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F-Permutations induce Some Graphs and Matrices

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Abstract. In this paper, by using the notion of fuzzy subsets, the concept of F-permutation is introduced. Then by applying this notion the concepts of presentation of an F-polygroup, graph of an F-permutation and F-permutation matrices are investigated.

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1. Introduction

Some researchers applied the notion of fuzzy subsets in many algebraic structures for example [1, 9, 5, 6]. Zadeh in 1965 [13] introduced the notion of fuzzy subsets of a non-empty set A as a function from A to $[0, 1]$. Rosenfeld in 1971 [9] defined fuzzy subgroup and obtained some basic results. The hyper algebraic structure theory was introduced in 1934 [8] by Marty at 8th Congress of Scandinavian Mathematicians. Since then many researchers have worked on this area for example [4, 10, 14]. A polygroup is a completely regular, reversible-in-itself multigroup in the sense of Dresher and Ore [3]. Ioulidis in 1987 [7] studied the concept of polygroup, which is a generalization of the concept of

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ordinary group. Zahedi, Bolurian and Hasankhani in 1995 [15] introduced the concept of a fuzzy subpolygroup. Zahedi and Hasankhani [5] defined the notion of F-polygroups, which is a generalization of polygroups.

In papers [2], [5], [6], by using the notion of fuzzy sets, the concept of groups, rings, modules and vector spaces are generalized. In paper [11], the notion of generalized permutation matrices is defined.

T. Vougiouklis in [12] by using the hyperalgebraic theory defined the notion of generalized permutation.

In this paper by using the notion of fuzzy sets, we defined notion of fuzzy permutation and proved some related results.

The motivations and applications of defining the notion of F-permutation are given for:

1. A presentation for any F-polygroup,
2. Some graphs and matrices.

2. Preliminaries.

A fuzzy subset of the non-empty set X is a function from X into $[0,1] \subseteq \mathbb{R}$. The set of all fuzzy subset of X is denoted by $F(X)$. Let $\mu \in F(X)$. Then by support of μ we mean the set, $supp(\mu) = \{x \in X | \mu(x) > 0\}$. We let $F_*(X) = \{\mu \in F(X) | supp(\mu) \neq \emptyset\}$. Each element of $F_*(X)$ is called a non-empty fuzzy subset of X. If $\mu, \eta \in F(X)$, then $\mu \leq \eta$ iff $\mu(x) \leq \eta(x)$, for all $x \in X$.

Definition 2.1. [13] Let $\{\mu_\alpha \mid \alpha \in \Lambda\}$ be a family of fuzzy subsets of X. Then the fuzzy sets $\bigvee_{\alpha\in\Lambda} \mu_\alpha$ and $\bigwedge_{\alpha\in\Lambda} \mu_\alpha$ are defined by, for all $x \in X$

i) $\left(\bigwedge_{\alpha\in\Lambda} \mu_\alpha\right)(x) = \bigwedge_{\alpha\in\Lambda} (\mu_\alpha(x)) = \inf_{\alpha\in\Lambda} \mu_\alpha(x)$

ii) $\left(\bigvee_{\alpha\in\Lambda} \mu_\alpha\right)(x) = \bigvee_{\alpha\in\Lambda} (\mu_\alpha(x)) = \sup_{\alpha\in\Lambda} \mu_\alpha(x)$.

Definition 2.2. [5] i) Let A be a non-empty set. A function “ $*$ ” from $A \times A$ into $F_*(A)$ is called an F-hyperoperation on A, and $(A, *)$ is called an F-hypergroupoid.

ii) Let $(A, *)$ be an F-hypergroupoid and $\mu, \eta \in F_*(A)$, then by $\mu * \eta$ we mean:

$$\mu * \eta = \bigvee_{\substack{x \in supp(\mu) \\ y \in supp(\eta)}} x * y$$

Notation 2.1. [5] *Let A be a non-empty set. If B and C are non-empty subsets of A, $a \in A$ and $\mu \in F_*(A)$, then*

i) $\chi_B(x) = \begin{cases} 1 & x \in B \\ 0 & x \notin B \end{cases}$, *for all* $x \in A$,
ii) by $a * \mu$ *and* $\mu * a$ *we mean* $\chi_{\{a\}} * \mu$ *and* $\mu * \chi_{\{a\}}$ *respectively,*
iii) by $a * B$, $B * a$, $B * \mu$ *and* $B * C$ *we mean* $\chi_{\{a\}} * \chi_B$, $\chi_B * \chi_{\{a\}}$, $\chi_B * \mu$ *and* $\chi_B * \chi_C$ *respectively.*

