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On the PUL-Stieltjes Integral on Smooth Manifolds

Greig Bates C. Flores

Department of Mathematics, Central Mindanao University, Maramag, Bukidnon, Philippines

E-mail: greigbates.flores@cmu.edu.ph

Abstract. The notion of the PU integral was first formulated by Kurzweil and Jarnik. It is a Henstock type that utilizes the concept of a partition of unity in it covering system. They mentioned, without much details, that this integration process may be used in the formulation of the Henstock integral in manifolds. In this paper, the above query will be revisit and the Henstock integral of a function defined on a manifold will be presented including some of its fundamental properties.

Keywords: Partition of unity, Henstock integral on manifolds, Saks-Henstock lemma, Change of variable formula.

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1. Introduction

The PU integral is an integration process that is adherent to the definition of the Henstock integral utilizing the partition of unity. In some sense, it is equivalent with the Henstock integral. Henstock integral is crafted with a covering system via "δ-fine", where δ is a positive function. A finite collection of point-interval pair $\{(t_i, I_i)\}_{i=1}^m$, where I_i is a compact interval, is of Perron type if $t_i \in I_i$ for all $i \leq m$. Moreover, given a gauge δ on $[a, b]$, a finite collection of point interval pairs, of Perron type, $\{(t_i, I_i)\}_{i=1}^m$ is said to be δ-fine Perron partition of $[a, b]$ if I_i is a partition of $[a, b]$ and $I_i \subseteq B(t_i, \delta(t_i))$. If for each $i \leq m$, the condition $t_i \in I_i$ is removed, then we obtain a McShane

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type and the McShane integral is of this type. On one hand, a partition of unity [1, 6] on a compact interval $\boldsymbol{E} \subseteq \mathbb{R}^n$ is a finite collection $\{\varphi_i\}_{i=1}^m$ smooth functions with the following property

(1) $\varphi_i \geq 0$ on $\boldsymbol{E}$; and
(2) $\displaystyle\sum_{k=1}^m \varphi_k = 1$ a.e. on $\boldsymbol{E}$.

If $\displaystyle\sum_{k=1}^m \varphi_k \leq 1$ a.e. on $\boldsymbol{E}$, then $\{\varphi_i\}_{i=1}^m$ is said to be a partial partition of unity. A finite collection of triples $\{(\boldsymbol{\xi}_i, \boldsymbol{I}_i, \varphi)\}_{i=1}^m$ is said to be δ-fine PUL-division of $\boldsymbol{E}$ if for each $i \leq m$, supp $\varphi_i \subseteq \boldsymbol{I}_i$ and $\boldsymbol{I}_i \subseteq B(\boldsymbol{\xi}_i, \delta(\boldsymbol{\xi}_i))$. Boonpogkrong, revisited the notion of the PUL integral and it's application in the integrals of a function defined on a manifold. In addition, Flores and Benitez [4, 5] extended the notion in its Stieltjes form and provided some convergence theorems. In this work, the definition of the PUL-Setieltjes integral will be discussed in a manifold setting including some of its fundamental properties.

2. Preliminaries

In this section, we introduce the PUL-Stieltjes integral of a function defined on a Manifold. Throughout the rest of this chapter, if no confusion arises, we denote M as a manifold.

Definition 2.1. Let $f : [\boldsymbol{a}, \boldsymbol{b}] \to X$ and $g : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$. Suppose that for each $k \in \{1, 2, \cdots, m\}$, the Riemann-Stieltjes integral $(\mathcal{R}) \displaystyle\int_{\boldsymbol{I}_k} \varphi_k dg$ exists. We define the *PUL-Stieljes sum* by

$$S(f, g, D) = \sum_{k=1}^m f(\boldsymbol{\xi}_k)\ (\mathcal{R}) \int_{I_k} \varphi_k dg$$

where D is a δ-fine division of $[\boldsymbol{a}, \boldsymbol{b}]$. For brevity, we write a δ-fine division of $[\boldsymbol{a}, \boldsymbol{b}]$ by $D = \{(\boldsymbol{\xi}, \boldsymbol{I}, \psi)\}$ and a PUL-Stieltjes sum of f with respect to g over D by

$$S(f, g, D) = (D) \sum f(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \psi\ dg = \sum_D f(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \psi\ dg.$$

Remark 2.2. For a δ-fine division $D = \{\boldsymbol{\xi}, \boldsymbol{I}, \varphi\}$, $\boldsymbol{I}$'s may be overlapping.

Definition 2.3. [4] Let X be a Banach space, $f : [\boldsymbol{a}, \boldsymbol{b}] \to X$ be a Banach-valued funtion, and $g : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$ be a function. We say that f is *PUL-Stieltjes integrable* to a vector A with respect to g on $[\boldsymbol{a}, \boldsymbol{b}]$ if for every $\epsilon > 0$, there exists a gauge $\delta > 0$ such that for every δ-fine division D of $[\boldsymbol{a}, \boldsymbol{b}]$, we have

$$\|S(f, g, D) - A\| < \epsilon.$$

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If A is the PUL-Stieltjes integral of f with respect to g, then we write

$$A = (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f \ dg.$$

Definition 2.4. [9] A topological space is *second countable* if it has a countable basis.

Definition 2.5. [9] A topological space M is *locally Euclidean of dimension n* if every point $p \in M$ has a neighborhood U_p such that there is a homeomorphism ϕ from U_p onto an open subset O_p of $\mathbb{R}^n$. We call the pair $(U, \phi : U \to \mathbb{R}^n)$ a *chart*, U a *coordinate neighborhood* or a *coordinate open set*, and ϕ a *coordinate map* or *coordinate system* on U. We say that a chart (U, ϕ) is centered at $p \in U$ if $\phi(p) = 0$. A chart (U, ϕ) about p simply means that (U, ϕ) is a chart and $p \in U$.

Definition 2.6. [9] A *topological manifold of dimension n* is a Hausdorff, second countable, locally Euclidean space of dimension n.

Exercise 2.7. [9] The Euclidean space $\mathbb{R}^n$ is covered by a single chart $(\mathbb{R}^n, 1_{\mathbb{R}^n})$, where $1_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$ is the identity map. It is a prime example of a topological manifold. Every open subset $\boldsymbol{U}$ of $\mathbb{R}^n$ is also a topological manifold, with chart $(\boldsymbol{U}, 1_{\mathbb{R}^n})$.

Lemma 2.8. *Let $\boldsymbol{U}$ be an open subset of a compact interval $\boldsymbol{E}^*$ in $\mathbb{R}^r$ and $\psi : \boldsymbol{U} \to \psi(\boldsymbol{U})$ be C^1-diffeomorphism. Suppose that $\boldsymbol{E}$ is a compact interval such that $\boldsymbol{E} \subseteq \psi(\boldsymbol{U}) \subseteq \mathbb{R}^n$ and $\varphi : \boldsymbol{E} \to \mathbb{R}$ is continuous and $g : \boldsymbol{E} \to \mathbb{R}$ be a function of bounded variation. Then*

$$(\mathcal{R}) \int_{\boldsymbol{E}} \varphi \ dg = (\mathcal{R}) \int_{\psi^{-1}(\boldsymbol{E})} (\varphi \circ \psi) d(g \circ \psi),$$

where $(\mathcal{R}) \int_{\boldsymbol{E}} \varphi \ dg$ is the Riemann-Stieltjes integral of φ with respect to g on $\boldsymbol{E}$.

Theorem 2.9. (Change of Variable Formula). *Let $f : [\boldsymbol{a}, \boldsymbol{b}] \to X$ be PUL-Stieltjes integrable with respect to $g : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$ on $[\boldsymbol{a}, \boldsymbol{b}]$. Let $\boldsymbol{U}$ be an open subset of a compact interval $\boldsymbol{E}$ in $\mathbb{R}^m$. Let $\Psi : \boldsymbol{U} \to \Psi(\boldsymbol{U})$ be C^1-diffeomorphism and $[\boldsymbol{a}, \boldsymbol{b}] \subseteq \Psi(\boldsymbol{U}) \subseteq \mathbb{R}^n$. Then $(f \circ \Psi) \cdot \chi_{\Psi^{-1}([\boldsymbol{a},\boldsymbol{b}])}$ is PUL-Stieltjes integrable with respect to $g \circ \Psi$ on $\boldsymbol{E}$ and*

$$(\mathcal{P}) \int_{\boldsymbol{E}} (f \circ \Psi) \cdot \chi_{\Psi^{-1}([\boldsymbol{a},\boldsymbol{b}])} d(g \circ \Psi) = (\mathcal{P}) \int_{\Psi^{-1}([\boldsymbol{a},\boldsymbol{b}])} (f \circ \Psi) d(g \circ \Psi) = (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f \ dg.$$

3. Results and discussions

Lemma 3.1. *Let $D_p = \{\boldsymbol{\xi}, \boldsymbol{I}, \varphi\}$ be a partial δ-fine division of $[\boldsymbol{a}, \boldsymbol{b}]$, $\sigma(\boldsymbol{x}) = \sum_{D_1} \varphi(\boldsymbol{x})$, and $D_f = \{\boldsymbol{\eta}, \boldsymbol{J}, \psi\}$ be δ-fine division of $[\boldsymbol{a}, \boldsymbol{b}]$. Then*

$$D_p \cup \{\boldsymbol{\eta}, \boldsymbol{J}, (1 - \sigma)\psi\}$$

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is a δ-fine division of $[\boldsymbol{a}, \boldsymbol{b}]$.

