $u(t) \in \mathbb{R}^q$  is control vector,  $E(t)$ ,  $F(t)$ ,  $G(t)$  and  $H(t)$  are certainty matrices of appropriate dimensions,  $x_0$  is a constant specified vector, and  $\varphi(t)$  and  $\psi(t)$  are arbitrary known functions. The time-delay  $\tau$  is positive, bounded and satisfies

$$(2.5) \quad 0 \leq \tau \leq h.$$

**Definition 1.** The uncertain time delay system (4) is said to be stabilizable if there exists a linear memoryless state feedback control law  $u(t) = Kx(t)$ ,  $K \in \mathbb{R}^{q \times l}$ , such that the resulting closed-loop system is stable [8].

By using the definition (1), the time varying system (4) is equivalent to the following system:

$$(2.6) \quad \dot{x}(t) = [E(t) + G(t)K]x(t) + [F(t) + H(t)K]x(t - \tau).$$

Suppose that

$$(2.7) \quad E(t) + G(t)K = A_0 + \Delta A_0(t), \quad F(t) + H(t)K = A_1 + \Delta A_1(t),$$

where  $A_0, A_1 \in \mathbb{R}^{l \times l}$  are constant matrices and the uncertainties  $\Delta A_0(t)$ ,  $\Delta A_1(t)$  are of the form

$$(2.8) \quad \Delta A_0(t) = D_0 F_0(t) E_0, \quad \Delta A_1(t) = D_1 F_1(t) E_1,$$

where  $D_0, E_0, D_1$ , and  $E_1$  are appropriate dimensional constant matrices, and  $\|F_0(t)\| \leq 1, \|F_1(t)\| \leq 1$  for all  $t$ . Then equation (9) can be rewritten as

$$(2.9) \quad \dot{x}(t) = [A_0 + D_0 F_0(t) E_0]x(t) + [A_1 + D_1 F_1(t) E_1]x(t - \tau).$$

In addition, the goal of this paper is to find the criteria for stability of the system (2.1) by using the Lyapunov method in conjunction with linear matrix inequality (LMI) techniques.

Since  $x(t)$  is continuously differentiable for  $t \geq 0$ , by adopting the Leibnitz-Newton formula, we have

$$(2.10) \quad \begin{aligned} x(t - \tau) &= x(t) - \int_{-\tau}^0 \dot{x}(t + \theta) d\theta = \\ &= x(t) - \int_{-\tau}^0 [A_0(\theta)x(t + \theta) + A_1(\theta)x(t + \theta - \tau)] d\theta. \end{aligned}$$

Therefore, the system (12) can be transferred into a distributed delay system

$$(2.11) \quad \begin{aligned} \dot{x}(t) &= (A_0 + D_0 F_0(t) E_0 + C)x(t) + (A_1 + D_1 F_1(t) E_1 - C)x(t - \tau) \\ &\quad - C \int_{t-\tau}^t [A_0(\theta)x(\theta) + A_1(\theta)x(\theta - \tau)] d\theta, \end{aligned}$$

where  $C$  is a parametric matrix which derives the stability result less restrictive to some degree. Since in this process only one integration over one delay interval is used, the process is called parameterized first-order model transformation [21]. The stability of the system (14) implies [4] the stability of the system (12).

Since the stability of the system (14) implies the stability of the system (12) and consequently the stability of the system (9), we mainly focus on the stability of the system (14) in the main theorem. The following lemmas will be essential for the proof of the main theorem.

**Lemma 1.** Let  $w(t) = \int_{a(t)}^{b(t)} \int_{t-\theta}^t f(s) ds d\theta$ . Then the following is satisfied [9]

$$\frac{d}{dt} w(t) = (b - a)f(t) - (1 - \dot{b}) \int_{t-b}^{t-a} f(s) ds + (\dot{b} - \dot{a}) \int_{t-a}^t f(s) ds.$$

**Lemma 2.** Let  $a(t) \leq b(t)$ . Then, the following inequality holds [9]

$$\left\| \int_a^b f(s) ds \right\|^2 \leq (b - a) \int_a^b \|f(s)\|^2 ds.$$

Now we present a sufficient condition that guarantees the stability of the system (2.1) satisfying the uncertainty in equation (11) and the time delay in equation (8).