Definition 2.3. [5] Let A be a non-empty set and "$*$" be an F-hyperoperation on A. Then $(A, *)$ is called an F-polygroup if for all $x, y, z \in A$,
i) $(x * y) * z = x * (y * z)$,
ii) there exists $e \in A$ such that

$$x \in supp(x * e \wedge e * x),$$

iii) for all $x \in A$, there exists the unique element x^{-1} in A such that

$$e \in supp(x * x^{-1} \wedge x^{-1} * x),$$

iv) $z \in supp(x * y) \Longrightarrow x \in supp(z * y^{-1}) \Longrightarrow y \in supp(x^{-1} * z)$.

Definition 2.4. [5] Let $(A, *)$ be an F-polygroup and H be a non-empty subset of A. Then H is said to be a sub F-polygroup of A if:

$$\begin{array}{lll} i) & x \in H & \Longrightarrow x^{-1} \in H, \\ ii) & x,\ y \in H & \Longrightarrow supp(x * y) \subseteq H. \end{array}$$

Lemma 2.1. *Let* $(A, *)$ *be an F-polygroup and* H *be a sub F-polygroup of* A. *Then for all* $x, y \in A$

$$\begin{array}{lll} i) & y \in supp(x * H) & \Longleftrightarrow x \in supp(y * H), \\ ii) & y \in supp(x * H) & \Longleftrightarrow x * H = y * H. \end{array}$$

Proof. (i) The proof is easy.
(ii) Let $x,\ y \in A$ and $y \in supp(x * H)$, then $y * H \leq x * H * H \leq x * H$. Similarly by (i), $x * H \leq y * H$. □

3. F-Permutation.

Definition 3.1. Let X be a non-empty set. The function $f : X \longrightarrow F_*(X)$ is called an F-permutation of X if

$$\bigcup_{x \in X} supp(f(x)) = X.$$

The set of all F-permutation of X is denoted by $FP(X)$.

The above definition is a generalization of Definition 6.1.1 of [13] (page 84).

Definition 3.2. Let $f \in FP(X)$. We define $f_I : X \longrightarrow F(X)$ by

$$(f_I(y))(x) = (f(x))(y); \quad \forall x, y \in X$$

and f_I is called the converse of f.

Theorem 3.1. *Let $f \in FP(X)$. Then $f_I \in FP(X)$, where*

$$(f_I(y))(x) = (f(x))(y), \ \forall x, y \in X.$$

Proof. Let $x \in X$. Then by Definition 3.1., $x \in supp(f(t))$ for some $t \in X$. In other words, $t \in supp(f_I(x))$. So $supp(f_I(x)) \neq \emptyset$ and hence f_I is a function from X into $F_*(X)$. Now let $y \in X$ be an arbitrary element. Since $supp(f(y)) \neq \emptyset$, then there exists $x \in X$ such that $y \in supp(f_I(x))$. Therefore $\bigcup_{x \in X} supp(f_I(x)) = X$. □

Definition 3.3. Let $f_1, f_2 \in FP(X)$. Then

(i) f_1 is called a sub F-permutation of f_2 if $f_1(x) \leq f_2(x)$, $\forall x \in X$. In this case we write $f_1 \subseteq f_2$.

(ii) If $f_1 \subseteq f_2$ and $supp(f_1(x_0)) \neq supp(f_2(x_0))$, for some $x_0 \in X$, then we say that f_1 is a proper sub F-permutation of f_2 and we write $f_1 \subset f_2$.

Note: If $f_1 \subseteq f_2$ then $supp(f_1(x)) \subseteq supp(f_2(x))$, for all $x \in X$.

Definition 3.4. Let $f \in FP(X)$. Then f is said to be minimal if it has no proper sub F-permutation. The set of all minimal elements of $FP(X)$ is denoted by $MFP(X)$.

Notation 3.1. *Let X be a countable set and $f \in FP(X)$. We write:*

$$f = \begin{pmatrix} x_1 & \cdots & x_{i-1} & x_i & x_{i+1} & \cdots \\ \cdots & \cdots & \cdots & \dfrac{t_{j_1}}{(f(x_i))(t_{j_1})}, \dfrac{t_{j_2}}{(f(x_i))(t_{j_2})}, \cdots & \cdots & \cdots \end{pmatrix};$$

where $x_j \in X$ and $t_{j_k} \in supp(f(x_j))$ for $k = 1, 2, \ldots$.