Proof. Let $\boldsymbol{x} \in [\boldsymbol{a}, \boldsymbol{b}]$. Since D_f is a δ-fine division of $[\boldsymbol{a}, \boldsymbol{b}]$, $\{\psi\}$ is a partition of unity. Thus,

$$\sum_{D_f} \psi(\boldsymbol{x}) = 1.$$

Observe that

$$\begin{aligned}\sum_{D_p} \varphi(\boldsymbol{x}) + \sum_{D_f}(1-\sigma(\boldsymbol{x})) \cdot \psi(\boldsymbol{x}) &= \sigma(\boldsymbol{x}) + \sum_{D_f} \psi(\boldsymbol{x}) - \sigma(\boldsymbol{x}) \sum_{D_f} \psi(\boldsymbol{x}) \\ &= \sigma(\boldsymbol{x}) + 1 - \sigma(\boldsymbol{x}) \cdot 1 = 1.\end{aligned}$$

This means that $\{\varphi\} \cup \{(1-\sigma)\psi\}$ is a partition of unity. Since D_p is a partial δ-fine division of $[\boldsymbol{a}, \boldsymbol{b}]$, it follows that

$$\text{supp } \varphi \subseteq \boldsymbol{I} \subseteq B(\boldsymbol{\xi}, \delta(\boldsymbol{\xi})). \tag{3.1}$$

Now, let $\boldsymbol{x} \in \text{supp } ((1-\sigma) \cdot \psi)$. Then $(1-\sigma(\boldsymbol{x})) \cdot \psi(\boldsymbol{x}) > 0$ and so $\psi(\boldsymbol{x}) > 0$. Hence, $\boldsymbol{x} \in \text{supp } \psi$. Thus, supp $((1-\varphi) \cdot \psi) \subseteq \text{supp } \psi$. So we have

$$\text{supp } ((1-\sigma) \cdot \psi) \subseteq \text{supp } \psi \subseteq \boldsymbol{J} \subseteq B(\boldsymbol{\eta}, \delta(\boldsymbol{\eta})). \tag{3.2}$$

By the inclusions in 3.1 and 3.2, evidently

$$D \cup \{\boldsymbol{\eta}, \boldsymbol{J}, (1-\sigma)\psi\}$$

is a δ-fine division of $[\boldsymbol{a}, \boldsymbol{b}]$. □

Lemma 3.2. *Let $f : [\boldsymbol{a}, \boldsymbol{b}] \to X$ be a PUL-Stieltjes integrable function with respect to $g : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$ and $\boldsymbol{G}$ be an open and bounded set such that supp $f \subseteq \boldsymbol{G} \subseteq \mathbb{R}^n$. Then PUL-Stiletjes integral*

$$(\mathcal{P}) \int_{[\boldsymbol{a}, \boldsymbol{b}]} f \cdot \phi \; dg$$

exists for any continuously differentiable function $\phi : \boldsymbol{G} \to \mathbb{R}$ in which

$$0 < \phi(\boldsymbol{x}) \leq 1, \text{ for all } \boldsymbol{x} \in \boldsymbol{G}.$$

Proof. Let $\epsilon > 0$. Then there exists a gauge δ_1 on $[\boldsymbol{a}, \boldsymbol{b}]$ such that for any δ_1-fine division $D = \{(\boldsymbol{\xi}, \varphi, \boldsymbol{I})\}$ of $[\boldsymbol{a}, \boldsymbol{b}]$, we have

$$\left\| \sum_D f(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \varphi dg - (\mathcal{P}) \int_{[\boldsymbol{a}, \boldsymbol{b}]} f dg \right\| < \epsilon.$$

Assume that $B(\boldsymbol{x}, \delta_1(\boldsymbol{x})) \subseteq G$, for each $\boldsymbol{x} \in \text{supp } f$. Since g is of bounded variation on $[\boldsymbol{a}, \boldsymbol{b}]$, $M = V(g; [\boldsymbol{a}, \boldsymbol{b}]) \in \mathbb{R}$. By continuity of ϕ at every $\boldsymbol{x} \in \text{supp } f \subseteq \boldsymbol{G}$, we may choose $\delta_2(\boldsymbol{x}) > 0$ such that for any $\boldsymbol{y} \in [\boldsymbol{a}, \boldsymbol{b}]$ with $\boldsymbol{y} \in B(\boldsymbol{x}, \delta_2(\boldsymbol{x}))$, we have

$$\|f(\boldsymbol{x})\| \cdot |\phi(\boldsymbol{y}) - \phi(\boldsymbol{x})| < \frac{\epsilon}{3(M+1)}. \tag{3.3}$$

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Take $\delta(\boldsymbol{x}) = \min\{\delta_1(\boldsymbol{x}), \delta_2(\boldsymbol{x})\}$, for all $\boldsymbol{x} \in$ supp f. Note that for each $\boldsymbol{x} \in$ supp f, if $\boldsymbol{y} \in B(\boldsymbol{x}, \delta(\boldsymbol{x}))$, the inequality 3.3 still holds. Thus, for each $\boldsymbol{x} \in$ supp f, we have

$$\|f(\boldsymbol{x})\| \cdot \sup\{|\phi(\boldsymbol{y}) - \phi(\boldsymbol{x})| : \boldsymbol{y} \in B(\boldsymbol{x}, \delta(\boldsymbol{x}))\} < \frac{\epsilon}{3(M+1)}. \tag{3.4}$$

Now, let $D_1 = \{(\boldsymbol{\xi}, \varphi, \boldsymbol{I})\}$ and $D_2 = \{(\boldsymbol{\eta}, \psi, \boldsymbol{J})\}$ be any δ-fine division of $[\boldsymbol{a}, \boldsymbol{b}]$. For every $\boldsymbol{x} \in$ supp f, since $0 < \phi(\boldsymbol{x}) \leq 1$, we have

$$\sum_{D_1} \phi(\boldsymbol{x}) \cdot \varphi(\boldsymbol{x}) = \phi(\boldsymbol{x}) \cdot \sum_{D_1} \varphi(\boldsymbol{x}) = \phi(\boldsymbol{x}) \leq 1.$$

This means that $\{\phi \cdot \varphi\}$ is a partial partition of unity on supp f. Since $\phi(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \in G$, it is evident that

$$\text{supp } (\phi \cdot \varphi) = \text{supp } (\varphi).$$

Since D_1 is a δ-fine division of $[\boldsymbol{a}, \boldsymbol{b}]$, we have

$$\text{supp } (\phi \cdot \varphi) = \text{supp } \varphi \subseteq \boldsymbol{I} \subseteq B(\boldsymbol{\xi}, \delta(\boldsymbol{\xi})).$$

Thus, $\{(\boldsymbol{\xi}, \phi \cdot \varphi, \boldsymbol{I})\}$ is a partial δ-fine division of $[\boldsymbol{a}, \boldsymbol{b}]$.
Since $0 < \phi(\boldsymbol{x}) \leq 1$ for all supp f,

$$\sum_{D_2} (1 - \phi(\boldsymbol{x})) \cdot \psi(\boldsymbol{x}) = (1 - \phi(\boldsymbol{x})) \cdot \sum_{D_2} \psi(\boldsymbol{x}) = 1 - \phi(\boldsymbol{x}) \leq 1.$$

So, $\{(1 - \phi) \cdot \psi)\}$ is a partial partition of unity on supp f.
Also,

$$\text{supp } (1 - \phi) \cdot \psi = \text{supp } \psi \subseteq \boldsymbol{J} \subseteq B(\boldsymbol{\eta}, \delta(\boldsymbol{\eta})).$$

Hence, $\{(\boldsymbol{\eta}, (1-\phi) \cdot \psi, \boldsymbol{J})\}$ is a partial δ-fine division of $[\boldsymbol{a}, \boldsymbol{b}]$ which, by Lemma 3.1, will further imply that $\{(\boldsymbol{\xi}, \phi \cdot \varphi, \boldsymbol{I})\} \cup \{(\boldsymbol{\eta}, (1-\phi) \cdot \psi, \boldsymbol{J})\}$ is δ-fine division of $[\boldsymbol{a}, \boldsymbol{b}]$. By PUL-Stieltjes integrability of f with respect to g on $[\boldsymbol{a}, \boldsymbol{b}]$, we have

$$\left\| \left(\sum_{D_1} f(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \phi \cdot \varphi \; dg + \sum_{D_2} f(\boldsymbol{\eta}) \int_{\boldsymbol{J}} (1-\phi) \cdot \psi \; dg \right) - (\mathcal{P}) \int_{[\boldsymbol{a}, \boldsymbol{b}]} f \; dg \right\| < \frac{\epsilon}{6}. \tag{3.5}$$

Since f is PUL-stieltjes integrable with respect to g on $[\boldsymbol{a}, \boldsymbol{b}]$ and D_2 is a δ-fine division of $[\boldsymbol{a}, \boldsymbol{b}]$,

$$\left\| \sum_{D_2} f(\boldsymbol{\eta}) \int_{\boldsymbol{J}} \psi \; dg - (\mathcal{P}) \int_{[\boldsymbol{a}, \boldsymbol{b}]} f \; dg \right\| < \frac{\epsilon}{6}. \tag{3.6}$$

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By (3.5) and (3.6), we have

$$\begin{aligned}
&\left\|\sum_{D_1} f(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \phi\cdot\varphi\ dg - \sum_{D_2} f(\boldsymbol{\eta}) \int_{\boldsymbol{J}} \phi\cdot\psi\ dg\right\| \\
&\qquad = \left\|\left(\sum_{D_1} f(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \phi\cdot\varphi\ dg + \sum_{D_2} f(\boldsymbol{\eta}) \int_{\boldsymbol{J}} (1-\phi)\cdot\psi\ dg\right) - (\mathcal{P})\int_{[\boldsymbol{a},\boldsymbol{b}]} f\ dg\right. \\
&\qquad\qquad \left. + (\mathcal{P})\int_{[\boldsymbol{a},\boldsymbol{b}]} f dg - \sum_{D_2} f(\boldsymbol{\eta}) \int_{\boldsymbol{J}} \psi\ dg\right\| \\
&\qquad \le \left\|\left(\sum_{D_1} f(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \phi\cdot\varphi\ dg + \sum_{D_2} f(\boldsymbol{\eta}) \int_{\boldsymbol{J}} (1-\phi)\cdot\psi\ dg\right) - (\mathcal{P})\int_{[\boldsymbol{a},\boldsymbol{b}]} f\ dg\right\| \\
&\qquad\qquad + \left\|(\mathcal{P})\int_{[\boldsymbol{a},\boldsymbol{b}]} f\ dg - \sum_{D_2} f(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \psi\ dg\right\| < \frac{\epsilon}{6} + \frac{\epsilon}{6} = \frac{\epsilon}{3},
\end{aligned}$$

that is,

$$\left\|\sum_{D_1} f(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \phi\cdot\varphi\ dg - \sum_{D_2} f(\boldsymbol{\eta}) \int_{\boldsymbol{J}} \phi\cdot\psi\ dg\right\| < \frac{\epsilon}{3}. \tag{3.7}$$

