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$\Re$-parts in hyperrings

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Abstract. In this article, first we generalize the concept of complete parts in hyperrings by introducing the concept $\Re$-parts in hyperrings and then we study $\Re$-closures in hyperrings. Finally we characterize $\Re$-closures in hyperfields.

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1. Introduction

The theory of hyperstructures was introduced in 1934 by Marty [12] at the 8th Congress of Scandinavian Mathematicians. This theory has been subsequently developed by Corsini and Leoreanu [1, 2] , Mittas [14, 15], Stratigopoulos [19], and by various authors [3, 5, 9]. Basic definitions and propositions about the hyperstructures are found in [1, 2, 20]. Krasner [11] has studied the notion of *hyperfields, hyperrings*, and then some researchers. Hyperrings are essentially rings with approximately modified axioms. There are different notions of hyperrings . If the addition + is a hyperoperation and the multiplication is a

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binary operation, then the hyperring is called Krasner (additive) hyperring [11]. Rota [17] introduced a multiplicative hyperring, where + is a binary operation and the multiplication is a hyperoperation. De Salvo [18] studied hyperrings in which the additions and the multiplications were hyperoperations. In 2007, Davvaz and Leoreanu-Fotea [6] published a book titled Hyperring Theory and Applications. Complete parts were introduced by Koskas [10] and studied then by Miglirato [13], Corsini and Sureau [1, 2]. Mousavi et al. [16] introduced the notion of $\Re$-parts in hypergroups as a generalization of complete parts in hypergroups. In this article we generalize the notion of complete parts by introducing *left* and *right* $\Re$-*parts* in hyperrings and we will study $\Re$-*closures* in hyperrings. Finally we characterize $\Re$-closures in *hyperfields*.

2. Preliminaries

A *hypergroupoid* $(H, \circ)$ is a non-empty set H together with a hyperoperation $\circ$ defined on H, that is a mapping of $H \times H$ into the family of non-empty subsets of H. If $(x, y) \in H \times H$, its image under $\circ$ is denoted by $x \circ y$ and for simplicity by xy. If A , B are non-empty subsets of H then $A \circ B$ is given by $A \circ B = \bigcup\{xy \mid x \in A, y \in B\}$. $x \circ A$ is used for $\{x\} \circ A$ (resp. $A \circ x$). A hypergroupoid $(H, \circ)$ is called a *hypergroup* in the sense of [12] if for all $x, y, z \in H$ the following two conditions hold: (i) $x(yz) = (xy)z$, (ii) $xH = Hx = H$, means that for any $x, y \in H$ there exist $u, v \in H$ such that $y \in xu$ and $y \in vx$. If $(H, \circ)$ satisfies only the first axiom, then it is called a *semi-hypergroup* an exhaustive review updated to 1992 of hypergroup theory appears in [1]. A recent book [2] contains a wealth of applications. A *hyperring* [20] is a triple $(R, +, \circ)$ which satisfies the ring-like axioms in the following way:(i) $(R, +)$ is a hypergroup , (ii) $(R, \circ)$ is a semi-hypergroup, (iii) the multiplication is distributive with respect to the hyperoperation $+$. The hyperrings were studied by many authors, for example see [8], [4], [20], [7] and [22]. In [23] and [21] Vougiouklis defines the relation Γ on hyperring as follows: $x\Gamma y$ if and only if $x, y \subseteq u$, where u is a finite sum of finite products of elements of R, in fact there exist $n, k_i \in \mathbb{N}$ and $x_{ij} \in R$ such that $u = \sum\limits_{i=1}^{n} \prod\limits_{j=1}^{k_i} x_{ij}$. He proved that the quotient R/Γ^*, where Γ^* is the transitive closure of Γ, is a ring and also Γ^* is the smallest equivalent relation on R such that the quotient R/Γ^* is a fundamental ring. The both $\oplus$ and $\odot$ on R/Γ^* are defined as follow:

$\forall z \in \Gamma^*(x) + \Gamma^*(y), \;\; \Gamma^*(x) \oplus \Gamma^*(y) = \Gamma^*(z);$

$$\forall z \in \Gamma^*(x) \circ \Gamma^*(y), \;\; \Gamma^*(x) \odot \Gamma^*(y) = \Gamma^*(z).$$

Let M be a non-empty subset of R. We say that M is a *complete part* if for every $n \in \mathbb{N}$, $i = 1, 2, ..., n$, $\forall k_i \in \mathbb{N}$, $\forall (z_{i1}, ..., z_{ik_i}) \in R^{k_i}$ we have:

$$\sum_{i=1}^{n} \prod_{j=1}^{k_i} z_{ij} \bigcap M \neq \emptyset \Rightarrow \sum_{i=1}^{n} \prod_{j=1}^{k_i} z_{ij} \subseteq M.$$

3. $\Re$-parts

Let $\mathcal{U}$ be the set of finite sums of finite products of elements of R and $\Re$ be a relation on $\mathcal{U}$. In this section first we generalize the notion of complete parts by introducing the notion of $\Re$-parts and then we study $\Re$-closures.

Definition 3.1. *Let R be a hyperring and $\mathcal{U}$ be the set of finite sum of finite products of elements of R and $\Re$ be a relation on $\mathcal{U}$. For a non-empty subset A of R we say:*

(i) A is a left $\Re$-part of R with respect to $\mathcal{U}$ (or briefly is $\mathcal{L}\Re_{\mathcal{U}}$-part) if for all $\sum_{i=1}^{n} \prod_{j=1}^{k_i} x_{ij}$ and $\sum_{i=1}^{m} \prod_{j=1}^{t_i} y_{ij}$ in $\mathcal{U}$ the following implication is valid:

$$[\sum_{i=1}^{n} \prod_{j=1}^{k_i} x_{ij} \bigcap A \neq \emptyset \ \text{ and } \ \sum_{i=1}^{m} \prod_{j=1}^{t_i} y_{ij} \ \Re \ \sum_{i=1}^{n} \prod_{j=1}^{k_i} x_{ij}] \Rightarrow \sum_{i=1}^{m} \prod_{j=1}^{t_i} y_{ij} \subseteq A;$$

(ii) A is a right $\Re$-part of R with respect to $\mathcal{U}$ (or briefly is $\mathcal{R}\Re_{\mathcal{U}}$-part) if for all $\sum_{i=1}^{n} \prod_{j=1}^{k_i} x_{ij}$ and $\sum_{i=1}^{m} \prod_{j=1}^{t_i} y_{ij}$ in $\mathcal{U}$ the following implication is valid:

$$[\sum_{i=1}^{n} \prod_{j=1}^{k_i} x_{ij} \bigcap A \neq \emptyset \ \text{ and } \ \sum_{i=1}^{n} \prod_{j=1}^{k_i} x_{ij} \ \Re \ \sum_{i=1}^{m} \prod_{j=1}^{t_i} y_{ij}] \Rightarrow \sum_{i=1}^{m} \prod_{j=1}^{t_i} y_{ij} \subseteq A;$$

(iii) A is a $\Re$-part of R with respect to $\mathcal{U}$ (or briefly is $\Re_{\mathcal{U}}$-part) if it is $\mathcal{L}\Re_{\mathcal{U}}$-part and $\mathcal{R}\Re_{\mathcal{U}}$-part.