Example 3.1. *Let $X = \{1, 2, 3\}$, and $\alpha_1, \ \alpha_2, \ \alpha_3, \ \alpha_4 \in (0, 1]$*

$$f = \begin{pmatrix} 1 & 2 & 3 \\ \dfrac{2}{\alpha_1}, \dfrac{3}{\alpha_2} & \dfrac{1}{\alpha_3} & \dfrac{3}{\alpha_4} \end{pmatrix}$$

(i.e. $(f(1))(2) = \alpha_1$, $(f(1))(3) = \alpha_2$, $(f(1))(1) = (f(2))(2) = (f(2))(3) = (f(3))(1) = (f(3))(2) = 0$, $(f(2))(1) = \alpha_3$, $(f(3))(3) = \alpha_4$), and let

$$f' = \begin{pmatrix} 1 & 2 & 3 \\ \dfrac{2}{\alpha_1} & \dfrac{1}{\alpha_3} & \dfrac{3}{\alpha_4} \end{pmatrix}.$$

Then $f, f' \in FP(X)$, $f' \subset f$ and $f' \in MFP(X)$.

Theorem 3.2. *Let $f \in FP(X)$. Then $f \in MFP(X)$ iff the following condition is valid: if $a \neq b$ and $supp(f(a)) \cap supp(f(b)) \neq \emptyset$, then $supp(f(a)) = supp(f(b))$ and $supp(f(a))$ is a singleton.*

Proof. Let $f \in MFP(X)$, $a, b \in X$, $a \neq b$ and $supp(f(a)) \cap supp(f(b)) \neq \emptyset$. It is enough to show that $supp(f(a))$ is a singleton. Suppose that $u \in supp(f(a)) \cap supp(f(b))$ and $supp(f(a))$ is not a singleton. We define

$$f' : X \longrightarrow F_*(X)$$

as follows

$$(f'(x))(y) = \begin{cases} (f(x))(y), & \text{if } (x, y) \neq (a, u) \\ 0 & \text{otherwise.} \end{cases}$$

Since $|supp(f(a))| \geq 2$, then $f'(x) \neq 0$ and hence $f'(x) \in F_*(X)$, so f' is well define. Let $x \in X$, then since $supp(f(x)) \neq \emptyset$, we obtain that $supp(f'(x)) \neq \emptyset$ by definition of f'. Now let $x = a$. Then since $supp(f(a))$ is not a singleton, there exists $t \in supp(f(a))$ such that $t \neq u$. By definition of f', $t \in supp(f'(a))$. In other word in this case $supp(f'(x)) \neq \emptyset$. On the other hand suppose that $y \in X$ be an arbitrary element. If $y = u$, then since $u \in supp(f(b))$ and $b \neq a$, $y \in supp(f'(b))$. Now let $y \neq u$. Then by Definition 3.1 for F and definition of f' we get that $y \in supp(f'(x))$, for some $x \in X$. Hence $f' \in FP(X)$ which is a contradiction, since $f' \subset f$ and $f \in MFP(X)$. Therefore $supp(f(a))$ is a singleton. Similarly $supp(f(b))$ is a singleton, consequently $supp(f(a)) = supp(f(b))$.

Conversely, suppose that $f'' \subset f$ for some $f'' \in FP(X)$. Let $a \in X$, choose $b \in supp(f(a)) - supp(f''(a))$, for some $a \in X$. By Definition 3.1. there exist $d \in X$ and $d \neq a$ such that:

$$b \in supp(f''(d)) \subseteq supp(f(d)),$$

for some $d \in X$ and $d \neq a$. Thus

$$b \in supp(f(a)) \cap supp(f(d)), d \neq a.$$

So $supp(f(a))$ is a singleton , which is a contradiction, since

$$\{b\} \cup supp(f''(a)) \subseteq supp(f(a)).$$

Hence $f \in MFP(X)$. □

Corollary 3.1. *Let $f \in FP(X)$. If $f \in MFP(X)$, then $P = \{supp(f(x)) \mid x \in X\}$ is a partition for X.*

Proof. The proof follows from Theorem 3.2. □

The following example shows that the converse of the Corollary 3.3 is not true.

