

A Note on Tensor Product of Graphs

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ABSTRACT. Let G and H be graphs. The tensor product $G \otimes H$ of G and H has vertex set $V(G \otimes H) = V(G) \times V(H)$, edge set $E(G \otimes H) = \{(a, b)(c, d) | ac \in E(G) \text{ and } bd \in E(H)\}$. In this paper, some results on this product are obtained by which it is possible to compute the Wiener and Hyper Wiener indices of $K_n \otimes G$.

Keywords: Tensor product, connected graph, bipartite graph, Wiener index, hyper Wiener index.

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1. INTRODUCTION

A graph G consists of a set of vertices $V(G)$ and a set of edges $E(G)$. For every vertices $a, b \in V(G)$, the edge connecting a and b is denoted by ab . The distance between two vertices in a connected graph G is the number of edges in a shortest path between them. For vertices a and b of G , their distance is shown by $d_G(a, b)$. This concept has been known for a very long time and has received considerable attention as a subject of research in metric graph theory. Graph operations play an important role in the study of graph decompositions into isomorphic subgraphs. For more details about graph operations see [1, 4, 6, 7, 8, 9, 10, 11, 15]. For any two simple graphs G and H , the tensor product of G and H has vertex set $V(G \otimes H) = V(G) \times V(H)$, edge set $E(G \otimes H) = \{(a, b)(c, d) | ac \in E(G) \text{ and } bd \in E(H)\}$. We refer the reader to

[2] for the proof of this fact that $|E(G \otimes H)| = 2|E(G)||E(H)|$. The distance between two vertices under tensor product of graphs is studied in [13]. Here we obtain useful equality for distance of vertices in tensor product of graphs by a new method and using simple concepts. Then we use these results for obtaining simple proofs for some classical theorems [4]. One of the concepts related to distance in graphs is the Wiener index [14]. It is not only an early index which correlates well with many physico-chemical properties of organic compounds but also a subject that has been studied by many mathematicians and chemists. The Wiener index is the sum of distances between all vertex pairs in a connected graph, $W(G) = \sum_{\{a,b\} \subseteq V(G)} d_G(a,b)$. The hyper-Wiener index of acyclic graphs was introduced by Milan Randić in 1993. Then Klein et al. [12], generalized Randić's definition for all connected graphs, as a generalization of the Wiener index. It is defined as $WW(G) = \frac{1}{2}W(G) + \frac{1}{2} \sum_{\{a,b\} \subseteq V(G)} d^2(a,b)$. In last section of this paper, formula for the Wiener and hyper-Wiener indices of the tensor product of complete graph K_n by a graph G are obtained. For more details about Wiener and hyper-Wiener index see for example [3, 5].

2. MAIN RESULT AND DISCUSSION

In this section some new concepts are presented by which it is possible to find new proof for some classical results.

Definition 1. Let G be a graph and $x, y \in V(G)$. Define $d'_G(x, y)$ as follows:

(i) If $d_G(x, y)$ is odd then $d'_G(x, y)$ is defined as the length of a shortest even walk joining x and y in G , and if there is no shortest even walk then $d'_G(x, y) = +\infty$.

(ii) If $d_G(x, y)$ is even then $d'_G(x, y)$ is defined as the length of a shortest odd walk joining x and y in G , and if there is no shortest odd walk then $d'_G(x, y) = +\infty$.

(iii) If $d_G(x, y) = +\infty$, then $d'_G(x, y) = +\infty$.

Example 1. Let K_n be a complete graph with $n \geq 3$. For each $a, b \in V(K_n)$, if $a \neq b$ then $d_{K_n}(a, b) = 1$ and $d'_{K_n}(a, b) = 2$ and if $a = b$ then $d_{K_n}(a, b) = 0$ and $d'_{K_n}(a, b) = 3$. Also let C_{2n+1} be a cycle of order $2n + 1$, then for each two vertices $a, b \in V(C_{2n+1})$, it is easy to see that, $d'_{C_{2n+1}}(a, b) = 2n + 1 - d_{C_{2n+1}}(a, b)$. In every even cycle C_{2n} , $d'_{C_{2n}}(a, b) = +\infty$, where $a, b \in V(C_{2n})$.