**Theorem 1.** The system (14) is asymptotically stable, if there exist four positive -definite matrices  $P, Q_{10}, Q_{11}, R > 0$ , positive constants  $\lambda, \eta_0, \eta_1$  and a constant matrix  $W \in \mathbb{R}^{n \times n}$  such that

$$\begin{aligned} \Omega_1 &= PA_0 + A_0^T P + W + W^T + \frac{1}{\lambda} P D_0 D_0^T P + \lambda E_0^T E_0 + \\ &\quad (PA_1 - W)R^{-1}(PA_1 - W)^T + P D_1 D_1^T P + W Q_{10}^{-1} W^T + \\ &\quad W Q_{11}^{-1} W^T + S_1 + h^2 S_{10}(Q_{10}, \eta_0) + h^2 S_{11}(Q_{11}, \eta_1) < 0, \end{aligned}$$

satisfying

$$(\eta_0 I - D_0^T Q_{10} D_0) > 0, \quad (\eta_1 I - D_1^T Q_{11} D_1) > 0, \quad S_1 = R + E_1^T E_1.$$

Also, the corresponding model transformation of the matrix in the system (14) is given by  $C = P^{-1}W$ .

**proof:** We consider the following Lyapanov-Krasovskii functional

$$(2.12) \quad V(x(t)) = V_1(x(t)) + hV_2(x(t)) + hV_3(x(t)),$$

where

$$V_2(x(t)) = \int_0^h \int_{t-\theta}^t x^T(s) S_{10}(Q_{10}, \eta_0) x(s) ds d\theta,$$

$$V_3(x(t)) = \int_\tau^{h+\tau} \int_{t-\theta}^t x^T(s) S_{11}(Q_{11}, \eta_1) x(s) ds d\theta.$$

The system is asymptotically stable if the derivative of the functional is strictly negative. By using Lemma 1, we have

$$\frac{d}{dt} V_2(x(t)) = hx^T(t) S_{10}(Q_{10}, \eta_0) x(t) - \int_{t-h}^t x^T(s) S_{10}(Q_{10}, \eta_0) x(s) ds,$$

$$\begin{aligned} \frac{d}{dt} V_3(x(t)) &= hx^T(t) S_{11}(Q_{11}, \eta_1) x(t) - \int_{t-h-\tau}^{t-\tau} x^T(s) S_{11}(Q_{11}, \eta_1) x(s) ds \\ &\leq hx^T(t) S_{11}(Q_{11}, \eta_1) x(t) - (1-d) \int_{t-h-\tau}^{t-\tau} x^T(s) S_{11}(Q_{11}, \eta_1) x(s) ds. \end{aligned}$$

Also, let:  $V_1(x(t)) = x^T(t) P x(t) + \int_{t-\tau}^t x^T(\lambda) S_1(\lambda) x(\lambda) d\lambda$ , with  $P > 0$ , then its time derivative is

$$\begin{aligned} \frac{d}{dt} V_1(x(t)) &= x^T(t) (P A_0 + A_0^T P + W + W^T) x(t) \\ &\quad + 2x^T(t) P D_0 F_0(t) E_0 x(t) + 2x^T(t) (P A_1 - W) x(t - \tau) \\ (2.13) \quad &\quad + 2x^T(t) P D_1 F_1(t) E_1 x(t - \tau) - 2x^T(t) P C \int_{t-\tau}^t A_0(\theta) x(\theta) d\theta \\ &\quad - 2x^T(t) W \int_{t-\tau}^t A_1(\theta) x(\theta - \tau) d\theta + x^T(t) S_1 x(t) \\ &\quad - x^T(t - \tau) S_1 x(t - \tau). \end{aligned}$$

Using the following inequalities for any positive real number  $\beta > 0$  and any positive definite matrix  $D$ , we have

$$-2u^T v \leq 2u^T v \leq \beta u^T D^{-1} u + \beta^{-1} v^T D v,$$

where  $u, v \in \mathbb{R}^n$ . We get

$$\begin{aligned} \frac{d}{dt} V_1(x(t)) &\leq x^T(t) (P A_0 + A_0^T P + W + W^T) x(t) \\ &\quad + \lambda x^T(t) E_0^T E_0 x(t) + \frac{1}{\lambda} x^T(t) P D_0 D_0^T P x(t) \\ (2.14) \quad &\quad + x^T(t) (P A_1 - W) R^{-1} (P A_1 - W)^T x(t) + x^T(t - \tau) R x(t - \tau) \\ &\quad + x^T(t) P D_1 D_1^T P x(t) + x^T(t - \tau) E_1^T E_1 x(t - \tau) \\ &\quad + 2 \| x^T(t) W Q_{10}^{-1/2} \| \cdot \| \int_{t-\tau}^t Q_{10}^{1/2} A_0(\theta) x(\theta) d\theta \| \\ &\quad + 2 \| x^T(t) W Q_{11}^{-1/2} \| \cdot \| \int_{t-\tau}^t Q_{11}^{1/2} A_1(\theta) x(\theta - \tau) d\theta \| \\ &\quad + x^T(t) S_1(x(t) - x(t - \tau)) S_1 x(t - \tau). \end{aligned}$$