Proposition 3.2. *Let $\Re$ be a relation on $\mathcal{U}$ and $\Re^{-1}$ be the inverse of $\Re$ then*

(i) A is $\mathcal{L}\Re_{\mathcal{U}}^{-1}$-part if and only if it is $\mathcal{R}\Re_{\mathcal{U}}$-part;

(ii) A is $\mathcal{R}\Re_{\mathcal{U}}^{-1}$-part if and only if it is $\mathcal{L}\Re_{\mathcal{U}}$-part.

Definition 3.3. *The intersection of $\mathcal{L}\Re_{\mathcal{U}}$-parts (or $\mathcal{R}\Re_{\mathcal{U}}$-parts, $\Re$-parts) which contain A is called $\mathcal{L}\Re_{\mathcal{U}}$-closure (or $\mathcal{R}\Re_{\mathcal{U}}$-closure, $\Re$-closure) of A in R and it will be denoted by $\overline{\mathcal{L}\Re_{\mathcal{U}}}(A)$ (or $\overline{\mathcal{R}\Re_{\mathcal{U}}}(A)$, $\overline{\Re_{\mathcal{U}}}(A)$).*

From now on R is a hyperring, $\mathcal{U}$ is the set of finite sum of finite products of elements of R, $u \in \mathcal{U}$ means $u = \sum_{i=1}^{n} \prod_{j=1}^{k_i} x_{ij}$ and A is a non-empty subset of R.

Proposition 3.4. *For a non-empty subset A of R we have:*

(i) $\overline{\mathcal{L}\Re_{\mathcal{U}}^{-1}}(A) = \overline{\mathcal{R}\Re_{\mathcal{U}}}(A)$;

(ii) $\overline{\mathcal{R}\Re_{\mathcal{U}}^{-1}}(A) = \overline{\mathcal{L}\Re_{\mathcal{U}}}(A)$.

Proof. Follows from Proposition 3.2. □

Lemma 3.5. *For a non-empty subset A of R define:*

$${}_A\sum^{\mathcal{U}} \stackrel{def}{:=} \{\Re \subseteq \mathcal{U}\times\mathcal{U} \mid \overline{\mathcal{L}\Re_{\mathcal{U}}}(A) = A\} and \sum\nolimits_A^{\mathcal{U}} \stackrel{def}{:=} \{\Re \subseteq \mathcal{U}\times\mathcal{U} \mid \overline{\mathcal{R}\Re_{\mathcal{U}}}(A) = A\}.$$

If ${}_A\sum^{\mathcal{U}} \neq \emptyset$ *(resp.* $\sum_A^{\mathcal{U}} \neq \emptyset$*), then* $({}_A\sum^{\mathcal{U}}, \circ)$ *(resp.* $(\sum_A^{\mathcal{U}}, \circ)$*) is a semigroup, where* $\circ$ *is the operation of relation composition.*

Proof. Suppose that $\Re, \Re' \in {}_A\sum^{\mathcal{U}}$ and $(\sum\limits_{i=1}^{n}\prod\limits_{j=1}^{k_i} y_{ij}, \sum\limits_{i=1}^{m}\prod\limits_{j=1}^{t_i} x_{ij}) \in \mathcal{U}\times\mathcal{U}$ are given. Let $\sum\limits_{i=1}^{m}\prod\limits_{j=1}^{t_i} x_{ij} \bigcap A \neq \emptyset$ and $\sum\limits_{i=1}^{n}\prod\limits_{j=1}^{k_i} y_{ij}\ \Re\circ\Re'\ \sum\limits_{i=1}^{n}\prod\limits_{j=1}^{k_i} x_{ij}$. So there exists $\sum\limits_{i=1}^{k}\prod\limits_{j=1}^{s_i} z_{ij}$ such that $\sum\limits_{i=1}^{k}\prod\limits_{j=1}^{s_i} z_{ij}\ \Re\ \sum\limits_{i=1}^{m}\prod\limits_{j=1}^{t_i} x_{ij}$ and $\sum\limits_{i=1}^{n}\prod\limits_{j=1}^{k_i} y_{ij}\ \Re'\ \sum\limits_{i=1}^{k}\prod\limits_{j=1}^{s_i} z_{ij}$. From $\sum\limits_{i=1}^{k}\prod\limits_{j=1}^{s_i} z_{ij}\ \Re\ \sum\limits_{i=1}^{m}\prod\limits_{j=1}^{t_i} x_{ij}$ and $\Re \in {}_A\sum^{\mathcal{U}}$, we have $\sum\limits_{i=1}^{k}\prod\limits_{j=1}^{s_i} z_{ij} \subseteq A$. Since $\Re' \in {}_A\sum^{\mathcal{U}}$ and $\sum\limits_{i=1}^{n}\prod\limits_{j=1}^{k_i} y_{ij}\ \Re'\ \sum\limits_{i=1}^{k}\prod\limits_{j=1}^{s_i} z_{ij}$, $\sum\limits_{i=1}^{n}\prod\limits_{j=1}^{k_i} y_{ij} \subseteq A$.

□

Theorem 3.6. *If $\Re$ is a permutation of finite order in $S_{\mathcal{U}}$ (the symmetric group on the set $\mathcal{U}$), then the following are equivalent:*

(i) A is $\mathcal{L}\Re_{\mathcal{U}}$-part;

(ii) A is $\mathcal{R}\Re_{\mathcal{U}}$-part;

(iii) A is $\Re_{\mathcal{U}}$-part.