Proposition 1. Let G be a graph. Then G is not bipartite graph if and only if there exist $x, y \in V(G)$ such that $d'_G(x, y) < +\infty$.

Proof. Let G is not bipartite, then there is an odd cycle C in G . Suppose x and y are two vertices on C . If $d_G(x, y)$ is even (odd) then there exists an odd (even) path between x and y in C , so $d'_G(x, y) < +\infty$. Conversely, let $x, y \in V(G)$ and $d'_G(x, y) < +\infty$. Therefore is odd and so there exists an odd

closed walk contains x . Thus there is an odd cycle in G and this proves G is non-bipartite. \square

Note. If G is a connected and non-bipartite graph, then $d'_G(x, y) < +\infty$, for all $x, y \in V(G)$.

Lemma 1. Let G and H be graphs and $(a, b), (c, d) \in V(G \otimes H)$. Then

$$(2.1) \quad \text{Max}\{d_G(a, c), d_H(b, d)\} \leq d_{G \otimes H}((a, b), (c, d)).$$

Proof. If $d_{G \otimes H}((a, b), (c, d)) = +\infty$, then inequality (2.1) holds. Suppose that $d_{G \otimes H}((a, b), (c, d)) = n$. So the following path between (a, b) and (c, d) in $G \otimes H$ exists:

$$P : (a, b) = (x_0, y_0), (x_1, y_1), \dots, (x_n, y_n) = (c, d).$$

Obviously $a = x_0, x_1, \dots, x_n = c$ is a walk with length n between a and c in G and $b = y_1, y_2, \dots, y_n = d$ is a walk with length n between b and d in H . Hence $d_G(a, c), d_H(b, d) \leq n$ and then,

$$\text{Max}\{d_G(a, c), d_H(b, d)\} \leq n = d_{G \otimes H}((a, b), (c, d)).$$

\square

Definition 2. Let G and H be two graphs and $(a, b), (c, d) \in V(G \otimes H)$. The relation \sim on the vertices of $G \otimes H$ is defined as follows:

$(a, b) \sim (c, d)$ if and only if $d_G(a, c), d_H(b, d) < +\infty$ and $d_G(a, c) + d_H(b, d)$ is even.

Proposition 2. Let G and H be graphs and $(a, b), (c, d) \in V(G \otimes H)$. If $(a, b) \sim (c, d)$, then (a, b) and (c, d) are in the same component of $G \otimes H$. Moreover

$$\text{Max}\{d_G(a, c), d_H(b, d)\} = d_{G \otimes H}((a, b), (c, d)).$$

Proof. Suppose $d_G(a, c) = m, d_H(b, d) = n$ and $m \leq n$. The following paths between a and c in G and b and d in H exist:

$$P_1 : a = x_0, x_1, \dots, x_m = c \quad \text{and} \quad P_2 : b = y_1, y_2, \dots, y_n = d.$$

Since $(a, b) \sim (c, d)$, then $m + n$ and $n - m$ are even. Therefore there exists the following path between (a, b) and (c, d) in $G \otimes H$, with length n :

$$P : (a, b) = (x_0, y_0), (x_1, y_1), \dots, (x_m, y_m) = (c, y_m), \\ (x_{m-1}, y_{m+1}), (x_m, y_{m+2}), \dots, (x_m, y_n) = (c, d).$$

So $d_{G \otimes H}((a, b), (c, d)) \leq n = d_H(b, d) = \text{Max}\{d_G(a, c), d_H(b, d)\}$. Similarly, if $m > n$, then $d_{G \otimes H}((a, b), (c, d)) \leq n = d_G(a, c) = \text{Max}\{d_G(a, c), d_H(b, d)\}$. By using Lemma 1, $\text{Max}\{d_G(a, c), d_H(b, d)\} \leq d_{G \otimes H}((a, b), (c, d))$, and this completes the proof. \square

