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Weak ϕ-Ideals and Weak $c^{\#}$-Ideals

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Abstract. In 2021, Ciloglu and Towers introduced the notion of a weak c-ideal of a Lie algebra, and used it give some characterizations of solvable and supersolvable Lie algebras. In this paper, analogously, we introduce the notions of weak ϕ-ideals and weak $c^{\#}$-ideals, and obtain some new characterizations of solvable and supersolvable Lie algebras by using these notions.

Keywords: Lie algebra, Solvable, Supersolvable, Weak ϕ-ideal, Weak $c^{\#}$-ideal.

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1. Introduction

Throughout, all Lie algebras will be of finite dimension over a fixed field F. We denote the largest ideal of L contained in all the maximal subalgebras of L, the Frattini ideal of L, by $\phi(L)$. For a subalgebra H of L, the core of H with respect to L, H_L, is the largest ideal of L contained in H. Also vector spaces direct sum will be denoted by $\dot{+}$. We say the factor algebra A/B is a chief factor of L if B is an ideal of L and A/B is a minimal ideal of L/B. Let H be a subalgebra of L. Then H is a subideal of L, if there exists a chain of subalgebras of L, $H = H_0 < H_1 < \cdots < H_n = L$ such that H_i is an ideal of H_{i+1}, for all $0 \leq i \leq n-1$. Also, a Lie algebra L is called supersolvable, if there is a chain of ideals $\{0\} \subseteq L_1 \subseteq L_2 \subseteq \cdots \subseteq L_n = L$ such that $\dim L_i = i$. In 2009, Towers [9] introduced the notion of a c-ideal of a finite dimensional

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Lie algebra, as follows:
A subalgebra H of L is a c-ideal of L, if there is an ideal K of L such that $L = H + K$ and $H \cap K \leq H_L$. He obtained some properties of c-ideals and used them to give some characterizations of solvable and supersolvable Lie algebras. Also, he [10] defined the notions of covering and avoiding properties and CAP-subalgebras of Lie algebras, as follows:
Let L be a Lie algebra and H a subalgebra of L and A/B a chief factor of L. We say that
(i) H covers A/B, if $H + A = H + B$; and
(ii) H avoids A/B, if $H \cap A = H \cap B$.
A subalgebra H of L is called a CAP-subalgebra of L, if H either covers or avoids every chief factor of L. These concepts have been explored in many articles. In particular, in [4] and [3], ϕ-ideals and $c^{\#}$-ideals were introduced and considered by using these concepts. In 2021, Ciloglu Sahin and Towers [2] introduced the concept of weak c-ideals of a Lie algebra, which encouraged us to define new concepts weak ϕ-ideals and weak $c^{\#}$-ideals of a Lie algebra.

Definition 1.1. (i) The subalgebra H of L is called a weak ϕ-ideal of L, if there is a subideal K of L such that $L = H + K$ and $H \cap K \leq \phi(H)$.
(ii) The subalgebra H of L is called a weak $c^{\#}$-ideal of L, if there is a subideal K of L such that $L = H + K$ and $H \cap K$ is a CAP-subalgebra of L.

Clearly, if H is a ϕ-ideal of L, then H is also a weak ϕ-ideal of L. But the converse is not necessarily true. For example, let L be three-dimensional Heisenberg algebra (with basis a, b, c and products given by $[a, b] = c, [a, c] = [b, c] = 0$). In [4, Example 3.3], it was shown that if M is a maximal subalgebra of L, then it is not a ϕ-ideal of L. Now, we consider the maximal subalgebra $M = span\{a, c\}$. Clearly, $L = M + span\{b\}$ and $M \cap span\{b\} = \{0\} = \phi(M)$. We have $span\{b\} \lhd span\{b, c\} \lhd L$. It shows that M is a weak ϕ-ideal of L.
In this paper, by the above notions, we obtain some characterizations for solvability and supersolvability of finite dimensional Lie algebras.

2. Preliminary results

This section is devoted to some basic results which are needed in our investigation.

Lemma 2.1. *Let L be a Lie algebra, H and K subalgebras of L and N an ideal of L.*
(i) *If H is a weak ϕ-ideal of L and $H \leq K \leq L$, then H is a weak ϕ-ideal of K.*
(ii) *If $N \leq H$ and H is a weak ϕ-ideal of L, then H/N is a weak ϕ-ideal of L/N. Furthermore, if $N \leq \phi(H)$, then the converse is true.*
(iii) *If $N \leq H$, then H is a weak $c^{\#}$-ideal of L if and only if H/N is a weak $c^{\#}$-ideal of L/N.*