Example 3.2. *Let* $X = \{1, 2, 3, 4\}$*. Define* $g,\ g' \in FP(X)$ *as follows:*

$$g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \\ \frac{2}{0.2}, \frac{3}{0.4} & \frac{1}{0.3} & \frac{4}{0.2} & \frac{2}{0.5}, \frac{3}{0.7} \end{pmatrix}, g' = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \\ \frac{2}{0.2} & \frac{1}{0.3} & \frac{4}{0.2} & \frac{2}{0.5}, \frac{3}{0.7} \end{pmatrix}.$$

It is easy to show that $P = \{supp(g(x)) \mid x \in X\} = \{\{2, 3\},\ \{1\},\ \{4\}\}$ *is a partition of* X *but* $g \notin MFP(X)$*, since* $g' \subset g$*.*

Theorem 3.3. *Let* $f \in FP(X)$*. For* $a \in X$*, if* $|supp(f(a))| = 1$*, then* $f \in MFP(X)$*.*

Proof. By contrary, let $f \notin MFP(X)$. Then there exists $f' \in FP(X)$ such that $f' \subset f$. Hence there is $a_0 \in X$ such that $supp(f'(a_0)) \subsetneqq supp(f(a_0))$ which implies that $supp(f'(a_0)) = \emptyset$ and it is a contradiction. □

The following example shows that the converse of the above theorem is not true.

Example 3.3. *Let* $X = \{1, 2, 3\}$ *and*

$$f = \begin{pmatrix} 1 & 2 & 3 \\ \\ \frac{2}{0.2}, \frac{3}{0.4} & \frac{1}{0.3} & \frac{1}{0.2} \end{pmatrix}.$$

Clearly $f \in MFP(X)$ *and* $|supp(f(1))| = 2 \neq 1$*.*

4. Presentation of an F-Polygroup.

Definition 4.1. Let $f, g \in FP(X)$, then the composition of f and g is denoted by $f \circ g$ and is defined by, for all $x \in X$

$$(f \circ g)(x) = \bigvee_{t \in supp(g(x))} f(t).$$

Theorem 4.1. *Let* X *be a non-empty set,* $f, g, h \in FP(X)$ *and* $J : X \longrightarrow F_*(X)$ *a function which is defined by* $J(x) = \chi_{\{x\}}$ *for all* $x \in X$*. Then*

i) $f \circ g \in FP(X)$
ii) $f \circ (g \circ h) = (f \circ g) \circ h$
iii) $x \in supp(f \circ f_I(x)) \cap supp(f_I \circ f(x))$
iv) $J \in MFP(X)$*, (it is called the identity F-permutation of* X*).*
v) $f \circ J = f$*.*

Proof. The proof follows from Definitions 4.1, 3.2 and some manipulations. □

Definition 4.2. Let $f \in FP(X)$. Then we say that f satisfies the condition S if, for all $x \in X$, $y \in supp(f(x))$, $supp(f(x)) = supp(f(y))$.
The set of all elements of $FP(X)$ which satisfy in condition S, is denoted by $FPS(X)$.

Example 4.1. *Let* $X = \{1, 2, 3\}$,

$$f = \begin{pmatrix} 1 & 2 & 3 \\ \\ \dfrac{1}{\alpha_1}, \dfrac{2}{\alpha_2} & \dfrac{1}{\alpha_3}, \dfrac{2}{\alpha_4} & \dfrac{3}{\alpha_5} \end{pmatrix}$$

where $\alpha_i \in (0, 1]$, $\forall i,\ 1 \leq i \leq 5$. *Then* $f \in FPS(X)$.

Definition 4.3. Let $(A, *)$ be an F-polygroup, H be a sub polygroup of A and $a \in A$. We define

$$\begin{array}{rl} \varphi_H : & A \longrightarrow F_*(A) \\ & x \mapsto x * H \end{array} \qquad \begin{array}{rl} \varphi_a : & A \longrightarrow F_*(A) \\ & x \mapsto a * x \end{array}$$

Theorem 4.2. *Let* $(A, *)$ *be an F-polygroup,* H *a sub F-polygroup of* A*. Then* $\varphi_H \in FPS(A)$.