Proposition 3. *Let G and H be graphs and $(a, b), (c, d) \in V(G \otimes H)$. If $(a, b) \approx (c, d)$ then,*

$$d_{G \otimes H}((a, b), (c, d)) = \text{Min} \left\{ \text{Max} \{d_G(a, c), d'_H(b, d)\}, \text{Max} \{d'_G(a, c), d_H(b, d)\} \right\}.$$

Proof. At first we show that the following inequality is holds:

$$(2.2) \quad d_{G \otimes H}((a, b), (c, d)) \leq \text{Min} \left\{ \begin{array}{l} \text{Max} \{d_G(a, c), d'_H(b, d)\}, \\ \text{Max} \{d'_G(a, c), d_H(b, d)\} \end{array} \right\}.$$

If $d_G(a, c) = +\infty$ ($d_H(b, d) = +\infty$), then $d'_G(a, c) = +\infty$ ($d'_H(b, d) = +\infty$), then the above inequality is satisfied.

We now assume that $d_G(a, c), d_H(b, d) < +\infty$. It is enough to show the following inequalities are hold:

$$(2.3) \quad d_{G \otimes H}((a, b), (c, d)) \leq \text{Max} \{d_G(a, c), d'_H(b, d)\},$$

$$(2.4) \quad d_{G \otimes H}((a, b), (c, d)) \leq \text{Max} \{d'_G(a, c), d_H(b, d)\}.$$

If $d'_H(b, d) = +\infty$ then inequality (2.3) holds. Suppose that $d'_H(b, d) = n$ and $d_G(a, c) = m$ and $m \leq n$. There exist the shortest path $P : a = x_0, x_1, \dots, x_m = c$ in G and the walk $W : b = y_0, y_1, \dots, y_n = d$ in H . Since $(a, b) \approx (c, d)$ then $n + m$ and $n - m$ are even. Therefore there exist the path P' between (a, b) and (c, d) in as follows:

$$P' : (a, b) = (x_0, y_0), (x_1, y_1), \dots, (x_m, y_m) = (c, y_m), \\ (x_{m-1}, y_{m+1}), (x_m, y_{m+2}), \dots, (x_m, y_n) = (c, d).$$

Hence, $d_{G \otimes H} \leq n = d'_H(b, d) = \text{Max} \{d_G(a, c), d'_H(b, d)\}$. Similarly, if $n < m$ then the inequality (2.3) is satisfied. By a similar argument, one can prove the inequality (2.4). By above argument we have shown inequality (2.2) is satisfied.

We now prove the following:

$$(2.5) \quad d_{G \otimes H}((a, b), (c, d)) \geq \text{Min} \left\{ \begin{array}{l} \text{Max} \{d_G(a, c), d'_H(b, d)\}, \\ \text{Max} \{d'_G(a, c), d_H(b, d)\} \end{array} \right\}.$$

If (a, b) and (c, d) are not in the same component of tensor product of G and H then, $d_{G \otimes H}((a, b), (c, d)) = +\infty$ and (2.5) is satisfied. Suppose (a, b) and (c, d) are in the same component. Then a and c (b and d) are in the same component of G (H). If $d_{G \otimes H}((a, b), (c, d)) = n$ then the path P'' exists, $P'' : (a, b) = (x_0, y_0), (x_1, y_1), \dots, (x_n, y_n) = (c, d)$. Since $(a, b) \approx (c, d)$ then $d_G(a, c) + d_H(b, d)$ is odd. We can suppose that $d_G(a, c)$ is odd and $d_H(b, d)$ is even. Our main proof consider the following cases:

Case 1: If n is odd, then the walk $W_1 : b = y_0, y_1, \dots, y_n = d$ is an odd walk with length n between b and d in H . So by definition of $d'_H(b, d)$,

it is clear that $d'_H(b, d) \leq n$. It is easy to see $d_G(a, c) \leq n$. Therefore $Max\{d_G(a, c), d'_H(b, d)\} \leq n$.