The following relations are obtained from [9]

$$\begin{aligned} A_0^T(t) Q_{10} A_0(t) &= [A_0 + D_0 F_0(t) E_0]^T Q_{10} [A_0 + D_0 F_0(t) E_0] \\ &\leq A_0^T Q_{10} A_0 + A_0^T Q_{10} D_0 (\eta_0 I - D_0^T Q_{10} D_0)^{-1} D_0^T Q_{10} A_0 \\ &\quad + \eta_0 E_0^T E_0 = S_{10}(Q_{10}, \eta_0), \end{aligned}$$

$$\begin{aligned}
(2.15) \quad A_1^T(t)Q_{11}A_1(t) &= [A_1 + D_1F_1(t)E_1]^T Q_{11} [A_1 + D_1F_1(t)E_1] \\
&\leq A_1^T Q_{11} A_1 + A_1^T Q_{11} D_1 (\eta_1 I - \\
&\quad D_1^T Q_{11} D_1)^{-1} D_1^T Q_{11} A_1 + \eta_1 E_1^T E_1 \\
&= S_{11}(Q_{11}, \eta_1).
\end{aligned}$$

Recall the following relations from [9]

$$\begin{aligned}
\int_{t-\tau}^t \|Q_{10}^{1/2} A_0(\theta)x(\theta)\|^2 d\theta &\leq \int_{t-\tau}^t x^T(\theta) S_{10}(Q_{10}, \eta_0) x(\theta) d\theta, \\
\int_{t-\tau}^t \|x^T(\theta - \tau) A_1^T(\theta) Q_{11}^{1/2}\|^2 d\theta &\leq \int_{t-\tau-h}^{t-\tau} x^T(\theta) S_{11}(Q_{11}, \eta_1) x(\theta) d\theta,
\end{aligned}$$

and apply this to equation (17) to obtain

$$\begin{aligned}
\frac{d}{dt} V_1(x(t)) &\leq x^T(t) (PA_0 + A_0^T P + W + W^T + \frac{1}{\lambda} P D_0 D_0^T P \\
&\quad + \lambda E_0^T E_0 + P D_1 D_1^T P + (PA_1 - W) R^{-1} (PA_1 - W)^T) x(t) \\
&\quad + x^T(t - \tau) R x(t - \tau) + x(t - \tau) (R + E_1^T E_1) x(t - \tau) \\
&\quad + x^T(t) W Q_{10}^{-1} W^T x(t) + \tau \int_{t-\tau}^t x^T(\theta) S_{10}(Q_{10}, \eta_0) x(\theta) d\theta \\
&\quad + x^T(t) W Q_{11}^{-1} W^T x(t) + h \int_{t-\tau-h}^{t-\tau} x^T(\theta) S_{11}(Q_{11}, \eta_1) x(\theta) d\theta \\
&\quad + x^T(t) S_1 x(t) - x^T(t - \tau) S_1 x(t - \tau).
\end{aligned}$$

Therefore

$$\begin{aligned}
\dot{V}(x(t)) &\leq x^T(t) (PA_0 + A_0^T P + W + W^T + \frac{1}{\lambda} P D_0 D_0^T P + \\
&\quad \lambda E_0^T E_0 + P D_1 D_1^T P + (PA_1 - W) R^{-1} (PA_1 - W)^T + W Q_{10}^{-1} W^T + \\
&\quad W Q_{11}^{-1} W^T + S_1 + h^2 S_{10}(Q_{10}, \eta_0) + h^2 S_{11}(Q_{11}, \eta_1)) x(t) < 0.
\end{aligned}$$

Thus, if the conditions of theorem hold then the derivative of the functional is strictly negative which implies asymptotic stability.

### 3. NUMERICAL SOLUTION

In this section the solution of time delay systems is obtained by using a hybrid functions and collocation method. The properties of the hybrid functions consisting of block-pulse functions and Legendre polynomials are presented.