Hence, by the linearity of the Riemann-stieltjes integral and by (3.4)

$$\begin{aligned}
&\left\|\sum_{D_1} f(\boldsymbol{\xi})\phi(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \varphi\ dg - \sum_{D_1} f(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \phi\cdot\varphi\ dg\right\| \\
&\qquad = \left\|\sum_{D_1} f(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \phi(\boldsymbol{\xi})\cdot\varphi\ dg - \sum_{D_1} f(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \phi\cdot\varphi\ dg\right\| \\
&\qquad = \left\|\sum_{D_1} f(\boldsymbol{\xi}) \left[\int_{\boldsymbol{I}} \phi(\boldsymbol{\xi})\cdot\varphi\ dg - \int_{\boldsymbol{I}} \phi\cdot\varphi\ dg\right]\right\| \\
&\qquad = \left\|\sum_{D_1} f(\boldsymbol{\xi}) \left[\int_{\boldsymbol{I}} \big(\phi(\boldsymbol{\xi})\cdot\varphi - \phi\cdot\varphi\big)\ dg\right]\right\| \\
&\qquad = \left\|\sum_{D_1} f(\boldsymbol{\xi}) \left[\int_{\boldsymbol{I}} \big(\phi(\boldsymbol{\xi}) - \phi\big)\cdot\varphi\ dg\right]\right\| \\
&\qquad \le \left\|\sum_{D_1} f(\boldsymbol{\xi}) \cdot \int_{\boldsymbol{I}} \sup\Big\{|\phi(\boldsymbol{\xi}) - \phi(\boldsymbol{x})| : \boldsymbol{x}\in B(\boldsymbol{\xi},\delta(\boldsymbol{\xi}))\Big\}\cdot\varphi\ dg\right\| \\
&\qquad \le \left\|\sum_{D_1} f(\boldsymbol{\xi}) \cdot \sup\Big\{|\phi(\boldsymbol{\xi}) - \phi(\boldsymbol{x})| : \boldsymbol{x}\in B(\boldsymbol{\xi},\delta(\boldsymbol{\xi}))\Big\}\cdot\int_{\boldsymbol{I}} \varphi\ dg\right\| \\
&\qquad \le \sum_{D_1} \|f(\boldsymbol{\xi})\| \cdot \sup\Big\{|\phi(\boldsymbol{\xi}) - \phi(\boldsymbol{x})| : \boldsymbol{x}\in B(\boldsymbol{\xi},\delta(\boldsymbol{\xi}))\Big\}\cdot\int_{\boldsymbol{I}} \varphi\ dg \\
&\qquad \le \frac{\epsilon}{3(M+1)} \cdot \sum_{D_1} \int_{\boldsymbol{I}} \varphi\ dg \le \frac{\epsilon}{3(M+1)}\cdot(M+1) = \frac{\epsilon}{3},
\end{aligned}$$

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that is,

$$\left\| \sum_{D_1} f(\boldsymbol{\xi})\phi(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \varphi \; dg - \sum_{D_1} f(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \phi \cdot \varphi \; dg \right\| < \frac{\epsilon}{3}. \tag{3.8}$$

Similarly,

$$\begin{aligned}
&\left\| \sum_{D_2} f(\boldsymbol{\eta})\phi(\boldsymbol{\eta}) \int_{\boldsymbol{J}} \psi \; dg - \sum_{D_2} f(\boldsymbol{\eta}) \int_{\boldsymbol{J}} \phi \cdot \psi \; dg \right\| \\
&\quad = \left\| \sum_{D_2} f(\boldsymbol{\eta}) \int_{\boldsymbol{J}} \phi(\boldsymbol{\eta}) \cdot \psi \; dg - \sum_{D_2} f(\boldsymbol{\eta}) \int_{\boldsymbol{J}} \phi \cdot \psi \; dg \right\| \\
&\quad = \left\| \sum_{D_2} f(\boldsymbol{\eta}) \left[\int_{\boldsymbol{J}} \phi(\boldsymbol{\eta}) \cdot \psi \; dg - \int_{\boldsymbol{J}} \phi \cdot \psi \; dg \right] \right\| \\
&\quad = \left\| \sum_{D_2} f(\boldsymbol{\eta}) \left[\int_{\boldsymbol{J}} \big(\phi(\boldsymbol{\eta}) \cdot \psi - \phi \cdot \psi \big) \; dg \right] \right\| \\
&\quad = \left\| \sum_{D_2} f(\boldsymbol{\eta}) \left[\int_{\boldsymbol{J}} \big(\phi(\boldsymbol{\eta}) - \phi \big) \cdot \psi \; dg \right] \right\| \\
&\quad \leq \left\| \sum_{D_2} f(\boldsymbol{\eta}) \cdot \int_{\boldsymbol{J}} \sup \Big\{ |\phi(\boldsymbol{\eta}) - \phi(\boldsymbol{x})| : \boldsymbol{x} \in B(\boldsymbol{\eta}, \delta(\boldsymbol{\eta})) \Big\} \cdot \psi \; dg \right\| \\
&\quad \leq \left\| \sum_{D_2} f(\boldsymbol{\eta}) \cdot \sup \Big\{ |\phi(\boldsymbol{\eta}) - \phi(\boldsymbol{x})| : \boldsymbol{x} \in B(\boldsymbol{\eta}, \delta(\boldsymbol{\eta})) \Big\} \cdot \int_{\boldsymbol{J}} \psi \; dg \right\| \\
&\quad \leq \sum_{D_2} \big\| f(\boldsymbol{\eta}) \big\| \cdot \sup \Big\{ |\phi(\boldsymbol{\eta}) - \phi(\boldsymbol{x})| : \boldsymbol{x} \in B(\boldsymbol{\eta}, \delta(\boldsymbol{\eta})) \Big\} \cdot \int_{\boldsymbol{J}} \psi \; dg \\
&\quad \leq \frac{\epsilon}{3\big(M+1\big)} \cdot \sum_{D_2} \int_{\boldsymbol{J}} \psi \; dg \leq \frac{\epsilon}{3\big(M+1\big)} \cdot (M+1) = \frac{\epsilon}{3},
\end{aligned}$$

that is,

$$\left\| \sum_{D_2} f(\boldsymbol{\eta})\phi(\boldsymbol{\eta}) \int_{\boldsymbol{J}} \psi \; dg - \sum_{D_2} f(\boldsymbol{\eta}) \int_{\boldsymbol{J}} \phi \cdot \psi \; dg \right\| < \frac{\epsilon}{3}. \tag{3.9}$$

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Therefore,by (3.7), (3.8), and (3.9)

$$\begin{aligned}
&\Big\|\sum_{D_1} f(\boldsymbol{\xi})\phi(\boldsymbol{\xi})\int_{\boldsymbol{I}}\varphi dg-\sum_{D_2} f(\boldsymbol{\eta})\phi(\boldsymbol{\eta})\int_{\boldsymbol{I}}\psi dg\Big\| \\
&\quad= \Big\|\sum_{D_1} f(\boldsymbol{\xi})\phi(\boldsymbol{\xi})\int_{\boldsymbol{I}}\varphi dg-\sum_{D_1} f(\boldsymbol{\xi})\int_{\boldsymbol{I}}\phi\cdot\varphi dg+\sum_{D_1} f(\boldsymbol{\xi})\int_{\boldsymbol{I}}\phi\cdot\varphi dg \\
&\qquad\quad -\sum_{D_2} f(\boldsymbol{\eta})\int_{\boldsymbol{J}}\phi\cdot\psi dg+\sum_{D_2} f(\boldsymbol{\eta})\int_{\boldsymbol{J}}\phi\cdot\psi dg-\sum_{D_2} f(\boldsymbol{\eta})\phi(\boldsymbol{\eta})\int_{\boldsymbol{J}}\psi dg\Big\| \\
&\quad\le \Big\|\sum_{D_1} f(\boldsymbol{\xi})\phi(\boldsymbol{\xi})\int_{\boldsymbol{I}}\varphi dg-\sum_{D_1} f(\boldsymbol{\xi})\int_{\boldsymbol{I}}\phi\cdot\varphi dg\Big\| \\
&\qquad\quad +\Big\|\sum_{D_1} f(\boldsymbol{\xi})\int_{\boldsymbol{I}}\phi\cdot\varphi dg-\sum_{D_2} f(\boldsymbol{\eta})\int_{\boldsymbol{J}}\phi\cdot\psi dg\Big\| \\
&\qquad\qquad +\Big\|\sum_{D_2} f(\boldsymbol{\eta})\int_{\boldsymbol{J}}\phi\cdot\psi dg-\sum_{D_2} f(\boldsymbol{\eta})\phi(\boldsymbol{\eta})\int_{\boldsymbol{J}}\psi dg\Big\| \\
&\quad< \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3}=\epsilon.
\end{aligned}$$

And the result follows. □

Theorem 3.3. (Saks-Henstock Lemma) *If $f:[\boldsymbol{a},\boldsymbol{b}]\to X$ is PUL-Stieltjes integrable with respect to $g:[\boldsymbol{a},\boldsymbol{b}]\to\mathbb{R}$ on $[\boldsymbol{a},\boldsymbol{b}]$, then for every $\epsilon>0$, there exists a gauge δ on $[\boldsymbol{a},\boldsymbol{b}]$ such that for any δ-fine partial division $P=\{(\boldsymbol{\xi},\varphi,\boldsymbol{I})\}$ of $[\boldsymbol{a},\boldsymbol{b}]$, we have*

$$\Big\|\sum_P\Big(f(\boldsymbol{\xi})\int_{\boldsymbol{I}}\varphi dg-(\mathcal{P})\int_{\boldsymbol{I}} f\varphi dg\Big)\Big\|<\epsilon.$$