Proof. (i)⇒ (ii). For this reason suppose that A is $\mathcal{L}\Re_{\mathcal{U}}$-part. So $\overline{\mathcal{L}\Re_{\mathcal{U}}}(A) = A$ and hence $\Re \in {}_A\sum^{\mathcal{U}}$. Since $\Re$ is a permutation of finite order in $S_{\mathcal{U}}$, $\langle\Re\rangle = \{\Re^n \mid n \in \mathbb{N}\}$ is a subgroup of ${}_A\sum^{\mathcal{U}}$ and so $\Re^{-1} \in {}_A\sum^{\mathcal{U}}$. Therefore by Proposition 3.4 we have $A = \overline{\mathcal{L}\Re_{\mathcal{U}}^{-1}}(A) = \overline{\mathcal{R}\Re_{\mathcal{U}}}(A)$, thus $\Re \in \sum_A^{\mathcal{U}}$ and hence A is $\mathcal{R}\Re_{\mathcal{U}}$-part. □

Theorem 3.7. *Suppose that $\Re \subseteq \mathcal{U}\times\mathcal{U}$.*

(i) We pose $K^{\mathcal{L}}_{1,\Re}(A) = A$ and

$$K^{\mathcal{L}}_{n+1,\Re}(A) = \{x \in R \mid \exists(u,v) \in \Re, x \in u \text{ and } v \cap K^{\mathcal{L}}_{n,\Re}(A) \neq \emptyset\},$$

if we consider $K^{\mathcal{L}}_{\Re}(A) = \cup_{n\geq 1} K^{\mathcal{L}}_{n,\Re}(A)$, then $K^{\mathcal{L}}_{\Re}(A) = \overline{\mathcal{L}\Re_{\mathcal{U}}}(A)$ and $K^{\mathcal{L}}_{\Re}(A)$ is the smallest $\mathcal{L}\Re_{\mathcal{U}}$-part containing A;

(ii) We pose $K^{\mathcal{R}}_{1,\Re}(A) = A$ and

$$K^{\mathcal{R}}_{n+1,\Re}(A) = \{x \in R \mid \exists(v,u) \in \Re, x \in u \text{ and } v \cap K^{\mathcal{R}}_{n,\Re}(A) \neq \emptyset\},$$

if we consider $K^{\mathcal{R}}_{\Re}(A) = \cup_{n\geq 1} K^{\mathcal{R}}_{n,\Re}(A)$, then $K^{\mathcal{R}}_{\Re}(A) = \overline{\mathcal{R}\Re_{\mathcal{U}}}(A)$ and $K^{\mathcal{R}}_{\Re}(A)$ is the smallest $\mathcal{R}\Re_{\mathcal{U}}$-part containing A;

(iii) We pose $K_{1,\Re}(A) = A$ *and*

$$K_{n+1,\Re}(A) = \{x \in R \mid \exists (u,v) \in \Re \cup \Re^{-1}, x \in u \text{ and } v \cap K_{n,\Re}(A) \neq \emptyset\},$$

if $K_{\Re}(A) = \cup_{n\geq 1} K_{n,\Re}(A)$*, then* $K_{\Re}(A) = \overline{\Re_{\mathcal{U}}}(A)$ *and* $K_{\Re}(A)$ *is the smallest* $\Re_{\mathcal{U}}$*-part containing* A*.*

Proof. (i) It is necessary to prove:
(1) $K^{\mathcal{L}}_{\Re}(A)$ is $\mathcal{L}\Re_{\mathcal{U}}$-part,
(2) if $A \subseteq B$ and B is $\mathcal{L}\Re_{\mathcal{U}}$-part, then $K^{\mathcal{L}}_{\Re}(A) \subseteq B$.

For the proof (1) suppose that $v \bigcap K^{\mathcal{L}}_{\Re}(A) \neq \emptyset$ and $u \,\Re\, v$. Therefore there exists $n \in \mathbb{N}$ such that $v \bigcap K^{\mathcal{L}}_{n,\Re}(A) \neq \emptyset$, from which follows $u \subseteq K^{\mathcal{L}}_{n+1,\Re}(A) \subseteq K^{\mathcal{L}}_{\Re}(A)$. Now we prove (2) by induction on n. We have $K^{\mathcal{L}}_{1,\Re}(A) = A \subseteq B$. Suppose that $K^{\mathcal{L}}_{n,\Re}(A) \subseteq B$. We prove that $K^{\mathcal{L}}_{n+1,\Re}(A) \subseteq B$. If $z \in K^{\mathcal{L}}_{n+1,\Re}(A)$, then there exists $(u,v) \in \mathcal{U} \times \mathcal{U}$ such that $z \in u$, $u \,\Re\, v$ and $v \bigcap K^{\mathcal{L}}_{n,\Re}(A) \neq \emptyset$. Therefore $v \bigcap B \neq \emptyset$ and hence $z \in u \subseteq B$. So $K^{\mathcal{L}}_{n+1,\Re}(A) \subseteq B$.

(ii) We have

$$\begin{aligned} K^{\mathcal{R}}_{\Re}(A) &= K^{\mathcal{L}}_{\Re^{-1}_{\mathcal{U}}}(A) \\ &= \overline{\mathcal{L}\Re^{-1}_{\mathcal{U}}}(A), \quad \text{by part (i)} \\ &= \overline{\mathcal{R}\Re_{\mathcal{U}}}(A), \quad \text{by Proposition 3.4.} \end{aligned}$$

(iii) Follows from (i) and (ii). □

Proposition 3.8. *Suppose that* B *is a non-empty subset of* R *and* $\Re$ *is a relation on* $\mathcal{U}$*. Then we have:*

(i) $\overline{\mathcal{L}\Re_{\mathcal{U}}}(B) = \bigcup_{b\in B} \overline{\mathcal{L}\Re_{\mathcal{U}}}(b)$;

(ii) $\overline{\mathcal{R}\Re_{\mathcal{U}}}(B) = \bigcup_{b\in B} \overline{\mathcal{R}\Re_{\mathcal{U}}}(b)$;

(iii) $\overline{\Re_{\mathcal{U}}}(B) = \bigcup_{b\in B} \overline{\Re_{\mathcal{U}}}(b)$.

Proof. (i) It is clear that for all $b \in B$, $\overline{\mathcal{L}\Re_{\mathcal{U}}}(b) \subseteq \overline{\mathcal{L}\Re_{\mathcal{U}}}(B)$. By Theorem 3.7(i), $\overline{\mathcal{L}\Re_{\mathcal{U}}}(B) = \bigcup_{n\geq 1} K^{\mathcal{L}}_{n,\Re}(B)$. We follow the proposition by induction on n. For $n = 1$, $K^{\mathcal{L}}_{1,\Re}(B) = B = \bigcup_{b\in B} \{b\} = \bigcup_{b\in B} K^{\mathcal{L}}_{1,\Re}(b)$. Supposing it is true for n, we show that $K^{\mathcal{L}}_{n+1,\Re}(B) \subseteq \bigcup_{b\in B} K^{\mathcal{L}}_{n+1,\Re}(b)$. If $z \in K^{\mathcal{L}}_{n+1,\Re}(B)$, then there exists $(u,v) \in \Re$ such that $z \in u$ and $v \bigcap K^{\mathcal{L}}_{n,\Re}(B) \neq \emptyset$. From this it follows, by the hypothesis of induction, $v \bigcap (\bigcup_{b\in B} K^{\mathcal{L}}_{n,\Re}(b)) \neq \emptyset$ and therefore $b' \in B$ exists such that $v \bigcap K^{\mathcal{L}}_{n,\Re}(b') \neq \emptyset$. So $z \in K^{\mathcal{L}}_{n+1,\Re}(b')$ and hence $\overline{\mathcal{L}\Re_{\mathcal{U}}}(B) \subseteq \bigcup_{b\in B} \overline{\mathcal{L}\Re_{\mathcal{U}}}(b)$. □