#### 3.1. Properties of hybrid functions

Hybrid functions  $b(n, m, t)$ ,  $n = 1, 2, \dots, N$ ,  $m = 0, 1, \dots, M - 1$ , have three arguments:  $n$  is the order of block-pulse functions,  $m$  is the order of Legendre

polynomials, and  $t$  is the normalized time [18]. They are defined on the interval  $[0, t_f]$  as

$$b(n, m, t) = \begin{cases} P_m\left(\frac{2N}{t_f}t - 2n + 1\right), & t \in \left[\frac{n-1}{N}t_f, \frac{n}{N}t_f\right), \\ 0, & \text{otherwise.} \end{cases}$$

Here  $P_m(t)$  are the well-known Legendre polynomials of order  $m$ . A function  $f(t)$  defined over the interval  $[0, t_f]$  may be expanded as

$$(3.1) \quad f(t) \simeq \sum_{n=1}^N \sum_{m=0}^{M-1} c_{nm} b(n, m, t) = c^T B(t),$$

where

$$(3.2) \quad c = [c_{10}, \dots, c_{1(M-1)} | c_{20}, \dots, c_{2(M-1)} | \dots | c_{N0}, \dots, c_{N(M-1)}]^T,$$

$$(3.3) \quad B(t) = [b(1, 0, t), \dots, b(1, M-1, t) | \dots | b(N, 0, t), \dots, b(N, M-1, t)]^T.$$

The integration of the vector  $B(t)$  defined in equation (21) can be approximated by

$$\int_0^t B(t') dt' \simeq PB(t),$$

where  $P$  is the  $NM \times NM$  operational matrix for integration and is given in [23]. Now, consider the linear time varying system (4), let

$$(3.4) \quad x(t) = [x_1(t), x_2(t), \dots, x_l(t)]^T,$$

$$(3.5) \quad u(t) = [u_1(t), u_2(t), \dots, u_q(t)]^T,$$

$$(3.6) \quad \hat{B}(t) = I_l \otimes B^T(t),$$

$$(3.7) \quad \hat{B}1(t) = I_q \otimes B^T(t),$$

where  $I_l$  and  $I_q$  are the  $l$  and  $q$  dimensional identity matrices respectively,  $\hat{B}(t)$  and  $\hat{B}1(t)$  are  $lMN \times l$  and  $qMN \times q$  matrices respectively as well, and  $\otimes$  denotes Kronecker product [18]. Assume that each of  $x_i(t)$  and  $u_j(t)$ ,  $i = 1, 2, \dots, l$ ,  $j = 1, 2, \dots, q$  can be written in terms of hybrid functions as

$$(3.8) \quad \dot{x}_i(t) = B^T(t)X_i,$$

$$(3.9) \quad u_i(t) = B^T(t)U_j.$$

By using equations (24)-(27) we have

$$(3.10) \quad \dot{x}(t) = \hat{B}(t)X,$$

$$(3.11) \quad u(t) = \hat{B}1(t)U,$$



where  $X$  and  $U$  are vectors of order  $lMN \times 1$  and  $qMN \times 1$ , respectively, given by

$$\begin{aligned} X &= [X_1, X_2, \dots, X_l]^T, \\ U &= [U_1, U_2, \dots, U_q]^T. \end{aligned}$$

From equation (26) we get

$$(3.12) \quad x(t) = \hat{B}(t)\hat{P}X + x(0),$$

where  $\hat{P} = I_l \otimes P^T$ . We can also approximate  $X(t - \tau)$  in terms of hybrid function as

$$(3.13) \quad x(t - \tau) = \begin{cases} \varphi(t - \tau), & 0 < t < \tau, \\ \hat{B}(t - \tau)\hat{P}X + x(0), & \tau < t < 1, \end{cases}$$

moreover

$$(3.14) \quad \int_0^t \hat{B}(t') dt' = (I_l \otimes B^T(t))(I_l \otimes P^T) = \hat{B}(t)\hat{P}.$$

### 3.2. Solution of time varying linear delay systems

Consider the time varying linear delay systems given in equation (4). To solve  $x(t)$ , we first choose  $N$  in the following manner:

$$(3.15) \quad N = \begin{cases} \frac{t_f}{\tau}, & \frac{t_f}{\tau} \in Z, \\ \lceil \frac{t_f}{\tau} \rceil + 1, & \text{otherwise,} \end{cases}$$

where  $\lceil \cdot \rceil$ , denotes greatest integer value. Define

$$I_n = [\frac{n-1}{N}t_f, \frac{n}{N}t_f), \quad n = 1, \dots, N,$$

the motivation for choosing such subintervals is to include the primary discontinuities of the solution in the boundaries of  $I_n$ .