Case 2: If n is even, then by a similar method we can see

$$Max\{d'_G(a, c), d_H(b, d)\} \leq n.$$

Hence

$$Min\{Max\{d_G(a, c), d'_H(b, d)\}, Max\{d'_G(a, c), d_H(b, d)\}\} \leq d_{G \otimes H}((a, b), (c, d)).$$

and the proof is complete. \square

Corollary 1. *Let G and H be graphs and $(a, b), (c, d) \in V(G \otimes H)$. Then*

$$d_{G \otimes H}((a, b), (c, d)) = \begin{cases} d_1((a, b), (c, d)) & (a, b) \sim (c, d) \\ d_1((a, b), (c, d)) & (a, b) \approx (c, d) \end{cases}$$

where,

$$d_1((a, b), (c, d)) = Max\{d_G(a, c), d_H(b, d)\}$$

and

$$d_2((a, b), (c, d)) = Min\{Max\{d_G(a, c), d'_H(b, d)\}, Max\{d'_G(a, c), d_H(b, d)\}\}.$$

3. APPLICATION

In this section at first we present new simple proofs for some classical theorems related to the tensor product of graphs and then as an application of previous results, we compute the Wiener and Hyper Wiener indices of the tensor product of K_n and G , where K_n is the complete graph of order n and G is a connected graph.

Proposition 4. ([4]) *Let G and H be connected graphs with at least two vertices. Then $G \otimes H$ is connected if and only if G or H is not bipartite.*

If G or H are not bipartite then by Proposition 4, $G \otimes H$ is connected and has just one component. Now suppose G and H are bipartite. If $|V(G)| = |V(H)| = 2$, then $G \otimes H$ also is connected and has just one component. Let $|V(G)| > 2$ or $|V(H)| > 2$. Since G is bipartite, there exists the path $P : a_1, a_2, a_3$ in G . Obviously $d_G(a_1, a_3) = 2$. Let b_1 and b_2 are adjacent vertices in H . Since $(a_1, b_1) \approx (a_2, b_1)$, then by using Proposition 3 or Corollary 1, $d_{G \otimes H}((a_1, b_1), (a_2, b_1)) = +\infty$. Then (a_1, b_1) and (a_2, b_1) are in different components. For each $(a, b) \in V(G \otimes H)$, two cases are considered:

Case 1: If $d_G(a, a_1) + d_H(b, b_1)$ is even, then $(a, b) \sim (a_1, b_1)$ and by Corollary 1, (a, b) and (a_1, b_1) are in the same component.

Case 2: If $d_G(a, a_2) + d_H(b, b_1)$ is even, then $(a, b) \sim (a_2, b_1)$ and by Corollary 1, (a, b) and (a_2, b_1) are in the same component. Notice that if $d_G(a, a_1) + d_H(b, b_1)$ and $d_G(a, a_2) + d_H(b, b_1)$ are odd, then both of $d_G(a, a_1)$ and $d_G(a, a_2)$ are even or odd. So $d_G(a, a_1) + d_G(a, a_2)$ is even, and hence there exists an even walk between a_1 and a_2 . So there exists an odd closed walk in G containing

which is a contradiction.

Therefore we can conclude the following well known result.

Proposition 5. ([4]) *Suppose G and H are connected graphs with at least two vertices, then $G \otimes H$ has at most two components.*

In the following propositions the Wiener and Hyper Wiener indices of the tensor product of K_n and a connected graph G are computed.

Proposition 6. *Let G be connected graph of order m . Then the Wiener index of the tensor product of K_n and G is given by:*

$$W(K_n \otimes G) = n^2 W(G) + 2 \binom{n}{2} |V(G)| + 2n|E(G)| - n|T_G|,$$

where $n \geq 3$ and $T_G = \{ab \in E(G) | ab \text{ is contained in a triangle}\}$.