Proof. Let $\epsilon>0$. Then there is a gauge δ on $[\boldsymbol{a},\boldsymbol{b}]$ such that whenever $D=\{(\boldsymbol{\xi},\sigma,\boldsymbol{I})\}$ is a δ-fine division of $[\boldsymbol{a},\boldsymbol{b}]$, we have

$$\Big\|\sum_D f(\boldsymbol{\xi})\int_{\boldsymbol{I}}\sigma dg-(\mathcal{P})\int_{[\boldsymbol{a},\boldsymbol{b}]} f dg\Big\|<\epsilon.$$

Let $P=\{(\boldsymbol{\xi},\varphi,\boldsymbol{I})\}$ be a δ-fine partial division of $[\boldsymbol{a},\boldsymbol{b}]$ and put $\phi=\sum_P\varphi$. By continuity of ϕ at every $\boldsymbol{x}\in\operatorname{supp} f$, there is a $\delta_1(\boldsymbol{x})\le\delta(\boldsymbol{x})$ such that for any $\boldsymbol{y}\in[\boldsymbol{a},\boldsymbol{b}]\cap B(\boldsymbol{x},\delta_1(\boldsymbol{x}))$, we have

$$\|f(\boldsymbol{x})\||\phi(\boldsymbol{y})-\phi(\boldsymbol{x})|<\frac{\epsilon}{3\Big(\int_{[\boldsymbol{a},\boldsymbol{b}]}dg+1\Big)}.$$

Thus, for each $\boldsymbol{x}\in\operatorname{supp} f$

$$\|f(\boldsymbol{x})\|\cdot\sup\Big\{|\phi(\boldsymbol{x})-\phi(\boldsymbol{y})|:\boldsymbol{y}\in B(\boldsymbol{x},\delta_1(\boldsymbol{x}))\Big\}<\frac{\epsilon}{3\Big(\int_{[\boldsymbol{a},\boldsymbol{b}]}dg+1\Big)}. \tag{3.10}$$

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Now, by Lemma 3.2, $f \cdot (1-\phi)$ is PUL-Stieltjes integrable with respect to g on $[\boldsymbol{a},\boldsymbol{b}]$. Thus, there is a gauge $\delta_2 \leq \delta_1$ on $[\boldsymbol{a},\boldsymbol{b}]$ such that for every δ_2-fine division $D = \{(\boldsymbol{\xi}', \psi, \boldsymbol{J})\}$ of $[\boldsymbol{a},\boldsymbol{b}]$, we have

$$\begin{aligned}
&\left\| \sum_D f(\boldsymbol{\xi}') \cdot (1-\phi(\boldsymbol{\xi}')) \int_{\boldsymbol{J}} \psi \ dg - (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f \cdot (1-\phi) \ dg \right\| \\
&= \left\| \sum_D f(\boldsymbol{\xi}') \cdot (1-\phi(\boldsymbol{\xi}')) \int_{\boldsymbol{J}} \psi \ dg - (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f \ dg + (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f \cdot \left(\sum_P \varphi \right) \ dg \right\| \\
&= \left\| \sum_D f(\boldsymbol{\xi}') \cdot (1-\phi(\boldsymbol{\xi}')) \int_{\boldsymbol{J}} \psi \ dg - (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f \ dg + \sum_P \int_{\boldsymbol{I}} f \cdot \varphi \ dg \right\| < \frac{\epsilon}{3},
\end{aligned}$$

that is,

$$\left\| \sum_D f(\boldsymbol{\xi}') \cdot (1-\phi(\boldsymbol{\xi}')) \int_{\boldsymbol{J}} \psi \ dg + \sum_P \int_{\boldsymbol{I}} f \cdot \varphi \ dg - (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f \ dg \right\| < \frac{\epsilon}{3}. \quad (3.11)$$

Now, since $P \cup \{(\boldsymbol{\xi}, \boldsymbol{J}, (1-\phi)\psi) : (\boldsymbol{\xi}, \boldsymbol{J}, \psi) \in D\}$ is also a δ-fine division of $[\boldsymbol{a},\boldsymbol{b}]$ and by integrability of f wrt g, we have

$$\left\| \sum_P f(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \varphi dg + \sum_D f(\boldsymbol{\xi}') \int_{\boldsymbol{J}} (1-\phi)\psi dg - (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f \ dg \right\| < \frac{\epsilon}{3}. \quad (3.12)$$

Observe that by (3.10),

$$\begin{aligned}
&\left\| \sum_D f(\boldsymbol{\xi}')(1-\phi(\boldsymbol{\xi}')) \int_{\boldsymbol{J}} \psi dg - \sum_D f(\boldsymbol{\xi}') \int_{\boldsymbol{J}} (1-\phi)\psi dg \right\| \\
&= \left\| \sum_D \left\{ f(\boldsymbol{\xi}') \cdot \int_{\boldsymbol{J}} \big[\phi - \phi(\boldsymbol{\xi}')\big] \psi dg \right\} \right\| \\
&\leq \left\| \sum_D \left\{ f(\boldsymbol{\xi}') \cdot \int_{\boldsymbol{J}} \sup \Big\{ |\phi(\boldsymbol{x}) - \phi(\boldsymbol{\xi}')| : \boldsymbol{x} \in B(\boldsymbol{\xi}', \delta_1(\boldsymbol{\xi}')) \Big\} \cdot \psi dg \right\} \right\| \\
&= \left\| \sum_D \left\{ f(\boldsymbol{\xi}') \cdot \sup \Big\{ |\phi(\boldsymbol{x}) - \phi(\boldsymbol{\xi}')| : \boldsymbol{x} \in B(\boldsymbol{\xi}', \delta_1(\boldsymbol{\xi}')) \Big\} \cdot \int_{\boldsymbol{J}} \psi dg \right\} \right\| \\
&\leq \sum_D \left\{ \big\| f(\boldsymbol{\xi}') \big\| \cdot \sup \Big\{ |\phi(\boldsymbol{x}) - \phi(\boldsymbol{\xi}')| : \boldsymbol{x} \in B(\boldsymbol{\xi}', \delta_1(\boldsymbol{\xi}')) \Big\} \cdot \int_{\boldsymbol{J}} \psi dg \right\} \\
&< \frac{\epsilon}{3\Big(\int_{[\boldsymbol{a},\boldsymbol{b}]} dg + 1\Big)} \cdot \sum_D \int_{\boldsymbol{J}} \psi dg \leq \frac{\epsilon}{3\Big(\int_{[\boldsymbol{a},\boldsymbol{b}]} dg + 1\Big)} \cdot \left(\int_{[\boldsymbol{a},\boldsymbol{b}]} dg + 1 \right) = \frac{\epsilon}{3};
\end{aligned}$$

that is,

$$\left\| \sum_D f(\boldsymbol{\xi}')(1-\phi(\boldsymbol{\xi}')) \int_{\boldsymbol{J}} \psi dg - \sum_D f(\boldsymbol{\xi}') \int_{\boldsymbol{J}} (1-\phi)\psi dg \right\| < \frac{\epsilon}{3}. \quad (3.13)$$

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Therefore, by (3.11), (3.12), and (3.13)

$$
\begin{aligned}
&\left\| \sum_P \left(f(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \varphi \ dg - \int_{\boldsymbol{I}} f\varphi \ dg \right) \right\| \\
&= \Big\| \sum_P f(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \varphi \ dg - (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f \ dg + (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f \ dg - \sum_D f(\boldsymbol{\xi}')(1-\phi(\boldsymbol{\xi}')) \int_{\boldsymbol{J}} \psi \ dg \\
&\quad + \sum_D f(\boldsymbol{\xi}')(1-\phi(\boldsymbol{\xi}')) \int_{\boldsymbol{J}} \psi \ dg - \sum_D f(\boldsymbol{\xi}') \int_{\boldsymbol{J}} (1-\phi)\psi \ dg \\
&\qquad + \sum_D f(\boldsymbol{\xi}') \int_{\boldsymbol{J}} (1-\phi)\psi \ dg - \sum_P \int_{\boldsymbol{I}} f\varphi \ dg \Big\| \\
&\leq \left\| \sum_P f(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \varphi dg + \sum_D f(\boldsymbol{\xi}') \int_{\boldsymbol{J}} (1-\phi)\psi \ dg - (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f \ dg \right\| \\
&\quad + \left\| (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f \ dg - \sum_P \int_{\boldsymbol{I}} f\varphi \ dg - \sum_D f(\boldsymbol{\xi}')(1-\phi(\boldsymbol{\xi}')) \int_{\boldsymbol{I}} \psi dg \right\| \\
&\qquad + \left\| \sum_D f(\boldsymbol{\xi}')(1-\phi(\boldsymbol{\xi}')) \int_{\boldsymbol{J}} \psi \ dg - \sum_D f(\boldsymbol{\xi}') \int_{\boldsymbol{J}} (1-\phi)\psi \ dg \right\| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}
$$

And the result follows. □

Corollary 3.4. *In view of the conditions of Theorem 3.3, if in addition that X is a finite dimensional Banach space, then for every $\epsilon > 0$, there exists a gauge δ on $[\boldsymbol{a},\boldsymbol{b}]$ such that for any δ-fine partial division $P = \{(\boldsymbol{\xi}, \varphi, \boldsymbol{I})\}$ of $[\boldsymbol{a},\boldsymbol{b}]$, we have*

$$\sum_P \left\| f(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \varphi dg - (\mathcal{P}) \int_{\boldsymbol{I}} f\varphi dg \right\| < \epsilon.$$

Definition 3.5. Let $f : M \to \mathbb{R}$ and $g : M \to \mathbb{R}$ be functions. For a δ-fine division $D = \{(x_i, I_i^{\alpha_{s_i}}, \varphi_i)\}_{i=1}^m$ of M, suppose that the Riemann-Stieltjes integral $\int_{I_i^{\alpha_{s_i}}} \varphi_i \ dg$ exists, for all $k \in \{1, 2, \cdots, m\}$. We define the *PUL-Stieltjes sum* on M by

$$\mathcal{S}(f, g, D) = \sum_{i=1}^m f(\xi_i) \int_{I_i^{\alpha_{s_i}}} \varphi_i \ dg.$$

Definition 3.6. Let M be a compact differentiable r-manifold, $f : M \to \mathbb{R}$ and $g : M \to \mathbb{R}$ be functions defined on M, and $\Theta = \{\alpha_i\}$ be an atlas of M. We say that f is *PUL-Stieltjes integrable* to real number A with respect to g on M if given an $\epsilon > 0$, there is a gauge δ on M such that for any δ-fine division $D = \{(x_i, I_i^{\alpha_{s_i}}, \varphi_i)\}_{i=1}^m$ of M, we have

$$|\mathcal{S}(f, g, D) - A| < \epsilon.$$

In this case, we write

$$A = (\mathcal{P}) \int_{M,\Theta} f \ dg.$$

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If no confussion arises, we may write

$$A = (\mathcal{P}) \int_M f \ dg.$$

In what follows, we will formulate the *Change of Variable formula*. Using this formula, we will establish the independence of this integral to the choice of the atlas.