Also from equations (4)-(7) and (22)-(32) the following is obtained

$$(3.16) \quad \begin{aligned} \hat{B}(t)X &= E(t)(\hat{B}(t)\hat{P}X + x(0)) + F(t)(\varphi(t - \tau) + \hat{B}(t - \tau)\hat{P}X \\ &+ x(0)) + G(t)\hat{B}1(t)U + H(t)(\psi(t - \tau) + \hat{B}1(t - \tau)\hat{P}U). \end{aligned}$$

For suitable collocation points we choose the points as

$$t_{nj} = (\frac{t_f}{2N})(t_j - 1) + \frac{n}{N}t_f, \quad j = 0, 1, \dots, M - 1, \quad n = 1, 2, \dots, N,$$

where  $t_j$  are the M Legendre nodes for  $[-1, 1]$  and the collocation points  $t_{nj}$  are the shifted of  $t_j$  into  $I_n$ ,  $n = 1, \dots, N$ . We now collocate equation (34) at MN points  $t_{nj}$  as

$$(3.17) \quad \hat{B}(t_{nj})X = E(t_{nj})(\hat{B}(t_{nj})\hat{P}X + x(0)) + F(t_{nj})(\varphi(t_{nj} - \tau) + \hat{B}(t_{nj} - \tau)\hat{P}X + x(0)) + G(t_{nj})\hat{B}1(t_{nj})U + H(t_{nj})(\psi(t_{nj} - \tau) + \hat{B}1(t_{nj} - \tau)\hat{P}U).$$

Equation (35) gives MN linear equations which can be solved for the elements of X using the well Newton's iterative method.  $x(t)$  is gained by substituting X in equation (30).

#### 4. ILLUSTRATIVE EXAMPLES

In this section, some numerical examples will be demonstrated to compare with the previous results.

**Example 1.** Let us consider the following easy case of the time delay uncertain system:

$$\dot{x}(t) = A_0x(t) + A_1x(t - \tau) + B_0u(t) + B_1u(t - \tau),$$

where

$$A_0 = \begin{pmatrix} -4 & -3.923076923 \\ 1 & 0 \end{pmatrix}, A_1 = \begin{pmatrix} -0.05 & -0.1 \\ -1.623076923 & -6.130177515 \end{pmatrix},$$

$$B_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, B_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \text{ Consequently the maximum delay associated}$$

$$\text{with } P = \begin{pmatrix} 1 & 1.923076923 \\ 1.923076923 & 6.130177515 \end{pmatrix}, W = \begin{pmatrix} 0.06 & 0 \\ 0 & 0.03 \end{pmatrix}, Q_{10} =$$

$$\begin{pmatrix} \frac{1}{6} & 0 \\ 0 & \frac{1}{6} \end{pmatrix}, Q_{11} = I_{2 \times 2}, K = B^T(t)P, R = \begin{pmatrix} 0.3 & 0 \\ 0 & 2 \end{pmatrix}, \Delta A_0(t) = \Delta A_1(t) =$$

$$0, \text{ is obtained as } \tau < 0.9523. \text{ Note that this example is made by authors.}$$

**Example 2.** Let us consider the uncertain time delay systems with time varying delay described by the following state equation:

$$\dot{x}(t) = A_0x(t) + A_1x(t - \tau) + \Delta A_0x(t) + \Delta A_1x(t - \tau),$$

where  $A_0 = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}, A_1 = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}, \|\Delta A_0(t)\| \leq 0.05$  and  $\|\Delta A_1(t)\| \leq 0.1$ . With  $P = \begin{pmatrix} 1 & -0.9 \\ -0.9 & 1 \end{pmatrix}, \lambda = 1, R = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, W =$

$$\begin{pmatrix} -0.1 & 0 \\ 0 & -0.5 \end{pmatrix}, Q_{10} = \begin{pmatrix} 0.1 & 0 \\ 0 & 1 \end{pmatrix}, Q_{11} = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.9 \end{pmatrix}, \eta_1 = 0.2$$
 and  $\eta_0 = 0.1$ , we conclude that the system is stable if  $\tau < 0.4661$ . This value is larger than  $\tau < 0.43$  in [4].