Theorem 3.9. *Suppose that* $\Re \subseteq \mathcal{U} \times \mathcal{U}$*. The relation* $K^{\mathcal{L}}_{\Re}$ *(resp.* $K^{\mathcal{R}}_{\Re}$*) on* R *defined by:*

$$x \, K^{\mathcal{L}}_{\Re} \, y \Leftrightarrow x \in K^{\mathcal{L}}_{\Re}(y) (x \in K^{\mathcal{R}}_{\Re}(y)),$$

where $K_{\Re}^{\mathcal{L}}(y) = K_{\Re}^{\mathcal{L}}(\{y\})$ *(resp.* $K_{\Re}^{\mathcal{R}}(y) = K_{\Re}^{\mathcal{R}}(\{y\})$*) is a preorder. Furthermore if* $\Re$ *is symmetric, then* $K_{\Re}^{\mathcal{L}}$ *(resp.* $K_{\Re}^{\mathcal{R}}$*) is an equivalence relation.*

Proof. It is easy to see that $K_{\Re}^{\mathcal{L}}$ is reflexive. Now suppose that $x\, K_{\Re}^{\mathcal{L}}\, y$ and $y\, K_{\Re}^{\mathcal{L}}\, z$. So $x \in K_{\Re}^{\mathcal{L}}(y)$ and $y \in K_{\Re}^{\mathcal{L}}(z)$. By Theorem 3.7(i) we have $K_{\Re}^{\mathcal{L}}(z)$ is $\mathcal{L}\Re_{\mathcal{U}}$-part thus $K_{\Re}^{\mathcal{L}}(y) \subseteq K_{\Re}^{\mathcal{L}}(z)$ and hence $x \in K_{\Re}^{\mathcal{L}}(z)$. Therefore $K_{\Re}^{\mathcal{L}}$ is preorder. Now let $\Re$ be symmetric. We prove that $K_{\Re}^{\mathcal{L}}$ is symmetric as well. To this end the following is premised:

(1) for all $n \geq 2$ and $x \in R$, $K_{n,\Re}^{\mathcal{L}}(K_{2,\Re}^{\mathcal{L}}(x)) = K_{n+1,\Re}^{\mathcal{L}}(x)$;

(2) $x \in K_{n,\Re}^{\mathcal{L}}(y)$ if and only if $y \in K_{n,\Re}^{\mathcal{L}}(x)$.

We prove (1) by induction on n. Suppose that $z \in K_{2,\Re}^{\mathcal{L}}(K_{2,\Re}^{\mathcal{L}}(x))$ so there exists $(u,v) \in \Re$ such that $z \in u$ and $v \bigcap K_{2,\Re}^{\mathcal{L}}(x) \neq \emptyset$. Thus $z \in K_{3,\Re}^{\mathcal{L}}(x)$. Let $K_{n,\Re}^{\mathcal{L}}(K_{2,\Re}^{\mathcal{L}}(x)) = K_{n+1,\Re}^{\mathcal{L}}(x)$ so we have:

$$\begin{aligned} z \in K_{n+1,\Re}^{\mathcal{L}}(K_{2,\Re}^{\mathcal{L}}(x)) &\Leftrightarrow \exists (u,v) \in \Re, z \in u, v \cap K_{n,\Re}^{\mathcal{L}}(K_{2,\Re}^{\mathcal{L}}(x)) \neq \emptyset \\ &\Leftrightarrow \exists (u,v) \in \Re, z \in u, v \cap K_{n+1,\Re}^{\mathcal{L}}(x) \neq \emptyset \\ &\Leftrightarrow z \in K_{n+2,\Re}^{\mathcal{L}}(x). \end{aligned}$$

We also prove (2) by induction on n. It is clear that $x \in K_{2,\Re}^{\mathcal{L}}(y)$ if and only if $y \in K_{2,\Re}^{\mathcal{L}}(x)$. Suppose $x \in K_{n,\Re}^{\mathcal{L}}(y)$ if and only if $y \in K_{2,\Re}^{\mathcal{L}}(x)$. Let $x \in K_{n+1,\Re}^{\mathcal{L}}(y)$ be given, so there exist $(u,v) \in \Re$ such that $x \in u$ and $v \bigcap K_{n,\Re}^{\mathcal{L}}(y) \neq \emptyset$. Therefore there exists $b \in v \bigcap K_{n,\Re}^{\mathcal{L}}(y)$ and hence $y \in K_{n,\Re}^{\mathcal{L}}(b)$. Since $\Re$ is symmetric and $(u,v) \in \Re$, $b \in v$ and $x \in u \bigcap K_{1,\Re}^{\mathcal{L}}(x)$ implies that $b \in K_{2,\Re}^{\mathcal{L}}(x)$ and hence $y \in K_{n,\Re}^{\mathcal{L}}(K_{2,\Re}^{\mathcal{L}}(x)) = K_{n+1,\Re}^{\mathcal{L}}(x)$. Similarly we can show if $y \in K_{n+1,\Re}^{\mathcal{L}}(x)$, then $x \in K_{n+1,\Re}^{\mathcal{L}}(x)$. □

Proposition 3.10. *Let* $\Re$ *be a relation on* $\mathcal{U}$ *and* A *be a non-empty subset of the hyperring* R*. The following conditions are equivalent:*

(i) A *is a (*$\mathcal{R}\Re_{\mathcal{U}}$*-part)* $\mathcal{L}\Re_{\mathcal{U}}$*-part of* R*;*

(ii) $x \in A, (x\, K_{\Re}^{\mathcal{L}}\, z) z\, K_{\Re}^{\mathcal{L}}\, x \Rightarrow z \in A$.

Proof. (i) $\Rightarrow$ (ii) If $x \in A$ and $z \in R$ such that $z\, K_{\Re}^{\mathcal{L}}\, x$, then there exists $(u,v) \in \Re$ such that $z \in u$ and $v \cap K_{n,\Re}^{\mathcal{L}}(A) \neq \emptyset$ for some $n \in \mathbb{N}$. Since A is a $\mathcal{L}\Re_{\mathcal{U}}$-part by Theorem 3.7, $K_{n,\Re}^{\mathcal{L}}(A) \subseteq A$ and so $v \cap A \neq \emptyset$. Therefore $u \subseteq A$ and hence $z \in A$.