The definition above is defined using an atlas Θ. First, we will verify that this definition is independent of the choice of the atlas. We begin with the following lemma:

Lemma 3.7. *Let* $D = \{(\xi_i, I_i^{\alpha_{s_i}}, \varphi_i)\}_{i=1}^m$ *be a partial* δ*-fine division of* M, $\varphi(x) = \sum_{i=1}^{m} \varphi_i(x)$, *and* $D_1 = \{(\eta_j, J_j^{\alpha_{s_j}}, \psi_j)\}_{j=1}^p$ *be* δ*-fine division of* M. *Then* $D \cup \{(\eta_j, J_j^{\alpha_{s_j}}, (1-\varphi)\cdot\psi_j)\}_{j=1}^p$ *is also a* δ*-fine division of* M.

Proof. Let $x \in M$. Observe that

$$\begin{aligned}\sum_{i=1}^{m} \varphi_i(x) + \sum_{j=1}^{p}(1-\varphi(x))\cdot\psi_j(x) &= \sum_{i=1}^{m} \varphi_i(x) + \sum_{j=1}^{p} \psi_j(x) - \varphi(x)\sum_{j=1}^{p}\psi_j(x) \\ &= \varphi(x) + 1 - \varphi(x) \\ &= 1.\end{aligned}$$

This means that $\{\varphi_i\}_{i=1}^m \cup \{(1-\varphi)\psi_j\}_{j=1}^p$ is a partition of unity. Since D is a partial δ-fine division of M, it follows that

$$\text{supp}\varphi_i \subseteq I^{\alpha_{s_i}} \subseteq B(\xi_i, \delta(\xi_i)) \tag{3.14}$$

for all $i \in \{1, 2, \cdots, m\}$.

Now, for each $j \in \{1, 2, \cdots, p\}$, let $x \in$ supp $((1-\varphi)\cdot\psi_j)$. Then $(1-\varphi(x))\cdot\psi_j(x) > 0$ and so $\psi_j(x) > 0$. Hence, $x \in$ supp ψ_j. Thus,

$$\text{supp } ((1-\varphi)\cdot\psi_j) \subseteq \text{supp } \psi_j$$

for all $j \in \{1, 2, \cdots, p\}$. So we have

$$\text{supp } ((1-\varphi)\cdot\psi_j) \subseteq \text{supp } \psi_j \subseteq J_j^{\alpha_{s_i}} \subseteq B(\eta_j, \delta(\eta_j)) \tag{3.15}$$

for all $j \in \{1, 2, \cdots, p\}$. By the inclusions in (3.14) and (3.15), it follows that $D \cup D_2$ is a δ-fine division of M. $\square$

The next results exhibits an important tool in the integration on manifolds.

Recall that for every $\beta > 0$ and for every interval $I = \prod_{k=1}^{n}[a_i, b_i]$ of $\mathbb{R}^n$, we defined the intervals

$$I + \beta = \prod_{k=1}^{n}[a_i - \beta, b_i + \beta]$$

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and

$$I - \beta = \prod_{k=1}^{n} [a_i + \beta, b_i - \beta].$$

Definition 3.8. Let $f : M \to \mathbb{R}$ be a real-valued function defined on a manifold M. We say that f is *continuous at* $x \in M$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that if $y \in M$ with $\|x - y\|_{\mathbb{R}^n} < \delta$, then

$$|f(x) - f(y)| < \epsilon.$$

We say that f is *continuous on* M if it is continuous at every $x \in M$.

Definition 3.9. Let $f : M \to \mathbb{R}$ be a real valued function defined on a manifold M. The *total variation of* g *over* M is given by

$$V(f, M) = \sup \left\{ \sum_{I^\alpha \in D} |\Delta_f(I^\alpha)| : D \text{ is a division of } M \right\},$$

where

$$|\Delta_f(I^\alpha)| = \sum_{t \in \mathcal{V}(\mathcal{I}^\alpha)} g(t) \prod_{k=1}^{m} (-1)^{\chi_{\{u_k\}}(t_k)}.$$

We say that f is a *function of bounded variation on* M if $V(f, M) < \infty$. We denote $BV(M)$ as the set of functions of bounded variations on M.

Theorem 3.10 (Change of Variable)**.** *Let M be a compact differentiable r-manifold with atlas Θ. Consider the functions $f : M \to \mathbb{R}$ and $g : M \to \mathbb{R}$, where $g \in BV(M)$. Suppose there is a chart $\alpha \in \Theta$ such that $\alpha : U \to V$ and $V \supseteq \text{supp } f$. If f is PUL-Stieltjes integrable with respect to g on M, then $(f \circ \alpha) \cdot \chi_U$ is PUL-Stieltjes integrable with respect to g on a compact interval $\boldsymbol{E}$, where $U \subseteq E \subseteq \mathbb{R}^n$, and*

$$(\mathcal{P}) \int_M f \ dg = (\mathcal{P}) \int_{\boldsymbol{E}} (f \circ \alpha) \cdot \chi_U \ d(g \circ \alpha).$$

Proof. Let $\epsilon > 0$. Since f is PUL-Stieltjes integrable with respect to g on M, there is a gauge δ on M such that for every δ-fine division $D = \{(\xi_i, I_i^{\alpha_{s_i}}, \varphi_i)\}_{i=1}^m$, we have

$$\left| \sum_{k=1}^{m} f(\xi) \int_{I_i^{\alpha_{s_i}}} \varphi_i \ dg - \int_M f \ dg \right| < \frac{\epsilon}{2}.$$

Choose $\beta > 0$ (sufficiently small) such that

$$U' = \alpha^{-1}(\text{supp } f) + \beta = \left\{ x \in U : \min_{x' \in \alpha^{-1}(\text{supp } f)} d(x', x) < \beta \right\} \subseteq U.$$

Note that $V \subseteq \text{supp } \ f$ and U' are open sets. Thus, we may assume that $B(x, \delta(x)) \subseteq V \smallsetminus \text{supp } f$ for all $x \in V \smallsetminus \text{supp } f$ and $B(x, \delta(x)) \subseteq U'$ for all $x \in \text{supp } f$. Now, define a gauge δ_1 on E such that if $x \in U'$, then

$$B(x, \delta_1(x)) \subseteq \alpha^{-1}(B(\alpha(x), \delta(\alpha(x)))); \tag{3.16}$$

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and if $x \in E \setminus U'$, then

$$B(x, \delta_1(x)) \cap E \subseteq E \setminus U'. \tag{3.17}$$

Since $g \in BV(M)$ and α is a homeomorphism, it follows that for any compact interval I of M $g \circ \alpha \in \mathrm{BV}(I)$; whence, $K = V(g \circ \alpha, I) \in \mathbb{R}$.

Now, let $D = \{(x_j, I_j, \sigma_j)\}_{j=1}^{s}$ be a δ_1-fine division of E. Suppose that $D = D_1 \cup D_2$, where $D_1 = \{(x, I, \sigma) \in D : x \in U'\}$ and $D_2 = D \setminus D_1$. For convenience, without loss of generality, we let $D_1 = \{(x_j, I_j, \sigma_j)\}_{j=1}^{p}$ and $D_2 = \{(x_j, I_j, \sigma_j)\}_{j=p+1}^{s}$. Let $x \in U'$. Then $x \in E$ and from the definition of D_1, we further have $\sigma_j(x) = 0$ for all $j \in \{p+1, p+2, \cdots, s\}$. Thus,

$$1 = \sum_{j=1}^{s} \sigma_j(x) = \sum_{j=1}^{p} \sigma_j(x) + \sum_{j=p+1}^{s} \sigma_j(x) = \sum_{j=1}^{p} \sigma_j(x).$$

Fix a tag x_k, where $k \in \{p+1, p+2, \cdots s\}$. So, $x_k \notin U'$. Since $\alpha^{-1}(\text{supp } f) \subseteq U'$ and $x_k \notin U'$, it follows that $x_k \notin \alpha^{-1}(\text{supp } f)$ which will further imply that $f(\alpha(x_k)) = 0$ for all $k = p+1, p+2, \cdots, s$. Hence,

$$\sum_{j=1}^{s} f(\alpha(x_j)) \cdot \chi_U(x_j) \cdot \int_{I_j} \sigma_j \; dg = \sum_{j=1}^{p} f(\alpha(x_j)) \cdot \chi_U(x_j) \cdot \int_{I_j} \sigma_j \; dg.$$

Put $y_j = \alpha(x_j)$ and $\tau_j = \sigma_j \circ \alpha^{-1}$ for all $j = 1, 2, \cdots, p$. Then $\tau_j : V \to [0, 1]$ and since the σ_j's and α^{-1} are C^1 maps on V, it follows that $\tau_j = \sigma_j \circ \alpha^{-1}$ remains a C^1 map on V. Fix $j \in \{1, 2, \cdots, p\}$. Let $x \in \text{supp } \tau_j$. Then $0 \neq \tau_j(x) = \sigma_j(\alpha^{-1}(x))$ and so $\alpha^{-1}(x) \in \text{supp } \sigma_j$. Thus, $x \in \alpha(\text{supp } \sigma_j) \subseteq V$. This means that $\text{supp } \tau_j \subseteq V$. Now, for each $j = 1, 2, \cdots, m$, extend the map τ_j on M in such a way that $\tau_j(x) = 0$ for all $x \in M \setminus V$. Then τ_j remains a C^1-diffeomorphism on M. Also, $\{\tau_j\}_{j=1}^{p}$ is a partial partition of unity on M. Put

$$\omega(x) = \sum_{j=1}^{p} \tau_j(x), \text{ for all } x \in M.$$

If $x \in \alpha(U')$, then $\alpha^{-1}(x) \in U'$. Since $\{\sigma_j\}_{j=1}^{p}$ is a partition of unity on $\boldsymbol{E}$, it follows that

$$\omega(x) = \sum_{j=1}^{p} \tau_j(x) = \sum_{j=1}^{p} \sigma_j(\alpha^{-1}(x)) = 1.$$

This means that $\omega(x) = 1$ for all $x \in \alpha(U')$ with $\omega(x) = 0$ for all $x \in M \setminus V$.