**Example 3:** Consider a time-varying delay system described by

$$\begin{aligned} \dot{x}(t) &= 16tx(t - \frac{1}{4}), \\ x(0) &= 1, \\ x(t) &= 0, \quad -\frac{1}{4} \leq t < 0, \end{aligned}$$

that is stable for  $\tau = 0.25$ . The exact solution is [18]

$$x(t) = \begin{cases} 1, & 0 \leq t < \frac{1}{4}, \\ 1 + 4(t - \frac{1}{4}) + 8(t - \frac{1}{4})^2, & \frac{1}{4} \leq t < \frac{1}{2}, \\ \frac{5}{2} + 8(t - \frac{1}{2}) + 24(t - \frac{1}{2})^2 \\ + \frac{128}{3}(t - \frac{1}{2})^3 + 32(t - \frac{1}{2})^4, & \frac{1}{2} \leq t < \frac{3}{4}, \\ \frac{163}{24} + 20(t - \frac{3}{4}) + 68(t - \frac{3}{4})^2 \\ + \frac{416}{3}(t - \frac{3}{4})^3 + 224(t - \frac{3}{4})^4 \\ + \frac{640}{3}(t - \frac{3}{4})^5 + \frac{256}{3}(t - \frac{3}{4})^6, & \frac{3}{4} \leq t < 1. \end{cases}$$

Since  $\tau = 0.25$ , we solve this problem with the method presented in Section 3 by choosing  $N=4$  and  $M=5$ . In Table 1 a comparison is made between the exact solution and the approximate solution of  $x(t)$  for  $0 \leq t \leq 1$ .

**Table 1.** Approximate and exact values of  $x(t)$ .

$t$	Numerical Solution	Exact
0.0	1.000000000000	1.000000000000
0.1	1.000000000000	1.000000000000
0.2	1.000000000000	1.000000000000
0.3	1.220000000000	1.220000000000
0.4	1.780000000000	1.780000000000
0.5	2.499999999999	2.500000000000
0.6	3.585866666666	3.585870000000
0.7	5.452533333333	5.452533333333
0.8	8.480308022689	8.480308022690
0.9	13.41991394574	13.41991394574
1.0	21.81125230994	21.81125230999

In this problem, the error of the approximation of solution is:

$$E^{N,M} = \|x^{exact} - x^{N,M}\|_{\infty} = \max_{0 \leq t \leq 1} |x^{exact}(t) - x^{N,M}(t)|, M = 0, 1, 2, \dots,$$

where  $x^{N,M}(t)$  is the approximate value of  $x(t)$  and  $N$  is a fixed number. In this example we have  $E^{4,5} \leq 5 \times 10^{-12}$

**Example 4:** Consider the same interval system as given in [30]:

$$\dot{x}(t) = A_0(t)x(t) + A_1(t)x(t - \tau),$$

where  $A_0(t) = A_0 + (\varepsilon_0 \sin^2 t)I_2$ ,  $A_1(t) = A_1 + (\varepsilon_1 \cos^2 t)I_2$ , where  $A_0, A_1$  are known  $2 \times 2$  matrices,  $\varepsilon_0$  and  $\varepsilon_1$  are uncertain but bounded as  $|\varepsilon_0| \leq$

0.35 and  $|\varepsilon_1| \leq 0.35$ . Here we assume that  $A_0 = \begin{pmatrix} -2 & 0 \\ 0 & -1.9 \end{pmatrix}$ ,  $A_1 = \begin{pmatrix} -0.5 & 0 \\ -0.1 & -0.5 \end{pmatrix}$ , with  $P = \begin{pmatrix} 172.2344 & 0 \\ 0 & 140 \end{pmatrix}$ ,  $R = \begin{pmatrix} 86 & 0 \\ 0 & 70 \end{pmatrix}$ ,  $W = \begin{pmatrix} -0.1172 & 0 \\ 0 & -0.1 \end{pmatrix}$ ,  $Q_{10} = Q_{11} = \begin{pmatrix} 0.1 & 0 \\ 0 & 1 \end{pmatrix} = D_0 = D_1$ ,  $E_0 = E_1 = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$ ,  $\lambda = 3$ ,  $d = 0$ ,  $\eta_0 = \eta_1 = 0.5$ . By considering the condition we get  $h < 0.961671992$  which is greater than  $h < 0.73$  of the example in [30] with the same condition. We applied the method presented in Section 3 for  $|\varepsilon_0| = 0.3$ ,  $|\varepsilon_1| = 0.2$ ,  $\tau = .25$  and  $x(t) = 1$  for  $-\tau \leq t < 0$ , the computational results are given in Tables 2 and 3:

**Table 2.** Approximate values of  $x_1(t)$ .

$t$	$N = 4, M = 6$	$N = 4, M = 7$	$N = 4, M = 8$	$N = 4, M = 9$
0.0	1.000000000383	1.000000000004	1.000000000000	1.000000000000
0.1	0.818812466605	0.818812466637	0.818812466640	0.818812466640
0.2	0.670852239548	0.670852239480	0.670852239482	0.670852239483
0.3	0.536019149150	0.536019153054	0.536019153220	0.536019153217
0.4	0.416389278469	0.416389276918	0.416389277062	0.416389277060
0.5	0.322279796990	0.322279820464	0.322279820950	0.322279820953
0.6	0.248760122354	0.248760121933	0.248760121765	0.248760121765
0.7	0.191777319156	0.191777324147	0.191777324963	0.191777324965
0.8	0.147679062343	0.147679065294	0.147679065385	0.147679065387
0.9	0.113600463339	0.113600462938	0.113600463024	0.113600463024
1.0	0.087294815424	0.087294828803	0.087294828476	0.087294828476