Proof. By Definition 2.4. $e \in H$ and also $x \in supp(x * e)$, for all $x \in A$, then $x \in supp(x * H)$. Hence $\varphi_H \in FP(A)$. Now let $y \in supp(\varphi_H(x))$. Then by Lemma 2.1, $\varphi_H(x) = \varphi_H(y)$. Hence $\varphi_H \in FPS(A)$. □

Lemma 4.1. *Let* $(A, *)$ *be an F-polygroup,* $a \in A$*. Then* $\varphi_a \in FP(A)$.

Proof. For all $a, x \in A$, since

$$x \in supp(e * x) \subseteq supp(a * (a^{-1} * x)),$$

then $x \in supp(a * t)$, for some $t \in supp(a^{-1} * x)$. Therefore $\varphi_a \in FP(A)$. □

Lemma 4.2. *Let* $\{f_\alpha | \alpha \in \Lambda\}$ *be a family of F-permutation of* X*. Then*

$$\bigvee_{\alpha \in \Lambda} f_\alpha \in FP(X).$$

Proof. Since $supp(f_\beta(x)) \subseteq supp(\bigvee_{\alpha \in \Lambda} f_\alpha(x))$, $\forall \beta \in \Lambda$, the proof follows. □

Definition 4.4. Let $T : X \longrightarrow FP(X)$ be a function, $\mu \in F_*(X)$. Then $T(\mu)$ is defined by

$$T(\mu) = \bigvee_{x \in supp(\mu)} T(x).$$

Definition 4.5. Let $(A, *)$ be an F-polygroup and $T : A \longrightarrow FP(A)$ be a function. Then T is called a presentation for A, if

$$T(a_1 * a_2) = T(a_1) \circ T(a_2), \quad \forall a_1, a_2 \in A.$$

Example 4.2. *In Definition 4.3,* φ_a *was introduced and in Theorem 4.5, we show that by using this notion, we can defined a presentation for any F-polygroup which is called left F-translation presentation.*
Similarly we can defined right F-translation presentation. Therefore the converse of Theorem 4.5 is not true.

Theorem 4.3. *Let $(A,*)$ be an F-polygroup. Define the function $T : A \longrightarrow FP(A)$ by $T(a) = \varphi_a$. Then T is a presentation for A.*

Proof. By Lemma 4.3, T is well-defined. Now for all $a, b, x \in A$ we have:

$$\begin{aligned}(T(a)\circ T(b))(x) &= \bigvee_{\omega\in supp(T(b))(x)} [(T(a))(\omega)] = \bigvee_{\omega\in supp(b*x)} a*\omega \\ &= a*(b*x) = (a*b)*x = \bigvee_{t\in supp(a*b)} t*x = \bigvee_{t\in supp(a*b)} \varphi_t(x) \\ &= (\bigvee_{t\in supp(a*b)} T(t))(x) = T(a*b)(x).\end{aligned}$$

□

5. Graph of an F-Permutation.

In the recent article [11], were suggested the graph of a generalized permutation and the new definition of generalized permutation matrices, associated with the generalized permutation. In this and next sections we extended these concepts to F-permutations.

Definition 5.1. Let X be a non empty set and f be an F-permutation on X. We consider X, the set of vertices and we define a weighted directed arc from "x" to "y", if $f(x)(y) > 0$. The weighted directed graph of f, is denoted by $WDG(f)$.

Example 5.1. *Let $X = \{1,2,3,4,5,6\}$ and f an F-permutation on X defined by follows:*

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \frac{2}{0.2}, \frac{3}{0.3} & \frac{3}{0.3} & \frac{4}{0.4} & \frac{5}{0.5} & \frac{3}{0.3} & \frac{1}{0.2}, \frac{6}{0.6} \end{pmatrix}$$

Hence $WDG(f)$ has a loop in the vertex 6, see figure 1,

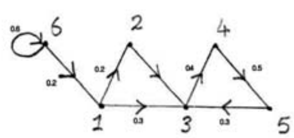

Figure 1. weighted directed graph of f.

and the converse of f is:

$$f_I := \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \frac{6}{0.2} & \frac{1}{0.2} & \frac{1}{0.3}, \frac{2}{0.3}, \frac{5}{0.3} & \frac{3}{0.4} & \frac{4}{0.5} & \frac{6}{0.6} \end{pmatrix}$$

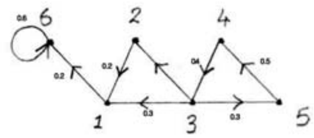

FIGURE 2. weighted directed graph of f_I.