(ii) $\Rightarrow$ (i) Let $u \cap A \neq \emptyset$ and $v\, \Re\, u$. So there exists $x \in A \cap u$ and $x \in u$, $u \cap K_{1,\Re}^{\mathcal{L}}(x) \neq \emptyset$. Now suppose that $z \in v$ is given. So

$$\begin{aligned} v\, \Re\, u &\Rightarrow z \in K_{2,\Re}^{\mathcal{L}}(x), \qquad \text{because } x \in u \\ &\Rightarrow z\, K_{\Re}^{\mathcal{L}}\, x \\ &\Rightarrow z \in A, \qquad\qquad \text{because } x \in A. \end{aligned}$$

Therefore $v \subseteq A$ and hence A is $\mathcal{L}\Re_{\mathcal{U}}$-part of R. □

4. Rings derived from hyperrings

In this section we give the notion of *(strongly) normal relation* on $\mathcal{U}$ and then we construct a ring from a hyperring.

Definition 4.1. *Suppose that* $\Re \subseteq \mathcal{U} \times \mathcal{U}$.

(i) for all $(x,y) \in R^2$ *define the relation* $\rho_{\mathcal{L},\Re}$ *on* R *by:*

$$x\ \rho_{\mathcal{L},\Re} y \Leftrightarrow [x = y \ \ or \ \ \exists (u,v) \in \Re \ \ such\ that\ \ x \in u \ \ and \ \ y \in v]$$

and $\rho^*_{\mathcal{L},\Re}$ *is the transitive closure of* $\rho_{\mathcal{L},\Re}$*;*

(ii) for all $(x,y) \in R^2$ *define the relation* $\rho_{\mathcal{R},\Re}$ *on* R *by:*

$$x\ \rho_{\mathcal{R},\Re} y \Leftrightarrow [x = y \ \ or \ \ \exists (v,u) \in \Re \ \ such\ that\ \ x \in u \ \ and \ \ y \in v]$$

and $\rho^*_{\mathcal{R},\Re}$ *is the transitive closure of* $\rho_{\mathcal{R},\Re}$*;*

(iii) for all $(x,y) \in R^2$ *define the relation* $\rho_{\Re}$ *on* R *by:*

$$x\ \rho_{\Re} y \Leftrightarrow [x = y \ \ or \ \ \exists (u,v) \in \Re \bigcup \Re^{-1} \ \ such\ that\ \ x \in u \ \ and \ \ y \in v]$$

and $\rho^*_{\Re}$ *is the transitive closure of* $\rho_{\Re}$.

Theorem 4.2. *Suppose that* $\Re \subseteq \mathcal{U} \times \mathcal{U}$. *For all* $(x,y) \in R^2$ *we have:*

(i) $x\ K^{\mathcal{L}}_{\Re}\ y$ *if and only if* $x\ \rho^*_{\mathcal{L},\Re}\ y$*;*

(ii) $x\ K^{\mathcal{R}}_{\Re}\ y$ *if and only if* $x\ \rho^*_{\mathcal{R},\Re}\ y$.

Proof. (i) It is easy to see that $\rho^*_{\mathcal{L},\Re} \subseteq K^{\mathcal{L}}_{\Re}$. Converesly suppose that $x\ K^{\mathcal{L}}_{\Re}\ y$ so by Theorem 3.9 we have $x \in K^{\mathcal{L}}_{n+1,\Re}(y)$ for some $n \in \mathbb{N}$. So there exists $(u_1, v_1) \in \Re$ such that $x \in u_1$ and $v_1 \bigcap K^{\mathcal{L}}_{n,\Re}(y) \neq \emptyset$ thus there exists $x_1 \in v_1 \bigcap K^{\mathcal{L}}_{n,\Re}(y)$ and hence $x\ \rho_{\mathcal{L},\Re}\ x_1$. Since $x_1 \in K^{\mathcal{L}}_{n,\Re}(y)$, there exists $(u_2, v_2) \in \Re$ such that $x_1 \in u_2$ and $v_2 \bigcap K^{\mathcal{L}}_{n-1,\Re}(y) \neq \emptyset$. Therefore $x_1\ \rho_{\mathcal{L},\Re}\ x_2$, where $x_2 \in v_2 \bigcap K^{\mathcal{L}}_{n-1,\Re}(y)$. As a consequence we conclude that $x_n \in v_n \bigcap K^{\mathcal{L}}_{n-(n-1),\Re}(y)$ exists such that $x_{n-1}\ \rho_{\mathcal{L},\Re}\ x_n$. Thus we have,

$$x\ \rho_{\mathcal{L},\Re}\ x_1\ \rho_{\mathcal{L},\Re}\ x_2\ \ldots\ x_n\ \rho_{\mathcal{L},\Re}\ y.$$

From this follows $K^{\mathcal{L}}_{\Re} \subseteq \rho^*_{\mathcal{L},\Re}$ and the proof is complete.

Similarly we have (ii). □

Proposition 4.3. *Suppose that* $\Re$ *is a permutation of finite order in* $S_{\mathcal{U}}$*, then* $\rho^*_{\mathcal{L},\Re} = \rho^*_{\Re}$.

Proof. Since $K^{\mathcal{L}}_{\Re}(y)$ is $\mathcal{L}\Re_{\mathcal{U}}$-part by Theorem 3.6, $K^{\mathcal{L}}_{\Re}(y)$ is $\mathcal{R}\Re_{\mathcal{U}}$-part and hence $K^{\mathcal{R}}_{\Re} \subseteq K^{\mathcal{L}}_{\Re}$. Analogously $K^{\mathcal{L}}_{\Re} \subseteq K^{\mathcal{R}}_{\Re}$ and so $K^{\mathcal{L}}_{\Re} = K^{\mathcal{R}}_{\Re}$. From this it follows that $\rho^*_{\mathcal{L},\Re} = \rho^*_{\Re}$. □

Definition 4.4. *If $(R,+,\circ)$ is a hyperring and $\rho \subseteq R\times R$ is an equivalence, then we set:*

$$A \overline{\overline{\rho}} B \Leftrightarrow a \rho b, \quad \forall a\in A, \forall b \in B,$$

for all pairs (A,B) of non-empty subsets of R. The relation ρ is said to be strongly regular to the left (resp. to the right) if (i) $x \rho y \Rightarrow a + x \overline{\overline{\rho}} a + y$ and (ii) $x \rho y \Rightarrow a\circ x \overline{\overline{\rho}} a\circ y$ (resp. (i) $x \rho y \Rightarrow x + a \overline{\overline{\rho}} y + a$ and (ii) $x \rho y \Rightarrow a\circ x \overline{\overline{\rho}} a \circ y$), for all $(x,y,a) \in R^3$. ρ is called strongly regular if it is (i) strongly regular to the right and to the left and moreover (ii) there exists e in R such that: $\rho(x) = \rho(t)$, for all $t \in x \circ e \bigcap e\circ x$.