We now show that $D^\alpha = \{(y_j, I_j^\alpha, \tau_j)\}_{j=1}^{p}$ is a partial δ-fine division of M, where $\alpha(I_j) = I_j^\alpha$. First, in arbitrary manner, we fix $j \in \{1, 2, \cdots, p\}$ and pick $x \in \text{supp } \tau_j$. Then $0 \neq \tau_j(x) = \sigma_j(\alpha^{-1}(x))$ and since D is a δ_1-fine division of $\boldsymbol{E}$, we further have $\alpha^{-1}(x) \in \text{supp } \sigma_j \subseteq I_j$. So, $x \in \alpha(I_j) = I_j^\alpha$ and $\text{supp } \tau_j \subseteq I_j^\alpha$. To show that $I_j^\alpha \subseteq B(y_j, \delta(y_j))$, we let $x \in I_j^\alpha$. Then there

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exists $z \in I_j$ such that $\alpha(z) = x$. Again, since D is a δ_1-fine division of $\boldsymbol{E}$ and by (3.16),

$$z \in I_j \subseteq B(x_j, \delta_1(x_j)) \subseteq \alpha^{-1}\big(B(\alpha(x_j), \delta(\alpha(x_j)))\big) = \alpha^{-1}\big(B(y_j, \delta(y_j))\big),$$

that is, $x = \alpha(z) \in B(y_j, \delta(y_j))$. Thus, $I_j^\alpha \subseteq B(y_j, \delta(y_j))$ for all $j = 1, 2, \cdots, p$. This shows that D^α is a partial δ-fine division of M. Let $D' = \{(\xi_k, J_k^{\alpha_{s_k}}, \psi_k)\}_{k=1}^m$ be δ-fine division of M. By Lemma 3.7, $D^\alpha \cup \{\xi_k, J_k^{\alpha_{s_k}}, (1-\omega)\psi_k\}_{k=1}^m$ is a δ-fine division of M. Thus, $1 - \omega(x) = 0$ for all $x \in J_k^{\alpha_{s_k}}$. From the definition of δ, if $\xi_k \in \operatorname{supp} f$, then for each $k \in \{1, 2, \cdots, m\}$

$$f(\xi_k) \int_{J_k^{\alpha_{s_k}}} (1 - \omega) \cdot \psi_k \; dg = 0.$$

It will further imply that

$$\sum_{k=1}^m f(\xi_k) \int_{J_k^{\alpha_{s_k}}} (1 - \omega) \cdot \psi_k \; dg = 0.$$

Hence

$$\sum_{j=1}^p f(y_j) \int_{I_j^\alpha} \omega_j dg + \sum_{k=1}^m f(\xi_k) \int_{J_k^{\alpha_{s_k}}} (1 - \omega) \cdot \psi_k dg = \sum_{j=1}^p f(y_j) \int_{I_j^\alpha} \omega_j dg.$$

Thus,

$$\begin{aligned}
\epsilon &> \Big| \sum_{j=1}^p f(y_j) \int_{I_j^\alpha} \omega_j dg + \sum_{k=1}^m f(\xi_k) \int_{J_k^{\alpha_{s_k}}} (1 - \omega) \cdot \psi_k dg - \int_M f \; dg \Big| \\
&= \Big| \sum_{j=1}^p f(y_j) \int_{I_j^\alpha} \omega_j dg - \int_M f \; dg. \Big|
\end{aligned}$$

Thus,

$$\begin{aligned}
&\Big| \sum_{j=1}^p f(\alpha(x_j)) \cdot \chi_U(\alpha(x_j)) \int_{I_j} \sigma_j \; d(g \circ \alpha) - \sum_{j=1}^p f(y_j) \int_{I_j^\alpha} \tau_j \; dg \Big| \\
&= \Big| \sum_{j=1}^p f(\alpha(x_j)) \int_{I_j} \sigma_j \; d(g \circ \alpha) - \sum_{j=1}^p f(y_j) \int_{I_j^\alpha} \tau_j \; dg \Big| \\
&= \Big| \sum_{j=1}^p f(\alpha(x_j)) \int_{I_j} \sigma_j \; d(g \circ \alpha) - \sum_{j=1}^p f(y_j) \int_{\alpha(I_j)} (\sigma_j \circ \alpha^{-1}) \; dg \Big| \\
&= \Big| \sum_{j=1}^p f(\alpha(x_j)) \int_{I_j} \sigma_j \; d(g \circ \alpha) - \sum_{j=1}^p f(\alpha(x_j)) \int_{I_j} \sigma_j \; d(g \circ \alpha) \Big| \\
&= 0.
\end{aligned}$$

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Therefore,

$$\begin{aligned}&\left|\sum_{j=1}^{p} f(\alpha(x_j)) \cdot \chi_U(\alpha(x_j)) \int_{I_j} \sigma_j \ d(g \circ \alpha) - \int_M f \ dg\right| \\ &= \left|\sum_{j=1}^{p} f(y_j) \int_{I_j^{\alpha}} \omega_j dg - \int_M f \ dg\right| < \epsilon\end{aligned}$$

implying that $(f \circ \alpha) \cdot \chi_U$ is PUL-Stieltjes integrable with respect to g to $\int_M f \ dg$ on $\boldsymbol{E}$. □

The Change of Variable formula in Manifold M enables us to show that the integral is independent to the choice of the associated atlas in M. To validate such claim, we formally present the next result.

Corollary 3.11. *Let M be a compact differentiable r-manifold, $f : M \to \mathbb{R}$, and $g : M \to \mathbb{R}$. If Θ_1 and Θ_2 are two atlas of M, then f is integrable with respect to g on M associated with atlas Θ_1 if and only if f is integrable on M associated with atlas Θ_2. Furthermore, the values of these two integrals are equal.*

Proof. Let $\Theta_1 = \{\alpha_j : U_j \to V_j\}_{j=1}^{s}$ and $\Theta_2 = \{\alpha_k : U_k \to V_k\}_{k=1}^{t}$ be two atlas of M. Set $V_{jk} = V_j \cap V_k$ for all j and k and let $\{\varphi_{jk}\}$ be its corresponding partition of unity on M such that supp $\varphi_{jk} \subseteq V_{jk}$. Let $x \in$ supp $(f \cdot \varphi_{jk})$. Then $f(x) \neq 0$ and $\varphi_{jk}(x) \neq 0$ and so $x \in$ supp $f \cap$ supp φ_{jk}. Since supp $\varphi_{jk} \subseteq V_{jk}$, $x \in$ supp $f \cap V_{jk}$. But supp $f \cap V_{jk} \subseteq V_{jk}$. Hence, $x \in V_{jk}$ and further, since x is arbitrarily picked, it follows that supp $f \cdot \varphi_{jk} \subseteq V_{jk}$. Thus, by Theorem 3.10, for each j and k, we have

$$\int_{\boldsymbol{E}} [(f \cdot \varphi_{jk}) \circ \alpha_j] \chi_{U_j} \ d(g \circ \alpha_j) = \int_M f \varphi_{jk} \ dg = \int_{\boldsymbol{E}} [(f \cdot \varphi_{jk}) \circ \alpha_k] \chi_{U_k} \ d(g \circ \alpha_k).$$

Hence we have

$$\begin{aligned}\int_{M,\Theta_1} f \ dg &= \sum_{j=1}^{s} \sum_{k=1}^{t} \int_{M,\Theta_1} f \cdot \varphi_{jk} \ dg \\ &= \sum_{j=1}^{s} \sum_{k=1}^{t} \int_{\boldsymbol{E}} [(f \cdot \varphi_{jk}) \circ \alpha_j] \chi_{U_j} \ d(g \circ \alpha_j) \\ &= \sum_{j=1}^{s} \sum_{k=1}^{t} \int_{\boldsymbol{E}} [(f \cdot \varphi_{jk}) \circ \alpha_k] \chi_{U_k} d(g \circ \alpha_j) \\ &= \sum_{j=1}^{s} \sum_{k=1}^{t} \int_{M,\Theta_2} f \cdot \varphi_{jk} \ dg \\ &= \int_{M,\Theta_2} f \ dg.\end{aligned}$$

□

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We give some of the standard properties of this integral in manifold such as the linearity of the integrand and the integrator, additivity of the integral over its domain, and the Cauchy criterion.

Theorem 3.12. *The value of the PUL-Stieltjes integral defined on M is unique.*

Proof. Suppose that $A_1 = \int_M f\ dg$ and $A_2 = \int_M f\ dg$. Let $\epsilon > 0$. Since f is PUL-Stieltjes integrable with respect to g on M to A_1, there is a gauge δ_1 on M such that for every δ_1-fine division D_1 of M, we have

$$\big|\mathcal{S}\mathcal{S})(f, g, D_1) - A_1\big| < \frac{\epsilon}{2}. \tag{3.18}$$

Similarly, there is a gauge δ_2 on M such that for any δ_2-fine division D_2 on M, we have

$$\big|\mathcal{S}(f, g, D_2) - A_2\big| < \frac{\epsilon}{2}. \tag{3.19}$$

Choose $\delta = \min\{\delta_1, \delta_2\}$. Then δ is a gauge on M. Let $D = \{\xi, I^{\alpha_s}, \varphi\}$ be a δ-fine division of M. Then

$$\text{supp } \varphi \subseteq I^{\alpha_s} \subseteq B(\xi, \delta(\xi)) \subseteq B(\xi, \delta_1)$$

and

$$\text{supp } \varphi \subseteq I^{\alpha_s} \subseteq B(\xi, \delta(\xi)) \subseteq B(\xi, \delta_2)$$

which imples D is both δ_1-fine and δ_2-fine division of M. Thus, by (3.18) and (3.19),

$$\begin{aligned} 0 &\leq \big|A_1 - A_2\big| \leq \big|A_1 - \mathcal{S}(f, g, D)\big| + \big|\mathcal{S}(f, g, D) - A_2\big| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Since ϵ is arbitrary, $\big|A_1 - A_2\big| = 0$; hence $A_1 = A_2$. $\square$

The next two results present the bilinear property of this integral, that is, linearity of both the integrand and the integrator holds in this integral.