Proof. By definition of the Wiener index,

$$W(K_n \otimes G) = \sum_{\{(a,b),(c,d)\} \subseteq V(K_n \otimes G)} d_{K_n \otimes G}((a,b), (c,d))$$

We consider a partition of $V(K_n \otimes G)$ into the following parts:

$$A_1 = \{\{(a,b), (c,d)\} | a \neq c, b \neq d, (a,b) \sim (c,d)\},$$

$$A_2 = \{\{(a,b), (c,d)\} | a \neq c, b \neq d, (a,b) \approx (c,d)\},$$

$$A_3 = \{\{(a,b), (c,d)\} | a \neq c, b = d\},$$

$$A_4 = \{\{(a,b), (c,d)\} | a = c, b \neq d, (a,b) \sim (c,d)\},$$

$$A_5 = \{\{(a,b), (c,d)\} | a = c, b \neq d, (a,b) \approx (c,d)\}.$$

Therefore,

$$\begin{aligned} W(K_n \otimes G) = & \sum_{\{(a,b),(c,d)\} \in A_1} d_{K_n \otimes G}((a,b), (c,d)) + \sum_{\{(a,b),(c,d)\} \in A_2} d_{K_n \otimes G}((a,b), (c,d)) + \\ & \sum_{\{(a,b),(c,d)\} \in A_3} d_{K_n \otimes G}((a,b), (c,d)) + \sum_{\{(a,b),(c,d)\} \in A_4} d_{K_n \otimes G}((a,b), (c,d)) + \\ & + \sum_{\{(a,b),(c,d)\} \in A_5} d_{K_n \otimes G}((a,b), (c,d)). \end{aligned}$$

We evaluate each summation separately. It is obvious that if $\{a, c\} \subseteq V(K_n)$, then $d_{K_n}(a, c) = 1$ and $d'_{K_n}(a, c) = 2$. Apply Proposition 3, if $(a, b) \sim (c, d)$ and $a \neq c, b \neq d$ then,

$$\text{Min}\{\text{Max}\{d_{K_n}(a, c), d'_G(b, d)\}, \text{Max}\{d'_{K_n}(a, c), d_G(b, d)\}\} = d_G(b, d)$$

and

$$\text{Max}\{d_{K_n}(a, c), d_G(b, d)\} = d_G(b, d).$$

Therefore,

$$\begin{aligned}
\sum_{\{(a,b),(c,d)\} \in A_1 \cup A_2} d_{K_n \otimes G}((a,b), (c,d)) &= \sum_{\{(a,b),(c,d)\} \in A_1} d_{K_n \otimes G}((a,b), (c,d)) \\
&+ \sum_{\{(a,b),(c,d)\} \in A_2} d_{K_n \otimes G}((a,b), (c,d)) \\
&= 2 \sum_{\{a,c\} \subseteq V(K_n)} \sum_{\{b,d\} \subseteq V(G)} d_G(b,d) \\
&= 2 \binom{n}{2} W(G).
\end{aligned}$$

By considering the set A_3 , we have:

$$\sum_{\{(a,b),(c,d)\} \in A_3} d_{K_n \otimes G}((a,b), (c,d)) = \sum_{\substack{\{a,c\} \subseteq V(K_n) \\ b \in V(G)}} 2 = 2|V(G)| \binom{n}{2}.$$

For computing our summation over the set A_4 , we notice that,

$$A_4 = \{ \{(a,b),(c,d)\} \mid a \in V(K_n), \{b,d\} \subseteq V(G) \text{ and } 2 \mid d_G(b,d) \}.$$

Hence,

$$\begin{aligned}
\sum_{\{(a,b),(c,d)\} \in A_4} d_{K_n \otimes G}((a,b), (c,d)) &= \sum_{\substack{\{b,d\} \subseteq V(G) \\ a \in V(K_n) \\ 2 \mid d_G(b,d)}} d_{K_n \otimes G}((a,b), (c,d)) \\
&= \sum_{\substack{\{b,d\} \subseteq V(G) \\ a \in V(K_n) \\ 2 \mid d_G(b,d)}} d_G(b,d) = n \sum_{\substack{\{b,d\} \subseteq V(G) \\ 2 \mid d_G(b,d)}} d_G(b,d).
\end{aligned}$$