**Table 3.** Approximate values of  $x_2(t)$ .

$t$	$N = 4, M = 6$	$N = 4, M = 7$	$N = 4, M = 8$	$N = 4, M = 9$
0.0	1.000000002135	1.000000000000	1.000000000000	1.000000000000
0.1	0.827041668543	0.827041668740	0.827041668741	0.827041668740
0.2	0.684404354022	0.684404353680	0.684404353680	0.684404353681
0.3	0.548167710187	0.548167713048	0.548167713172	0.548167713169
0.4	0.423073764020	0.423073762387	0.423073762493	0.423073762492
0.5	0.324644523787	0.324644549418	0.324644549921	0.324644549924
0.6	0.248030323842	0.248030323419	0.248030323245	0.248030323244
0.7	0.189129765919	0.189129771414	0.189129771224	0.189129771225
0.8	0.143963941347	0.143963944851	0.143963944952	0.143963944954
0.9	0.109409425501	0.109409425089	0.109409425185	0.109409425185
1.0	0.083017439576	0.083017455163	0.083017454776	0.083017454776

To show the convergence behavior of the approximate solution of this problem, we define the maximum errors for  $x_i$  as,

$$E_i^{N,M+1} = \|x_i^{N,M+1} - x_i^{N,M}\|_\infty = \max_{0 \leq t \leq 1} |x_i^{N,M+1}(t) - x_i^{N,M}(t)|,$$

for  $M = 0, 1, 2, \dots$  and  $i = 1, 2$ .

$N$  is a fixed number and  $x_i^{N,M}(t)$  is the approximate value of  $x_i(t)$ . In Tables 2 and 3 it is shown that by increasing  $M$ , the number of fixed decimal digits increases and so does the maximum errors of  $E_i^{N,M+1} \rightarrow 0$  for  $i = 1, 2$ . In this example we have  $E_i^{4,9} \leq 3 \times 10^{-12}$ .

## 5. CONCLUSION

This paper has presented a state- and input-delay dependent stabilization criterion for a system with both state and input delays that employs a memoryless state feedback control law. The stability is obtained by using the Lyapunov-Krasovskii functional approach. A new sufficient condition for delay-dependent systems is given in matrix inequality form. Then, we present a highly accurate method to solve the time delay systems. The hybrid of block-pulse functions and Legendre polynomials and the associated operational matrix of integration  $P$  are applied to solve the time-delay systems. The matrix  $P$  has many zeros; hence, the method is computationally attractive. The method is based upon reducing the system into a set of algebraic equations. By using collocation method, the coefficients of the hybrid functions are obtained. The method proposed in this paper can be easily applied to several engineering problems.

**Acknowledgments.** The authors are very grateful to both reviewers for carefully reading the paper and for their comments and suggestions.

## REFERENCES

1. B. Balachandran, T. Kalmar-Nagy and D. E. Gilsinn, *Delay Differential Equations, Recent Advances and New Directions*, Springer, 2009.
2. S. A. J. Campbell, Bélair, Multiple-delayed differential equations as model for biological control systems, *In Proceeding World Congress of Nonlinear Analysis*, **92** (1992), 3110-3117, Tampa.
3. Y. Y. Cao, Z. Lin and T. Hu, Stability analysis of linear time-delay systems subject to input saturation, *IEEE Transactions on Circuits and Systems: Fundamental Theory and Applications*, **49** (2002), 233-240.
4. J. Cao and J. Wang, Delay-dependent robust stability of uncertain nonlinear systems with time delay, *Applied Mathematics and Computation*, **154** (2004), 289-297.
5. W. L. Chen and Y. P. Shih, Shift walsh matrix and delay differential equations, *IEEE Trans. Automatic Control*, **23** (1978), 1023-1028 .
6. E. Fridman and S. I. Niculescu, On complete Lyapunov-Krasovskii functional techniques for uncertain systems with fast-varying delays, *International Journal of Robust and Nonlinear Control*, **18**(3) (2008), 364-374.