In general the weighted directed graph of a converse fuzzy permutation f is the same weighted directed graph of f which directed are opposite.

Remark 5.1. We recall that the number of the input or output arcs to a vertex α is said to be input or output degree in α and denoted by $id(\alpha)$ or $od(\alpha)$, respectively. In this article, for each loop in a vertex, we consider one degree for input and one degree for output. Therefore in Example 5.1, $id(6) = 1$ and $od(6) = 2$.

Lemma 5.1. *Let $f \in FP(X)$, then for all positive integer number n,*
$f^n := \underbrace{f \circ f \circ \cdots \circ f}_{n-times}$ *is an F-permutation on X.*

Proof. The proof follows from Theorem 4.1(i). □

Theorem 5.1. *Let f be an F-permutation on the non empty and finite set X with the weighted directed graph $WDG(f)$. Then there exists an element $x \in X$, and there exists a positive integer number n, such that $x \in supp(f^n(x))$, in other words, $f^n(x)(x) > 0$.*

Proof. If $WDG(f)$ have a loop in vertex x, then $x \in supp(f(x))$, and the problem solved. Let $WDG(f)$ have no loop and $y \in X$, for every positive integer numbers k, we consider the sets $supp(f^k(y))$. Since $f^k \in FP(X)$, the cardinal number of the set $supp(f^k(y))$ is positive. Since $\bigcup\limits_{k\in N} supp(f^k(y))$ is a subset of the finite set X, hence there are natural numbers m and l, such that $m > l$ and $supp(f^l(y)) \cap supp(f^m(y))$ is non empty set. Let m and l be the smallest number with the above property. Since f have no loop, there is $x \neq y$, where $x \in supp(f^l(y)) \cap supp(f^m(y))$, therefore we have a sequence of elements X, $y = y_0,\ y_1,\ y_2, \ldots,\ y_l = x,\ y_{l+1},\ y_{l+2}, \ldots,\ y_m = x$, such that $y_i \in supp(f(y_{i+1})),\ i = 1, 2, \ldots, m$, therefore we have $x = y_m \in supp(f(y_{m-1})) \cap supp(f^2(y_{m-2})) \cap \cdots \cap supp(f^{m-l}(y_l))$, hence by choose $n = m - l > 0$ we have $x \in supp(f^{m-l}(x))$ or $f^n(x)(x) > 0$ and the proof is complete. □

Remark 5.2. We can not extend the above theorem to all members of X. For instance in Example 5.1, for all positive integer n, $f^n(1)(1) = 0$.

Remark 5.3. We recall that the directed graph $WDG(f)$ is said to be strongly-connected if there is a directed path from each vertex in the directed graph to every other vertex.

Theorem 5.2. *Let f be an F-permutation on the set X, then for every $x \in X$, there is a positive integer (number) n, such that $x \in \operatorname{supp}(f^n(x))$ if and only if each components of the directed graph $WDG(f)$ is strongly-connected.*

Proof. By Remark 5.3, the proof is straightforward. □

Definition 5.2. Let X and Y be two non empty sets and $f : X \longrightarrow F_*(X)$ and $g : Y \longrightarrow F_*(Y)$ be two F-permutations. We say f is equivalent to g (denoted by $f \sim g$) if there exists a bijection function $h : X \longrightarrow Y$, such that $K \circ f = g \circ h$, where $K : F_*(X) \longrightarrow F_*(Y)$ defined by $K(\mu)(y) = \mu(h^{-1}(y))$, for all $y \in Y$ and $\mu \in F_*(X)$.

$$\begin{array}{ccc} X & \longrightarrow^{f} & F_*(X) \\ h\downarrow & \circlearrowright & \downarrow K \\ Y & \longrightarrow_{g} & F_*(Y) \end{array}$$

Definition 5.3. Let X be a non empty set, $B = \{Y|\ card(Y) = card(X)\}$ and $A = \{f|f \in FP(Y),\ Y \in B\}$, where by $card(Y)$ we mean the cardinal number of the set Y.