Definition 4.5. *Let R be a hyperring, then*

(i) a relation $\Re$ on $\mathcal{U}$ is called normal if for all $x \in R$, one has $K^{\mathcal{L}}_{\Re}(x) = K^{\mathcal{R}}_{\Re}(x)$,

(ii) a normal relation $\Re$ on $\mathcal{U}$ is called strongly normal to the left (resp. to the right) if $\rho^*_{\mathcal{L},\Re}$ (resp. $\rho^*_{\mathcal{R},\Re}$) is strongly regular to the left (resp. to the right),

(iii) a normal relation $\Re$ on $\mathcal{U}$ is called strongly normal if $\rho^*_{\Re}$ is strongly regular.

Suppose that $\Re \subseteq \mathcal{U}\times\mathcal{U}$. For every element x of a hyperring R, set:

$P^n_{\mathcal{L},\Re}(x) = \bigcup\{v \mid v \Re u_n, u_n = \sum\limits_{i=1}^{n}\prod\limits_{j=1}^{k_i} x_{ij}, x\in u_n\}$;

$P_{\mathcal{L},\Re}(x) = \bigcup\limits_{n\geqslant 1} P^n_{\mathcal{L},\Re}(x) \bigcup \{x\}$;

$\rho^*_{\mathcal{L},\Re}(x) = \{y \in R \mid y \rho^*_{\mathcal{L},\Re} x\}$.

Theorem 4.6. *Let R be a hyperring and $\Re$ be a relation on $\mathcal{U}$. The following conditions are equivalent:*

(i) $\rho_{\mathcal{L},\Re}$ is transitive;

(ii) for every $x\in R$, $\rho^*_{\mathcal{L},\Re}(x) = P_{\mathcal{L},\Re}(x)$;

(iii) for every $x \in R$, $P_{\mathcal{L},\Re}(x)$ is a $\mathcal{L}\Re_{\mathcal{U}}$-part of R.

Proof. (i) $\Rightarrow$ (ii) For every pair (x,y) of elements of R we have:

$$y \in \rho^*_{\mathcal{L},\Re}(x) \Leftrightarrow y \rho^*_{\mathcal{L},\Re} x \Leftrightarrow y \rho_{\mathcal{L},\Re} x \Leftrightarrow y \in P_{\mathcal{L},\Re}(x).$$

(ii) $\Rightarrow$ (iii) Let $(v,u) \in \Re$ such that $u \cap P_{\mathcal{L},\Re}(x) \neq \emptyset$ be given. So $u\cap \rho^*_{\mathcal{L},\Re}(x) \neq \emptyset$ and hence there exists $z \in R$ such that $z\in u$ and $z \in \rho^*_{\mathcal{L},\Re}(x)$, thus $z \in K^{\mathcal{L}}_{\Re}(x)$, by Theorem 4.2. On the other hand, $z \in K^{\mathcal{L}}_{\Re}(z)$, so $u \cap K^{\mathcal{L}}_{\Re}(z) \neq \emptyset$ and hence $v \subseteq K^{\mathcal{L}}_{\Re}(z)$, because $v \Re u$ and $K^{\mathcal{L}}_{\Re}(z)$ is a $\mathcal{L}\Re_{\mathcal{U}}$-part of R, by Theorem 3.7. Now suppose that $t\in v$ is an arbitrary element, thus $t \in K^{\mathcal{L}}_{\Re}(x)$ and hence $t \rho^*_{\mathcal{L},\Re} x$. Therefore $t \in \rho^*_{\mathcal{L},\Re}(x) = P_{\mathcal{L},\Re}(x)$ and so $v \subseteq P_{\mathcal{L},\Re}(x)$.

(iii) $\Rightarrow$ (i) Let x, y and z in R be given such that $x \rho_{\mathcal{L},\Re} y$ and $y \rho_{\mathcal{L},\Re} z$. Since $x \rho_{\mathcal{L},\Re} y$, there exists $(u,v) \in \Re$ such that $x\in u$ and $y \in v$. Therefore $v \cap P_{\mathcal{L},\Re}(y) \neq \emptyset$ and since $P_{\mathcal{L},\Re}(y)$ is a $\mathcal{L}\Re_{\mathcal{U}}$-part, $u \subseteq P_{\mathcal{L},\Re}(y)$ and hence $x \in P_{\mathcal{L},\Re}(y)$. We can see that $P_{\mathcal{L},\Re}(y) \subseteq P_{\mathcal{L},\Re}(z)$, because $y \rho_{\mathcal{L},\Re} z$ and so by above $y \in P_{\mathcal{L},\Re}(z)$. Therefore $x \in P_{\mathcal{L},\Re}(z)$ and hence $x \rho_{\mathcal{L},\Re} z$. □

Proposition 4.7. *If $\Re$ is a normal relation on $\mathcal{U}$, then:*
(i) $\Re^{-1}$ is a normal relation;
(ii) $\rho^*_{\mathcal{L},\Re} = \rho^*_{\Re}$ and $\rho^*_{\mathcal{L},\Re}$ is an equivalence relation.

Proof. The proof follows from Proposition 3.4 and Theorem 4.2. □

Theorem 4.8. *Suppose that $(R, +, \circ)$ is a hyperring and $\Re$ is a strongly normal relation on $\mathcal{U}$. A ring structure turns out to be define on $R/\rho^*_{\Re}$ with respect to the operations:*

$$\rho^*_{\Re}(x) \oplus \rho^*_{\Re}(y) = \rho^*_{\Re}(z), \quad \textit{where } z \in x + y.$$

$$\rho^*_{\Re}(x) \odot \rho^*_{\Re}(y) = \rho^*_{\Re}(z), \quad \textit{where } z \in x \circ y.$$

Proof. We will prove that the operation $\oplus$ is well defined. Let $\rho^*_{\Re}(x_0) = \rho^*_{\Re}(x_1)$ and $\rho^*_{\Re}(y_0) = \rho^*_{\Re}(y_1)$. It is necessary to verify that $\rho^*_{\Re}(x_0) \oplus \rho^*_{\Re}(y_0) = \rho^*_{\Re}(x_1) \oplus \rho^*_{\Re}(y_1)$. By hypothesis $(m, n) \in \mathbb{N}^2$, $(z_0, z_1, ..., z_m) \in R^{m+1}$ and $(t_0, t_1, ..., t_n) \in R^{n+1}$ exist such that

$$x_0 = z_0 \ \rho_{\Re} \ z_1 \ \rho_{\Re} \ z_2 \ \ldots \ z_{m-1} \ \rho_{\Re} \ z_m = x_1$$

and

$$y_0 = t_0 \ \rho_{\Re} \ t_1 \ \rho_{\Re} \ t_2 \ \ldots \ t_{n-1} \ \rho_{\Re} \ t_n = y_1$$