Compute our summation over A_5 , we have:

$$\sum_{\{(a,b),(c,d)\} \in A_5} d_{K_n \otimes G}((a,b), (c,d)) = \sum_{\substack{\{b,d\} \subseteq V(G) \\ a \in V(K_n) \\ 2 \nmid d_G(b,d)}} d_{K_n \otimes G}((a,b), (c,d)).$$

If $d_G(b,d)$ is odd then by Proposition 3,

$$\begin{aligned}
d_{K_n \otimes G}((a,b), (c,d)) &= \text{Min} \left\{ \text{Max} \{ d_{K_n}(a,a), d'_G(b,d) \}, \right. \\
&\quad \left. \text{Max} \{ d'_{K_n}(a,a), d_G(b,d) \} \right\} \\
&= \text{Min} \left\{ d'_G(b,d), \text{Max} \{ 3, d_G(b,d) \} \right\}.
\end{aligned}$$

By a case by case calculations, one can see that

$$\text{Min} \left\{ d'_G(b,d), \text{Max} \{ 3, d_G(b,d) \} \right\} = \begin{cases} d_G(b,d) & d_G(b,d) \geq 3 \\ 2 & d_G(b,d) = 1 \text{ and } d'_G(b,d) = 2 \\ 3 & d_G(b,d) = 1 \text{ and } d'_G(b,d) \geq 4 \end{cases}$$

Define:

$$\begin{aligned} A'_5 &= \left\{ \{(a, b), (c, d)\} \mid a \in V(K_n), d_G(b, d) \geq 3 \text{ and } d_G(b, d) \text{ is odd} \right\}, \\ A''_5 &= \left\{ \{(a, b), (c, d)\} \mid a \in V(K_n), d_G(b, d) = 1 \text{ and } d'_G(b, d) = 2 \right\}, \\ A'''_5 &= \left\{ \{(a, b), (c, d)\} \mid a \in V(K_n), d_G(b, d) = 1 \text{ and } d'_G(b, d) \geq 3 \right\}. \end{aligned}$$

Such that $A_5 = A'_5 \cup A''_5 \cup A'''_5$. Hence,

$$\begin{aligned} \sum_{\{(a,b),(c,d)\} \in A_5} d_{K_n \otimes G}((a, b), (c, d)) &= \sum_{\{(a,b),(c,d)\} \in A'_5} d_{K_n \otimes G}((a, b), (c, d)) + \\ & \sum_{\{(a,b),(c,d)\} \in A''_5} d_{K_n \otimes G}((a, b), (c, d)) + \sum_{\{(a,b),(c,d)\} \in A'''_5} d_{K_n \otimes G}((a, b), (c, d)) = \\ & \sum_{\{(a,b),(c,d)\} \in A'_5} d_G(b, d) + \sum_{\{(a,b),(c,d)\} \in A''_5} 2 + \sum_{\{(a,b),(c,d)\} \in A'''_5} 3 = \\ n \left(\sum_{\substack{\{(b,d)\} \subseteq V(G) \\ 2 \nmid d_G(b,d)}} d_G(b, d) - |E(G)| \right) &+ 2n|T_G| + 3n(|E(G)| - |T_G|). \end{aligned}$$

By the above calculations, one can see that,

$$W(K_n \otimes G) = n^2 W(G) + 2 \binom{n}{2} |V(G)| + 2n|E(G)| - n|T_G|.$$

□

Corollary 2. *The following statements are hold:*

a) *Let G be a triangle-free graph then,*

$$W(K_n \otimes G) = n^2 W(G) + 2 \binom{n}{2} |V(G)| + 2n|E(G)|.$$

b) *The Wiener index of two complete graphs of orders m and n is computed as follows:*

$$W(K_n \otimes K_m) = n(n+1) \binom{m}{2} + n(n-1)m.$$

Proof. a) It is immediate by Proposition 6.