Theorem 5.3. *The relation $\sim$ defined in the Definition 5.2 is an equivalence relation on A.*

Proof. The proof is straightforward. □

Theorem 5.4. *If f and g be two equivalent F-permutations then their graphs are isomorphic.*

Proof. Let f and g be equivalent F-permutations on the non empty sets X and Y respectively. Therefore, there is one to one correspondence functions $h : X \longrightarrow Y$ such that $K \circ f = g \circ h$, where $K : F_*(X) \longrightarrow F_*(Y)$ defined by $K(\mu)(y) = \mu(h^{-1}(y))$, for all $y \in Y$ and $\mu \in F_*(X)$. Hence h is a one to one correspondence between vertices $WDG(f)$ and $WDG(g)$. Now consider $\hat{ab}$ be an arc in $WDG(f)$ with weight $w(a,b)$ then $f(a)(b) = w(a,b)$ is positive. we have:

$$\begin{aligned} g(h(a))(h(b)) = ((g \circ h)(a))(h(b)) &= ((K \circ f)(a))(h(b)) \\ &= (K(f(a)))(h(b)) \\ &= f(a)(h^{-1}(h(a))) \\ &= f(a)(b) = w(a,b) > 0 \end{aligned}$$

Therefore $h(a)\hat{}h(b)$ is a directed arc in $WDG(f)$ with weight $w(a,b)$ and the proof is completed. □

6. F-permutation Matrices.

Let n be a positive integer number and $X = \{1, 2, \ldots, n\}$. By definition of F-permutation, we can define an F-permutation matrix as follows:

Definition 6.1. The fuzzy set $\mu : \{0, 1\} \longrightarrow [0, 1]$, such that $supp(\mu) = \{1\}$, is called a fuzzy unit and is denoted by $\widetilde{1}$.

Definition 6.2. Let FM be a square matrix with entries in $F_*(\{0,\ 1\})$, which entries of each row and column of FM have at least one or more elements one $\widetilde{1}$ and all other entries are zero with degree 1. Then FM is called an F-permutation matrix.

Theorem 6.1. *Let $X = \{1, 2, \ldots, n\}$ be a non empty set and $\mu : X \longrightarrow F_*(X)$ be an F-permutation on X. The matrix FM_μ is defined as:*

$$FM_\mu := \begin{pmatrix} e_{\mu(1)} \\ e_{\mu(2)} \\ \vdots \\ e_{\mu(n)} \end{pmatrix}_{n\times n},$$

with for non empty fuzzy subset $\mu(i)$ of X, $e_{\mu(i)} = (\gamma_{i1} \quad \gamma_{i2} \quad \cdots \quad \gamma_{in})_{1\times n}$, $i = 1, 2, \ldots, n$; where:
Case 1: If $supp(\mu(i)) \neq X$, then for

$$\alpha_j = \begin{cases} 1 & \text{if } j \in supp(\mu(i)) \\ 0 & \text{if } j \notin supp(\mu(i)) \end{cases}, \ \gamma_{ij}(\alpha_j) = \begin{cases} \mu(i)(1), & j = i \\ 0 & j \neq i \end{cases}, \ j = 1, 2, \ldots, n.$$

Case 2: If $supp(\mu(i)) = X$, $\gamma_{ij}(1) = \mu(i)(j)$, $\gamma_{ij}(0) = 0$, $j = 1, 2, \ldots, n$. Then FM_μ is an F-permutation matrix.

Proof. Since μ is an F-permutation, hence for each $i \in X$, $\mu(i)$ is a non empty fuzzy subset of X, therefore $supp\ (\mu(i))$ is non empty. Let $k_i \in supp(\mu(i))$, we show that $\gamma_{ik_i} = \widetilde{1}$. We have $\alpha_{k_i} = 1$ and hence $\gamma_{ik_i}(1) = \mu(i)(k_i) > 0$ is positive. Moreover $\gamma_{ik_i}(0) = 0$. So in each row, there is at least one $\widetilde{1}$. On the other hand, since $\bigcup_{i=1}^{n} supp(\mu(i)) = X$, hence for each column j, there is an $i \in X$, such that $j \in supp(\mu(i))$ for $i = 1, 2, \ldots, n$, and hence $\gamma_{ij} = \widetilde{1}$. So there is at least one $\widetilde{1}$ in each column. Then FM_μ is an F-permutation matrix. $\square$

Example 6.1. *Let g be an F-permutation on $\{1, 2, 3\}$ defined by follows:*

$$f = \begin{pmatrix} 1 & 2 & 3 \\ \\ \frac{2}{0.2} & \frac{3}{0.3} & \frac{1}{0.1}, \frac{2}{0.2}, \frac{3}{0.3} \end{pmatrix}$$