Proof. (i) Let H be a weak ϕ-ideal of L. Then there is a subideal C of L with $L = H + C$ and $H \cap C \leq \phi(H)$. So we have $K = K \cap (H + C) = H + (K \cap C)$. Since C is a subideal of L, so there exists a chain of subalgebras of L, as follows:

$$C = C_0 < C_1 < ... < C_n = L,$$

such that for $0 \leq i \leq n-1$, $C_i \lhd C_{i+1}$. Thus there is a chain of subalgebras of K,

$$C \cap K = C_0 \cap K < C_1 \cap K < ... < C_n \cap K = K.$$

Clearly $C_i \cap K \lhd C_{i+1} \cap K$. Therefore $C \cap K$ is a subideal of K. Also $H \cap (C \cap K) = (H \cap C) \cap K \leq \phi(H) \cap K = \phi(H)$.
(ii) Let H be a weak ϕ-ideal of L. Then there is a subideal C of L with $L = H + C$ and $H \cap C \leq \phi(H)$. Hence we have $L/N = (H + C)/N = H/N + (C + N)/N$ such that $(C + N)/N$ is a subideal of L/N and by [8, Proposition 4.3],

$$H/N \cap (C + N)/N = ((H \cap C) + N)/N \leq (\phi(H) + N)/N \leq \phi(H/N).$$

Therefore H/N is a weak ϕ-ideal of L/N. If $N \leq \phi(H)$, then $\phi(H/N) = \phi(H)/N$, thanks to [8, Proposition 4.3] and so the converse clearly holds.
(iii) We suppose that H is a weak $c^{\#}$-ideal of L. Then there is a subideal C of L with $L = H + C$ and $H \cap C$ is a CAP-subalgebra of L. So $L/N = H/N + (C + N)/N$ and $H/N \cap (C + N)/N = ((H \cap C) + N)/N$. Now, since $H \cap C$ is a CAP-subalgebra of L, then $(H \cap C) + N$ is a CAP-subalgebra of L, by [10, Lemma 2.5], so $((H \cap C) + N)/N$ is a CAP-subalgerbra of L/N, thanks to [10, Lemma 2.1(v)]. Therefore H/N is a weak $c^{\#}$-ideal of L/N.
Conversely, if H/N is a weak $c^{\#}$-ideal of L/N, then there is a subideal C/N of L/N with $L/N = H/N + C/N = (H + C)/N$ and $H/N \cap C/N = (H \cap C)/N$ is a CAP-subalgebra of L/N. Therefore $L = H + C$ and $H \cap C$ is a CAP-subalgebra of L, by [10, Lemma 2.1(v)]. □

Lemma 2.2. *Let L be a Lie algebra, N an abelian minimal ideal of L and M a maximal subalgebra of N. If M is a weak ϕ-ideal or weak $c^{\#}$-ideal of L, then $dimN = 1$.*

Proof. Since M is a weak ϕ-ideal or a weak $c^{\#}$-ideal of L, there is an subideal C of L such that $L = M + C$ and $M \cap C \leq \phi(M)$ or $M \cap C$ is a CAP-subalgebra of L. Now by using [2, Lemma 4.2], we have $N \subseteq C$ or $[N, L] = 0$. In the former case, $M \subseteq N \subseteq C$ and so $M = M \cap C \leq \phi(M)$, which is impossible, or $M = M \cap C$ is a CAP-subalgebra of L and so covers or avoids $N/\{0\}$. But M cannot cover N. Therefore $M \cap N = M \cap \{0\}$ which concludes that $dimN = 1$. If $[N, L] = 0$, then the result holds, trivially. □

Theorem 2.3. *If K is a subalgebra of L and H is a weak $c^{\#}$-ideal of L with $H \leq \phi(K)$, then H is a CAP-subalgebra of L.*

Proof. Since H is a weak $c^{\#}$-ideal of L, there exists a subideal N of L such that $L = H+N$ and $H \cap N$ is a CAP-subalgebra of L. Also, $K = H+(K \cap N)$. Now, by using of [8, Lemma 2.1], we conclude that $K = K \cap N$ and so $H \subseteq K \subseteq N$. Hence $L = N$ and $H = H \cap N$ is a CAP-subalgebra of L. □

Theorem 2.4. *Let L be a Lie algebra. Then the one-dimensional subalgebra Fx of L is a weak ϕ-ideal of L if and only if it is a ϕ-ideal of L.*

Proof. Let Fx be a weak ϕ-ideal of L. Then there is a subideal K of L with $L = Fx + K$ and $Fx \cap K \leq \phi(Fx) = 0$. Thus K has codimension one in L. Therefore K is an ideal of L and so Fx is a ϕ-ideal of L. The coverse follows from the fact that every ϕ-ideal of L is a weak ϕ-ideal of L. □

In [2, Corollary 3.3], Ciloglu and Towers proved that over a field of characteristic zero, L is solvable if and only if every maximal subalgebra of L is a weak c-ideal. Is the same theorem true for weak ϕ-ideals? In the following, we give an example of a solvable Lie algebra that has a maximal subalgebra that is not a weak ϕ-ideal.