Theorem 3.13. *Suppose that $f_1 : M \to \mathbb{R}$ and $f_2 : M \to \mathbb{R}$ are PUL-Stieltjes integrable with respect to a real-valued function $g : M \to \mathbb{R}$ on M. Then cf_1 and $f_1 + f_2$ are also PUL-Stieltjes integrable with respect to g on M for all $c \in \mathbb{R}$. Moreover,*

$$\int_M cf_1\ dg = c\int_M f_1\ dg$$

and

$$\int_M (f_1 + f_2)\ dg = \int_M f_1\ dg + \int_M f_2\ dg.$$

Theorem 3.14. *Suppose that $f : M \to \mathbb{R}$ is PUL-Stieltjes integrable with respect to real-valued functions $g_1 : M \to \mathbb{R}$ and $g_2 : M \to \mathbb{R}$ on M. Then for all $c \in \mathbb{R}$, f is PUL-Stieltjes integrable with respect to both cg_1 and $g_1 + g_2$ on M. Moreover*

$$\int_M f\ d(cg_1) = c\int_M f\ dg$$

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and

$$\int_M f\ d(g_1+g_2)=\int_M f\ dg_1+\int_M f\ dg_2.$$

Remark 3.15. If $f:[\boldsymbol{a},\boldsymbol{b}]\to\mathbb{R}$ is non-negative Riemann-Stieltjes integrable with respect to $g:[\boldsymbol{a},\boldsymbol{b}]\to\mathbb{R}$ on $[\boldsymbol{a},\boldsymbol{b}]$ with $g\geq 0$ on $[\boldsymbol{a},\boldsymbol{b}]$, then

$$\int_{[\boldsymbol{a},\boldsymbol{b}]} f\ dg\geq 0.$$

Theorem 3.16. *Let f_1 and f_2 be PUL-Stieltjes integrable with respect to $g\geq 0$ on M. Suppose that $f_1(x)\leq f_2(x)$ for all $x\in M$. Then*

$$(\mathcal{P})\int_M f_1\ dg\leq(\mathcal{P})\int_M f_2\ dg.$$

Theorem 3.17. (Cauchy Criterion) *A function $f:M\to\mathbb{R}$, where M is a manifold, is PUL-Stieltjes integrable with respect to g on M if and only if for every $\epsilon>0$, there is a gauge δ on M such that for any δ-fine divisions D_1 and D_2 of M, we have*

$$\Big|\mathcal{S}(f,g,D_1)-\mathcal{S}(f,g,D_2)\Big|<\epsilon.$$

Proof. $(\Rightarrow)$ Let $\epsilon>0$. Then there is a gauge δ on M such that for any δ-fine division D of M, we have

$$\Big|\mathcal{S}(f,g,D)-\int_M f\ dg\Big|<\frac{\epsilon}{2}. \tag{3.20}$$

Let D_1 and D_2 be δ-fine divisions of M. Then by (3.21),

$$\begin{aligned}\Big|\mathcal{S}(f,g,D_1)-\mathcal{S}(f,g,D_2)\Big|&=\Big|\mathcal{S}(f,g,D_1)-\int_M f\ dg+\int_M f\ dg-\mathcal{S}(f,g,D_2)\Big|\\&\leq\Big|\mathcal{S}(f,g,D_1)-\int_M f\ dg\Big|+\Big|\int_M f\ dg-\mathcal{S}(f,g,D_2)\Big|\\&<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.\end{aligned}$$

$(\Leftarrow)$ for each $n\in\mathbb{N}$, there is a gauge δ_n on M such that for any pair of δ_n-fine divisions D_n and D'_n of M, we have

$$\Big|\mathcal{S}(f,g,D_n)-\mathcal{S}(f,g,D'_n)\Big|<\frac{1}{n}. \tag{3.21}$$

We may assume that for each $x\in M$, $\langle\delta_n(x)\rangle_{n\in\mathbb{N}}$ is decreasing. For each $n\in\mathbb{N}$, fix a δ_n-fine division D_n of M and we denote $s_n=\mathcal{S}(f,g,D_n)$ be its corresponding PUL-Stieltjes sum. We will show that $\langle s_n\rangle_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$. Let $\epsilon>0$. Then there is an $N\in\mathbb{N}$ such that $\frac{1}{N}<\epsilon$. Let $m,n\geq N$. Then $\delta_N(x)\geq\delta_m(x)$ and $\delta_N(x)\geq\delta_n(x)$. So, D_m and D_n are both δ_N-fine divisions of M. By the hyphothesis,

$$|s_n-s_m|=\big|\mathcal{S}(f,g,D_n)-\mathcal{S}(f,g,D_m)\big|<\frac{1}{N}<\epsilon$$

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and $\langle s_n \rangle_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$. Hence, $\langle s_n \rangle_{n\in\mathbb{N}}$ converges to, say, $A \in \mathbb{R}$, that is, $\lim_{n\to\infty} s_n = A$.

We now show that f is a PUL-Stieltjes integrable with respect to g on M and

$$\int_M f \; dg = A.$$

Let $\epsilon > 0$. Since $\lim_{n\to\infty} s_n = A$, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$

$$\big|\mathcal{S}(f, g, D_n) - A\big| = \big|s_n - A\big| < \frac{\epsilon}{2}. \tag{3.22}$$

Thus, there exists $N_2 \in \mathbb{N}$ such that $\dfrac{1}{N_2} < \dfrac{\epsilon}{2}$. Take $N = \max\{N_1, N_2\}$ and put $\delta(x) = \delta_N(x)$ for all $x \in M$. Let D be any δ-fine division of M. Then D is δ_N-fine division of M. Since $N \geq N_2$, by (3.21) we have

$$\big|\mathcal{S}(f, g, D) - \mathcal{S}(f, g, D_N)\big| < \frac{1}{N} \leq \frac{1}{N_2} < \frac{\epsilon}{2}. \tag{3.23}$$

Also, since $N \geq N_1$, inequality (3.22) for $n = N$; i.e.

$$\big|\mathcal{S}(f, g, D_N) - A\big| < \frac{\epsilon}{2}. \tag{3.24}$$

Hence, by (3.23) and (3.24) we have

$$\begin{aligned} \big|\mathcal{S}(f, g, D) - A\big| &\leq \big|\mathcal{S}(f, g, D) - \mathcal{S}(f, g, D_N)\big| \\ &\quad + \big|\mathcal{S}(f, g, D_N) - A\big| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

This shows that f is a PUL-Stieltjes integrable with respect to g on M and

$$(\mathcal{P}) \int_M f \; dg = A. \qquad \square$$

Corollary 3.18. *If $f : M \to \mathbb{R}$ is PUL-Stieltjes integrable with respect to $g : M \to \mathbb{R}$ on M and $I \subseteq M$ is a compact differentiable r-manifold, then f is PUL-Stieltjes integrable with respect to g on I.*

Corollary 3.19. *Let f be any real-valued funtion on M. Suppose that f is PUL-stieltjes integrable with respect to g on a compact differentiable r-manifolds F_1 and F_2 with F_1 and F_2 is a partition of M. Then f is PUL-Stieltjes integrable with respect to g on $F_1 \bigcup F_2$ and*

$$(\mathcal{P}) \int_{F_1 \bigcup F_2} f dg = (\mathcal{P}) \int_{F_1} f dg + (\mathcal{P}) \int_{F_2} f dg.$$

Theorem 3.20. *Let $f : M \to \mathbb{R}$ and $g : M \to \mathbb{R}$ be two real-valued functions defined on a manifold M. If $f \in C(M)$ and $g \in BV(M)$, then f is PUL-Stieltjes integrable with respect to g on M.*

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Proof. Let $\epsilon > 0$. Note that the Riemann-stieltjes integral $\int_I \varphi \, dg$ exists, whenever φ is a partition of unity and g is of bounded variation. Since $g \in BV(M, \mathbb{R}^n)$, $M = V(g, M) \in \mathbb{R}$. Since f is continuous on M, then for each $x \in M$, there exists a $\delta(x) > 0$ such that for any $y \in M$ with $\|x - y\|_M < \delta(x)$, we have

$$|f(x) - f(y)| < \frac{\epsilon}{2[M+1]}.$$

Let $D_1 = \{(\xi, I^{\alpha_s}, \varphi)\}$ and $D_2 = \{(\zeta, J^{\alpha_t}, \psi)\}$ be any two δ-fine divisions of M. Let $D_3 = \{\gamma, K^{\alpha_r}, \sigma\}$ be a δ-fine division of M, where $K^{\alpha_r} = I^{\alpha_s} \cap J^{\alpha_t}$ with $I^{\alpha_s} \in D_1$ and $J^{\alpha_t} \in D_2$.