The F-permutation matrix associated to g is :

$$FM_g := \begin{pmatrix} \frac{0}{1}, \frac{1}{0} & \frac{0}{0}, \frac{1}{0.2} & \frac{0}{1}, \frac{1}{0} \\ \\ \frac{0}{1}, \frac{1}{0} & \frac{0}{1}, \frac{1}{0} & \frac{0}{0}, \frac{1}{0.3} \\ \\ \frac{0}{0}, \frac{1}{0.2} & \frac{0}{0}, \frac{1}{0.2} & \frac{0}{1}, \frac{1}{0.3} \end{pmatrix}_{3\times 3}$$

For simplicity the matrix FM_g is denoted by

$$FM_g := \begin{pmatrix} \frac{0}{1} & \frac{1}{0.2} & \frac{0}{1} \\ \\ \frac{0}{1} & \frac{0}{1} & \frac{1}{0.3} \\ \\ \frac{1}{0.1} & \frac{1}{0.2} & \frac{1}{0.3} \end{pmatrix}_{3\times 3}$$

Example 6.2. *By the simplicity of Example 6.1, if f is an F-permutation defined in Example 5.1. The F-permutation matrix associated to f is :*

$$FM_f := \begin{pmatrix} \frac{0}{1} & \frac{1}{0.2} & \frac{1}{0.3} & \frac{0}{1} & \frac{0}{1} & \frac{0}{1} \\ \\ \frac{0}{1} & \frac{0}{1} & \frac{1}{0.3} & \frac{0}{1} & \frac{0}{1} & \frac{0}{1} \\ \\ \frac{0}{1} & \frac{0}{1} & \frac{0}{1} & \frac{1}{0.5} & \frac{0}{1} & \frac{0}{1} \\ \\ \frac{0}{1} & \frac{0}{1} & \frac{0}{1} & \frac{0}{1} & \frac{1}{0.5} & \frac{0}{1} \\ \\ \frac{0}{1} & \frac{0}{1} & \frac{1}{0.3} & \frac{0}{1} & \frac{0}{1} & \frac{0}{1} \\ \\ \frac{1}{0.2} & \frac{0}{1} & \frac{0}{1} & \frac{0}{1} & \frac{0}{1} & \frac{1}{0.6} \end{pmatrix}_{6\times 6}$$

Theorem 6.2. *There is a one-to-one corresponding between all F-permutations on $X = \{1, 2, \ldots, n\}$ and all F-permutation matrices on X.*

Proof. If μ be an F-permutation, in Theorem 6.1, we obtained an F-permutation matrix correspond FM_μ. Conversely, let FM be an F-permutation matrix, we defined map μ_{FM} as follows:

$$\begin{array}{llllll} \mu_{FM}: & X & \to & F_*(X) & & \\ & i & \mapsto & \mu_{FM}(i): & X & \to & [0\ ,\ 1] \\ & & & & j & \mapsto & \begin{cases} \gamma_{ij}(1), & \gamma_{ij} = \widetilde{1} \\ 0 & otherwise \end{cases} \end{array}$$

Now, since in each row at least one $\widetilde{1}$ appear, hence $supp(\mu_{FM}(i)) \neq \emptyset$, for all $i \in X$. First we show that $\bigcup_{i=1}^{n} supp(\mu_{FM}(i)) = X$. To do this, let $j \in X$, hence there exists $k \in X$, such that $\gamma_{kj} = \widetilde{1}$ which implies that $(\mu_{FM})(k)(j) > 0$.
It is easy to show that $FM = FM_{\mu_{FM}}$ and $\mu = \mu_{FM_\mu}$ and proof completes. $\square$

Theorem 6.3. *Let μ be an F-permutation, then $FM_{\mu_I} = FM_\mu^T$, where FM_μ^T is the transpose of FM_μ.*

Proof. The proof is straightforward. □

7. Conclusion

In this paper we have given the following results:
1. By using F-permutation, all minimal F-permutations are characterized (Theorem 3.2).
2. An useful presentation for any F-polygroup is introduced (Theorem 4.5).
3. Some F-permutations induce directed graphs which each of its components is strongly-connected (Theorem 5.3).
4. There is a one-to-one corresponding between all F-permutations on n-letters and all F-permutation matrices (Theorem 6.2).

8. Further research

By using level subsets, can we find some relations between hyper permutations [12] and F-permutations?

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