7. I. R. Horng and J. H. Chou, Analysis, parameter estimation and optimal control of time delay systems via Chebyshev series, *International Journal of Control*, **41** (1985), 1221-1234.
8. S. M. Karbassi and F. Soltanian, A new approach to the solution of sensitivity minimization in linear state feedback control, *Iranian Journal of Mathematical Sciences and Informatics*, **2** (2007), 1-13.
9. J. H. Kim, Delay and time-derivative dependent robust stability of time-delay linear systems with uncertainty, *IEEE Trans, Autom. Contr.*, **46**(5) (2001) 789-792.
10. Y. Kuang, *Delay Differential Equations with Applications in Population Dynamics*, Academic Press, Boston, 1993.
11. F. C. Kung and H. Lee, Solution and parameter estimation of linear time-invariant delay systems using laguerre polynomial expansion, *Trans. ASME Ser. G J. Dynamic Systems Measurement Control*, **16** (1985), 1249-1256.
12. H. Lee and F. C. Kung, Shifted Legendre series solution and parameter estimation of linear delayed systems, *International Journal of Systems Science*, **16** (1985), 1221-1234.
13. C. D. Li and X. Liao, A global exponential robust Stability Criterion for NN with variable delays, *Neurocomputing*, **69** (2006) 80-89.
14. X. Li and C. E. de Souza, LMI approach to delay -dependent robust stability of uncertain linear systems, *Proc. of the 34th CDC*, New Orleans, (1995), 3614-3619.
15. X. Li and C. E. de Souza, Delay dependent robust stability and stabilization of uncertain linear delay system, A linear matrix inequality approach, *IEEE Trans, on Automatic Control*, **40** (1997) 1144-1148.
16. N. Macdonald, *Biological Delay Systems, Linear Stability Theory*, Cambridge University Press, Cambridge, 1989.
17. H. R. Marzban and M. Razzaghi, Optimal control of linear delay systems via hybrid of Block Pulse and Legendre polynomials , *Journal of the Franklin Institute*, **341** (2004), 279-293.
18. H.R. Marzban and M. Razzaghi, Solution of time-varying delay systems by hybrid functions, *Mathematics and Computers in Simulation*, **64** (2004), 597-607.
19. S. G. Mouroutsos and P. D. Sparis, Shift and product Fourier matrices and linear delay differential equations, *International Journal of Systems Science*, **17** (1986), 1335-1348.
20. S.-I. Niculescu, J.-M. Doin, L. Dugard and H. Li, Stability of linear S systems with several delays: An L.M.I. Approach, *JESA, Spicial Issue on Analysis and Control of Time Delay Systems*, **31** (1997) 955-970.
21. S.-I. Niculescu, *Delay Effects on Stability, A Robust Approach*, Springer, Berlin, 2001.
22. A. Papachristodoulou, M. Peet and S. Lall, *Constructing Lyapunov-Krasovskii Functionals For Linear Time Delay Systems*, American Control Conference June 8-10, Portland, OR, USA 2005.
23. M. Razzaghi and H.R. Marzban, A Hybrid analysis direct method in the calculus of variations, *Int. J. Comput. Math.*, **75** (2000), 259-269.
24. J. P. Richard, Time-delay systems:an Overview of some recent advances and open problems, *Automatica*, **39** (2003), 1667-694.
25. Y. P. Shih, C. Hwang and W. K. Chia, Parameter estimation of delay systems via block pulse functions, *Trans. ASME Ser. G J. Dynamic Systems Measurement Control*, **102** (1980), 159-162.
26. G. Stepan, *Retarded dynamical system: stability and characteristic function*, *Reaserch Notes in Mathematics Series*, New Yourk, John Wiley, 1998, pp. 210.
27. J. H. Su, Further results on the robust stability of linear systems with a single time delay, *Syst. Contr. Lett.*, **23** (1994), 374-379.

28. Y. Xia, M. Fu and P. Shi, *Analysis and synthesis of dynamical systems with time-delays: Lecture notes in control and information sciences*, Springer, **387**, 2009, pp. 99-118.
29. X. M. Zhang, M. Wu, J. H. She and Y. He, Delay-dependent stabilization of linear systems with time-varying state and input delays, *Automatica*, **41** (2005), 1405-1412.
30. Z. Zhang, Ch. Li and X. Liao, Delay-dependent robust stability analysis for interval linear time-variant system with delays and application to delayed neural networks, *Neuro computing*, **70** (2007), 2980-2995.
31. J. Zhou and C. Wen, *Adaptive backstepping control of uncertain systems, Lecture notes in control and information sciences*, Springer, **372**, 2008, pp. 241-263.