b) In complete graph K_n , we can see that $T_{K_n} = E(K_n)$ and so by Proposition 6,

$$\begin{aligned} W(K_n \otimes K_m) &= n^2 W(K_m) + 2 \binom{n}{2} m + 2n \binom{m}{2} - n \binom{m}{2} \\ &= n(n+1) \binom{m}{2} + n(n-1)m. \end{aligned}$$

□

Proposition 7. *Let G be connected graph of order m . Then the hyper Wiener index of the tensor product of K_n and G is given by:*

$$WW(K_n \otimes G) = \binom{n+1}{2} WW(G) + 3 \binom{n}{2} |V(G)| + 5n|E(G)| - 3n|T_G|.$$

Proof. The proof is similar to the proof of Proposition 6. \square

Corollary 3. *The hyper Wiener index of two complete graphs of orders m and n is computed as follows:*

$$WW(K_n \otimes K_m) = \binom{n+1}{2} m(m-1) + 3 \binom{n}{2} m + 5n \frac{m(m-1)}{2} - 3nm(m-1).$$

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REFERENCES

1. J. Bosak, *Decompositions of Graphs*, Kluwer Academic Publishers, 1990.
2. A. Bottreou and Y. Metivier, Some remarks on the Kronecker product of graphs, *Inform. Process. Lett.*, **63** (1998), 55–61.
3. G. H. Fath-Tabar and A. R. Ashrafi, The Hyper-Wiener Polynomial of Graphs, *Iranian Journal of Mathematical Sciences and Informatics*, **6**(2) (2011), 67–74.
4. W. Imrich and S. Klavsar, *Product Graphs, Structure and Recognition*, Discrete Appl. Math., John Wiley and Sons, New York, USA, 2000.
5. A. Iranmanesh and Y. Alizadeh, Computing Wiener Index of $HAC_5C_7[p, q]$ Nanotubes by Gap Program, *Iranian Journal of Mathematical Sciences and Informatics*, **3**(1) (2008), 1–12.
6. M. H. Khalifeh, H. Yousefi-Azari, A. R. Ashrafi and I. Gutman, The edge Szeged index of product graphs, *Croat. Chem. Acta*, **81**(2) (2008), 277–281.
7. M. H. Khalifeh, H. Yousefi-Azari and Ashrafi, The Hyper-Wiener index of graph operations, *Comput. Math. Appl.*, **56** (2008), 1402–1407.
8. M. H. Khalifeh, H. Yousefi-Azari, A. R. Ashrafi and S. Wagner, Some new results on distance-based graph invariants, *Europ. J. Combin.*, **30**(5) (2009), 1149–1163.
9. M. H. Khalifeh, H. Yousefi-Azari and A. R. Ashrafi, Vertex and edge PI indices of Cartesian product graph, *Discrete Appl. Math.*, **156** (2008), 1780–1789.
10. M. H. Khalifeh, H. Yousefi-Azari and A. R. Ashrafi, The first and second Zagreb indices of graph operations, *Discrete Appl. Math.*, **157**(4) (2009), 804–811.
11. M. H. Khalifeh, H. Yousefi-Azari and A. R. Ashrafi, A matrix method for computing Szeged and vertex PI indices of join and composition of graphs, *Linear Alg. Appl.*, **429**(11,12) (2008), 2702–2709.
12. D. J. Klein and I. Lukovits, On the definition of the hyper-Wiener index for cycle-containing structures, *J. Chem. Inf. Comput. Sci.*, **35** (1995), 50–52.
13. D. Stevanovic, Distance regularity of compositions of graphs, *Appl. Math. Lett.*, **17** (2004), 337–343.
14. H. Wiener, Structural determination of the paraffin boiling points, *J. Am. Chem. Soc.*, **69** (1974), 17–20.
15. H. Yousefi-Azari, B. Manoochehrian and A. R. Ashrafi, The PI index of product graphs, *Appl. Math. Lett.*, **21**(6) (2008), 624–627.