Since $\Re$ is normal, for all $u \in z_{s-1} + t_{s-1}$ and $v \in z_s + t_s$, where $1 \leqslant s \leqslant k$ and $k = min\{m, n\}$, we have $u \ \rho^*_{\Re} v$. Therefore $\rho^*_{\Re}(x_0) \oplus \rho^*_{\Re}(y_0) = \rho^*_{\Re}(z_1) \oplus \rho^*_{\Re}(t_1) = ... = \rho^*_{\Re}(z_k) \oplus \rho * (t_k) = \rho^*_{\Re}(a_{k+i}) \oplus \rho^*_{\Re}(b_{k+i})$, where $k + 1 \leqslant k + i \leqslant max\{m, n\}$ and:

$$(a_{k+i}, b_{k+i}) = \begin{cases} (x_1, t_{k+i}) & \text{if } \ k = m; \\ (z_{k+i}, y_1) & \text{if } k = n. \end{cases}$$

Hence $\oplus$ is well defined. Similarly the operation $\odot$ is well defined and Theorem 31 of [2] shows that $(R/\rho^*_{\Re}, \oplus)$ is a group. By strongly normality of $\Re$ we conclude that $(R/\rho^*_{\Re}, \odot)$ is a monoid with unit $\rho^*_{\Re}(e)$. The commutativity of $\oplus$ is related with the existence of the unit in multiplication. Since $\Re$ is strong, there exists e in R such that $\rho(x) = \rho(t)$ for all $t \in x \circ e \bigcap e \circ x$ which means $\rho^*_{\Re}(e)$ is the unit of multiplication so we have:

$[\rho^*_{\Re}(x) \oplus \rho^*_{\Re}(y)] \odot [\rho^*_{\Re}(e) \oplus \rho^*_{\Re}(e)] = (\rho^*_{\Re}(x) \odot [\rho^*_{\Re}(e) \oplus \rho^*_{\Re}(e)]) \oplus (\rho^*_{\Re}(y) \odot [\rho^*_{\Re}(e) \oplus \rho^*_{\Re}(e)]) = (\rho^*_{\Re}(x) \oplus \rho^*_{\Re}(x)) \oplus (\rho^*_{\Re}(y) \oplus \rho^*_{\Re}(y))$ and also $[\rho^*_{\Re}(x) \oplus \rho^*_{\Re}(y)] \odot [\rho^*_{\Re}(e) \oplus \rho^*_{\Re}(e)] = ([\rho^*_{\Re}(x) \oplus \rho^*_{\Re}(y)] \odot \rho^*_{\Re}(e)) \oplus ([\rho^*_{\Re}(x) \oplus \rho^*_{\Re}(y)] \odot \rho^*_{\Re}(e)) = (\rho^*_{\Re}(x) \oplus \rho^*_{\Re}(y)) \oplus (\rho^*_{\Re}(x) \oplus \rho^*_{\Re}(y))$.
So $(\rho^*_{\Re}(x) \oplus \rho^*_{\Re}(x)) \oplus (\rho^*_{\Re}(y) \oplus \rho^*_{\Re}(y)) = (\rho^*_{\Re}(x) \oplus \rho^*_{\Re}(y)) \oplus (\rho^*_{\Re}(x) \oplus \rho^*_{\Re}(y))$ gives, $\rho^*_{\Re}(x) \oplus \rho^*_{\Re}(y) = \rho^*_{\Re}(y) \oplus \rho^*_{\Re}(x)$. □

Let $(R, +, \circ)$ and $(R', +', \circ')$ be two hyperrings. We say that $f : R \to R'$ is a homomorphism if for every $(x, y) \in R^2$ we have $f(x + y) = f(x) +' f(y)$ and $f(x \circ y) = f(x) \circ' f(y)$.

Definition 4.9. *Let R is a hyperring and $\Re$ be a strongly normal relation on $\mathcal{U}$. If $\varphi_{\Re} : R \to R/\rho^*_{\Re}$ be the canonical projection, we set $\omega_{\Re} = \varphi_{\Re}^{-1}(1_{R/\rho^*_{\Re}})$, and called the heart of $\varphi_{\Re}$.*

Theorem 4.10. *Let $(R, +, \circ)$ is a hyperfield (i.e, $(R, +, \circ)$ be a hyperring and $(R, \circ)$ is a hypergroup) and B is a non-empty subset of R, then we have $\omega_{\Re} \circ B = B \circ \omega_{\Re} = \varphi_{\Re}^{-1}(\varphi_{\Re}(B))$.*

Proof. Clearly $\varphi_{\Re}^{-1}(\varphi_{\Re}(B)) = \{x \in R \mid \exists b \in B : \varphi_{\Re}(b) = \varphi_{\Re}(x)\}$. Let $y \in \varphi_{\Re}^{-1}(\varphi_{\Re}(B))$, thus for some $b \in B$, $\varphi_{\Re}(b) = \varphi_{\Re}(y)$. Since $(R, \circ)$ is a hypergroup, $u \in R$ exists such that $y \in b \circ u$, so $\varphi_{\Re}(y) = \varphi_{\Re}(b) \odot \varphi_{\Re}(u)$. Since $(R/\rho^*_{\Re}, \odot)$ is a group and $\varphi_{\Re}(b) = \varphi_{\Re}(y)$, we obtain $\varphi_{\Re}(u) = 1_{R/\rho^*_{\Re}}$ and so $u \in \varphi_{\Re}^{-1}(1_{R/\rho^*_{\Re}}) = \omega_{\Re}$. Therefore, $\varphi_{\Re}^{-1}(\varphi_{\Re}(B)) \subseteq B \circ \omega_{\Re}$.
Converesly if $z \in B \circ \omega_{\Re}$, then $\varphi_{\Re}(z) \in \varphi_{\Re}(B)$ and so $z \in \varphi_{\Re}^{-1}(\varphi_{\Re}(B))$. It is proved that $\omega_{\Re} \circ B = \varphi_{\Re}^{-1}(\varphi_{\Re}(B))$ by a similar way and we obtain $\varphi_{\Re}^{-1}(\varphi_{\Re}(B)) = \omega_{\Re} \circ B = B \circ \omega_{\Re}$. □

Theorem 4.11. *If $(R, +, \circ)$ is a hyperfield and B is a non-empty subset of R, then we have $\omega_{\Re} \circ B = B \circ \omega_{\Re} = \overline{\Re_{\mathcal{U}}}(B)$.*

Proof. If $\varphi_{\Re}(b) = \varphi_{\Re}(x)$ then $x \in \overline{\Re_{\mathcal{U}}}(b)$. Therefore $\varphi_{\Re}^{-1}(\varphi_{\Re}(B)) = \bigcup_{b \in B} \overline{\Re_{\mathcal{U}}}(b) = \overline{\Re_{\mathcal{U}}}(B)$. □