Observe that

$$\begin{aligned}
\mathcal{S}(f, g, D_1) &= \sum_{I^{\alpha_s} \in D_1} f(\xi) \int_{I^{\alpha_s}} \varphi \, dg \\
&= \sum_{I^{\alpha_s} \in D_1} f(\xi) \Big[\sum_{J^{\alpha_t} \in D_2} \int_{I^{\alpha_s} \cap J^{\alpha_t}} \varphi \, dg \Big] \\
&= \sum_{K^{\alpha_r} \in D_3} f(\xi) \int_{K^{\alpha_r}} \sigma \, dg
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{S}(f, g, D_2) &= \sum_{J^{\alpha_t} \in D_2} f(\zeta) \int_{J^{\alpha_t}} \psi \, dg \\
&= \sum_{J^{\alpha_t} \in D_2} f(\zeta) \Big[\sum_{I^{\alpha_s} \in D_1} \int_{J^{\alpha_t} \cap I^{\alpha_s}} \psi \, dg \Big] \\
&= \sum_{K^{\alpha_r} \in D_3} f(\zeta) \int_{K^{\alpha_r}} \sigma \, dg.
\end{aligned}$$

Then

$$\begin{aligned}
&\big\|\mathcal{S}(f, g, D_1) - \mathcal{S}(f, g, D_2)\big\| \\
&\quad = \big\|\mathcal{S}(f, g, D_1) - \mathcal{S}(f, g, D_3) + \mathcal{S}(f, g, D_3) - \mathcal{S}(f, g, D_2)\big\| \\
&\quad \le \big\|\mathcal{S}(f, g, D_1) - \mathcal{S}(f, g, D_3)\big\| + \big\|\mathcal{S}(f, g, D_3) - \mathcal{S}(f, g, D_2)\big\| \\
&\quad = \Big\| \sum_{K^{\alpha_r} \in D_3} f(\xi) \int_{K^{\alpha_r}} \sigma \, dg - \sum_{K^{\alpha_r} \in D_3} f(\gamma) \int_{K^{\alpha_r}} \sigma \, dg \Big\| \\
&\qquad\quad + \Big\| \sum_{K \in D_3} f(\gamma) \int_K \sigma dg - \sum_{K^{\alpha_r} \in D_3} f(\zeta) \int_{K^{\alpha_r}} \sigma \, dg \Big\|
\end{aligned}$$

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$$\begin{aligned}
&\leq \sum_{K^{\alpha_r} \in D_3} \left[\|f(\xi) - f(\gamma)\| \cdot \left| \int_{K^{\alpha_r}} \sigma \ dg \right| \right] \\
&\qquad + \sum_{K^{\alpha_s} \in D_3} \left[\|f(\gamma) - f(\zeta)\| \cdot \left| \int_{K^{\alpha_r}} \sigma \ dg \right| \right] \\
&< \sum_{K^{\alpha_s} \in D_3} \left[\frac{\epsilon}{2[M+1]} \cdot \left| \int_{K^{\alpha_r}} \sigma \ dg \right| \right] + \sum_{K^{\alpha_r} \in D_3} \left[\frac{\epsilon}{2[M+1]} \cdot \left| \int_{K^{\alpha_r}} \sigma \ dg \right| \right] \\
&= \frac{\epsilon}{M+1} \cdot \sum_{K^{\alpha_r} \in D_3} \left| \int_{K^{\alpha_s}} \sigma \ dg \right| \\
&\leq \frac{\epsilon}{M+1} \cdot \sum_{K^{\alpha_r} \in D_3} \left| \int_{K^{\alpha_r}} dg \right| \\
&= \frac{\epsilon}{M+1} \cdot \sum_{K^{\alpha_r} \in D_3} \Delta_g(K^{\alpha_r}) \\
&\leq \frac{\epsilon}{M+1} \cdot M < \epsilon.
\end{aligned}$$

Therefore, f is PUL-Stieltjes integrable on M with respect to g. □

Lemma 3.21. *Let $f : M \to \mathbb{R}$ be bounded and $g \in BV(M)$. Suppose that the PUL-Stieltjes integral of f with respect to g exists on M. Then*

$$\left| (\mathcal{P}) \int_M f dg \right| \leq \|f\|_\infty \cdot V(g, M),$$

where $\| \cdot \|_\infty = \inf\{K > 0 : |f(x)| \leq K \text{ for all } x \in M\}$.

Proof. Let $\epsilon > 0$. Since f is PUL-Stieltjes integrable with respect to g on M, there exists a gauge δ on M such that for any δ-fine division D of M, we have

$$\left| \mathcal{S}(f, g, D) - (\mathcal{P}) \int_M f \ dg \right| < \epsilon.$$

Since f is bounded on M, there is a positive $K_0 \in \mathbb{R}$ such that $|f(x)| \leq K_0$ for all $x \in M$. Thus, $|f(x)| \leq \|f\|_\infty$ for all $x \in M$.

Let D' be a δ-fine division of M. Observe that

$$\begin{aligned}
|\mathcal{S}(f, g, D')| &= \left| (D') \sum f(\xi) \int_{I^\alpha} \varphi \ dg \right| \\
&\leq (D') \sum \left| f(\xi) \int_{I^\alpha} \varphi \ dg \right| \\
&= (D') \sum |f(\xi)| \left| \int_{I^\alpha} \varphi \ dg \right| \\
&\leq (D') \sum \|f\|_\infty \left| \int_{I^\alpha} \ dg \right| \\
&\leq \|f\|_\infty V(g; M).
\end{aligned}$$

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Thus,

$$\left|(\mathcal{P})\int_M f\ dg\right| \leq \left|(\mathcal{P})\int_M f\ dg - \mathcal{S}(f,g,D')\right| + |\mathcal{S}(f,g,D')| < \epsilon + \|f\|_\infty \cdot V(g;M).$$

Since ϵ is arbitrary,

$$\left|(\mathcal{P})\int_M f\ dg\right| \leq \|f\|_\infty \cdot V(g;M) \qquad \square$$

Theorem 3.22. (The Uniform Convergence Theorem)
Let $g \in BV(M)$. Suppose that $\langle f_n\rangle_{n=1}^\infty$ is a sequence of bounded and integrable functions with respect to g on M. If $f_n \to f$ uniformly on M, then f is PUL-Stieltjes integrable with respect to g on M and

$$\lim_{n\to\infty}(\mathcal{P})\int_M f_n\ dg = (\mathcal{P})\int_M f\ dg.$$

Proof. Let $\epsilon > 0$. Since $f_n \to f$ uniformly on M, we can choose $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$ and for all $x \in M$, we have

$$|f_n(x) - f(x)| < \frac{\epsilon}{3V(g;M)}. \tag{3.25}$$

If $m, n \geq N_1$ and $x \in M$, then by (3.25)

$$\begin{aligned}|f_n(x) - f_m(x)| &= |[f_n(x) - f(x)] + [f(x) - f_m(x)]|\\ &\leq |[f_n(x) - f(x)]| + |[f(x) - f_m(x)]|\\ &< \frac{\epsilon}{3V(g;M)} + \frac{\epsilon}{3V(g;M)}\\ &= \frac{2\epsilon}{3V(g;M)}.\end{aligned}$$

This means that, for all $m, n \geq N_1$,

$$\|f_n - f_m\|_\infty \leq \frac{2\epsilon}{3V(g;M)}.$$

Thus,

$$\begin{aligned}\left|(\mathcal{P})\int_M f_m\ dg - (\mathcal{P})\int_M f_n\ dg\right| = \left|(\mathcal{P})\int_M (f_m - f_n)\ dg\right| &\leq \|f_m - f_n\|_\infty \cdot V(g;M)\\ &< \frac{2\epsilon}{3V(g;M)} \cdot V(g;M)\\ &= \frac{2\epsilon}{3} < \epsilon\end{aligned}$$

for all $m, n \geq N_1$. This means that $\left\langle (\mathcal{P})\int_M f_n\ dg\right\rangle$ is Cauchy in $\mathbb{R}$. Since $\mathbb{R}$ is, indeed, a Banach Space, $(\mathcal{P})\int_M f_n\ dg$ converges to, say, A. Choose $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$,

$$\left|(\mathcal{P})\int_M f_n\ dg - A\right| < \frac{\epsilon}{3}. \tag{3.26}$$

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Take $N = \max\{N_1, N_2\}$. Since, in particular, f_N is integrable with respect to g on M, there exists a gauge δ on M such that for any δ-fine division D of M, we have

$$\left|\mathcal{S}(f_N, g, D) - (\mathcal{P})\int_M f_N \; dg\right| < \frac{\epsilon}{3}. \tag{3.27}$$

Observe that from (3.25),

$$\begin{aligned}
|\mathcal{S}(f,g,D) - \mathcal{S}(f_N,g,D)| &= \left|(D)\sum f(\xi)\int_{I^\alpha}\varphi \; dg - f_N(\xi)\int_{I^\alpha}\varphi \; dg\right| \\
&= \left|(D)\sum (f(\xi) - f_N(\xi))\int_{I^\alpha}\varphi \; dg\right| \\
&\le (D)\sum |f(\xi) - f_N(\xi)|\left|\int_{I^\alpha}\varphi \; dg\right| \\
&< \frac{\epsilon}{3V(g;M)}(D)\sum\int_{I^\alpha} dg \\
&\le \frac{\epsilon}{3V(g;M)}V(g;M) \\
&= \frac{\epsilon}{3}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&|\mathcal{S}(f,g,D) - A| \\
&\quad \le |\mathcal{S}(f,g,D) - \mathcal{S}(f_N,g,D)| + \left|\mathcal{S}(f_N,g,D) - (\mathcal{P})\int_M f_N \; dg\right| \\
&\qquad + \left|(\mathcal{P})\int_M f_N \; dg - A\right| \\
&\quad < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&\quad = \epsilon.
\end{aligned}$$

This shows the integrablity of f with respect to g on M. Moreover,

$$\lim_{n\to\infty}(\mathcal{P})\int_M f_n \; dg = A = (\mathcal{P})\int_M f \; dg. \qquad \square$$

4. Conclusions and recommendations

The PUL integral is a McShane type and upto date, such definition of integral that is defined on some manifolds is yet to be established. In some sense, the PUL integral is equivalent to the Lebesgue integral and the latter is equivalent to the McShane integral. Thus, results on this paper may provide a link in the formulation of the McShane integral, and at the same time, a Lebesgue integral, on manifolds. On the other hand, integration process with vector domains is worthwhile to revisit. One good result for such is the Stoke's Theorem. Thus, it is recommended that in view of the PUL-Stieltjes integral,

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a version of the Stoke's Theorem (Green's Theorem in 2-dimensional spaces) of this integral may be established.

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