5. $\Re$-parts and A_R-hyperrings

We recall that a K_H hypergroup is a hypergroup constructed from a hypergroup $(H, \circ)$ and a family $\{A(x)\}_{x \in H}$ of non-empty subsets that are mutually disjoint. Put $K_H = \bigcup_{x \in H} A(x)$ and define the hyperoperation $*$ on K_H as following,

$$\forall (a, b) \in K_H^2, \quad a \in A(x), b \in A(y), \quad a * b :\stackrel{def}{=} \bigcup_{z \in x \circ y} A(z).$$

$(H, \circ)$ is a hypergroup if and only if $(K_H, *)$ is a hypergroup. In this case K_H is said to be a *K_H-hypergroup generated by H.*

Now let $(R, \dagger, \star)$ be a commutative hyperring, S_r , $r \in R$ be a family of non-empty sets indexed in R such that for all $r_1, r_2 \in R$, $r_1 \neq r_2$, $S_{r_1} \bigcap S_{r_2} = \emptyset$. We set $A = \bigcup_{r \in R} S_r$ and we define the hyperoperations $\uplus$ and $\odot$ in A in the following way:

$$\forall (x, y) \in S_{r_1} \times S_{r_2}, \quad x \uplus y = \bigcup_{t \in r_1 \dagger r_2} S_t \quad and \quad x \odot y = \bigcup_{u \in r_1 \star r_2} S_u.$$

It is easy to see that the structure $(A, \uplus, \odot)$ is a hyperring. The hyperring $(A, \uplus, \odot)$ is called a A_R-*hyperring with suport* A or A_R-*hyperring generated by*

R.

For all $P \in P^*(R)$, let $S(P) = \bigcup\limits_{x \in P} S_x$.

Theorem 5.1. *Let $\Re$ be a relation on $\mathcal{U}$. Then P is $\mathcal{L}\Re_{\mathcal{U}}$-part of R if and only if $S(P)$ is $\mathcal{L}\widehat{\Re}_{\mathcal{U}}$-part of A_R, where the relation $\widehat{\Re}$ is defined as follows:*

$$\sum_{i=1}^{n}\prod_{j=1}^{t_i} x_{ij} \Re \sum_{i=1}^{m}\prod_{j=1}^{k_i} y_{ij} \Leftrightarrow \bigcup_{v \in \sum\limits_{i=1}^{n}\prod\limits_{j=1}^{t_i} x_{ij}} S_v \ \widehat{\Re} \bigcup_{u \in \sum\limits_{i=1}^{m}\prod\limits_{j=1}^{k_i} y_{ij}} S_u.$$

Proof. Let $S(P)$ be a $\mathcal{L}\widehat{\Re}_{\mathcal{U}}$-part of A_R and $(\prod\limits_{i=1}^{n} x_i, \prod\limits_{i=1}^{m} y_i) \in \Re$ such that $\prod\limits_{i=1}^{m} y_i \bigcap P \neq \emptyset$ be given. So $\bigcup\limits_{v \in \prod\limits_{i=1}^{n} x_i} S_v \ \widehat{\Re} \bigcup\limits_{u \in \prod\limits_{i=1}^{m} y_i} S_u$ and we have,

$$\begin{aligned}
\prod_{i=1}^{m} y_i \cap P \neq \emptyset &\Rightarrow \exists p \in P, \text{ such that } p \in \prod_{i=1}^{m} y_i \\
&\Rightarrow \exists p \in P, \text{ such that } S_p \subseteq \bigcup_{u \in \prod\limits_{i=1}^{m} y_i} S_u \\
&\Rightarrow \bigcup_{u \in \prod\limits_{i=1}^{m} y_i} S_u \cap S(P) \neq \emptyset \\
&\Rightarrow \bigcup_{v \in \prod\limits_{i=1}^{n} x_i} S_v \subseteq S(P), \text{ because } S(P) \text{ is a } \mathcal{L}\widehat{\Re}_{\mathcal{U}} - \text{part.}
\end{aligned}$$

Now suppose that $t \in \prod\limits_{i=1}^{n} x_i$ is given. Then $S_t \subseteq S(P)$ and so there exists $q \in P$ such that $S_t \cap S_q \neq \emptyset$. Therefore $t = q$ and hence $t \in P$, thus $\prod\limits_{i=1}^{n} x_i \subseteq P$. For the proof of the converse implication let $\sum\limits_{i=1}^{n}\prod\limits_{j=1}^{t_i} z_{ij} \bigcap S(P) \neq \emptyset$ and $\sum\limits_{i=1}^{s}\prod\limits_{j=1}^{l_i} t_{ij} \ \widehat{\Re} \ \sum\limits_{i=1}^{n}\prod\limits_{j=1}^{t_i} z_{ij}$ be given. Therefore there exists $x_{ij} \in A$ such that for all $1 \leqslant i \leqslant m'$,$1 \leqslant j \leqslant k'_i$, $z_{ij} \in S_{x_{ij}}$. Suppose that $u \in \bigcup\limits_{y \in \sum\limits_{i=1}^{n}\prod\limits_{j=1}^{t_i} x_{ij}} S_y$, thus $u \in S_{y_0}$ for some $y_0 \in \prod\limits_{i=1}^{n} x_i$. Since $u \in S(P)$, then there exists $y_1 \in P$ such that $u \in S_{y_1}$. Therefore $S_{y_0} \cap S_{y_1} \neq \emptyset$, which implies $y_0 = y_1 \in \prod\limits_{i=1}^{n} x_i \cap P$. Since

P is $\mathcal{L}\Re_{\mathcal{U}}$-part of R and $\sum\limits_{i=1}^{s}\prod\limits_{j=1}^{l_i} x'_{ij} \ \Re \ \prod\limits_{i=1}^{n} x_i$, where $t_{ij} \in S_{x'_{ij}}$ for all $1 \leqslant i \leqslant s$, then $\sum\limits_{i=1}^{s}\prod\limits_{j=1}^{l_i} x'_{ij} \subseteq P$. So $\sum\limits_{i=1}^{s}\prod\limits_{j=1}^{l_i} t_{ij} = \bigcup\limits_{w \in \sum\limits_{i=1}^{s}\prod\limits_{j=1}^{l_i} x'_{ij}} S_w \subseteq \bigcup\limits_{u \in P} S_u = S(P)$. □

6. Conclusion

In this paper we introduce and analyze a generalization of the notion of a complete part in a hyperring. We call this generalization $\Re$-part of a hyperring. Several properties are investigated, such as the structure of $\Re$-closures of a subset. This research can be continuated, for instance in the study of some particular classes of hyperrings.

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