Example 2.5. Consider the real Lie algebra $A_{5,12} = span\{e_1, e_2, e_3, e_4, e_5\}$, with nonzero multiplications $[e_1, e_5] = e_1$, $[e_2, e_5] = e_1 + e_2$, $[e_3, e_5] = e_2 + e_3$, $[e_4, e_5] = e_3 + e_4$. (See [5]). If we put $M = span\{e_1, e_2, e_3, e_4\}$, then M is a maximal subalgebra of $A_{5,12}$ and M is not a weak ϕ-ideal of L.

3. Main results

Lemma 3.1. ([9, Lemma 4.1]) *Let L be a Lie algebra over any field F, let A be an ideal of L, and let U/A be a maximal nilpotent subalgebra of L/A. Then $U = C + A$, where C is a maximal nilpotent subalgebra of L.*

Proposition 3.2. *Let L be a Lie algebra, in which all maximal subalgebras of each maximal nilpotent subalgebra of L are weak ϕ-ideals or weak $c^{\#}$-ideals of L. If A is a minimal ideal of L, then all maximal subalgebras of each maximal nilpotent subalgebra of L/A are weak ϕ-ideals or weak $c^{\#}$-ideals of L/A.*

Proof. We suppose that U/A is a maximal nilpotent subalgebra of L/A. Then $U = C + A$, where C is a maximal nilpotent subalgebra of L, by Lemma 3.1. If B/A is a maximal subalgebra of U/A, then $B = B \cap (C + A) = (B \cap C) + A = D + A$, where D is a maximal subalgebra of C and $B \cap C \leq D$. Now
(i) If D is a weak ϕ-ideal of L, then there exists a subideal K of L with $L = D + K$ and $D \cap K \leq \phi(D)$. If $A \leq K$, $L/A = (D+A)/A + K/A = B/A + K/A$ and $(D+A)/A \cap K/A = ((D \cap K) + A)/A \leq (\phi(D) + A)/A \cong \phi(B/A)$. So the result holds.
If $A \nleq K$, then $[L, A] = 0$ by [2, Lemma 4.2]. Hence $A \subseteq Z(L)$ and so $A \leq C$ and $B = D$. Consequently, we have $L = B + K$ and $B \cap K \leq \phi(B)$. Hence

$L/A = B/A + (K + A)/A$ and

$$B/A \cap (K + A)/A = ((B \cap K) + A)/A \leq (\phi(B) + A)/A \leq \phi(B/A).$$

Therefore B/A is a weak ϕ-ideal of L/A.
(ii) If D is a weak $c^{\#}$-ideal of L, then there exists a subideal K of L with $L = D + K$ and $D \cap K$ is a CAP-subalgebra of L. Therefore $D \cap K$ covers or avoids $A/\{0\}$. If $D \cap K + A = D \cap K$, then $A \subseteq D \cap K \subseteq D$ and so $B = D$. It follows from Lemma 2.1(iii) that B/A is a weak $c^{\#}$-ideal of L/A and so the result holds. If $D \cap K \cap A = 0$, then by [2, Lemma 4.2] we consider two cases:
1. $A \leq K$: In this case, $L/A = (D + A)/A + K/A = B/A + K/A$ and $(D + A)/A \cap K/A = ((D \cap K) + A)/A$. Since $D \cap K$ is a CAP-subalgebra of L and A is an ideal of L, then by [10, Lemma 2.5], $(D \cap K) + A$ is a CAP-subalgebra of L and so $((D \cap K) + A)/A$ is a CAP-subalgebra of L/A, thanks to [10, Lemma 2.1(v)]. Thus B/A is a weak $c^{\#}$-ideal of L/A.
2. $[A, L] = 0$: In this case, $A \subseteq Z(L)$. Consequently, $U = C + A$ is a nilpotent subalgebra of L and so we must have $C + A = C$. Therefore $A \leq C$ and $B = D$ and B/A is a weak $c^{\#}$-ideal of L/A, thanks to Lemma 2.1(iii). □

Finally, we obtain a condition implying a Lie algebra L to be supersolvable.

Theorem 3.3. *Let L be a solvable Lie algebra, in which all maximal subalgebras of each maximal nilpotent subalgebra of L are a weak ϕ-ideal or a weak $c^{\#}$-ideal of L. Then L is supersolvable.*

Proof. Let L be a minimal counterexample and A be a minimal ideal of L. Then by the previous proposition, L/A satisfies the hypothesis of this theorem and so L/A is supersolvable. It is sufficient to show that $dimA = 1$. If there is another ideal I of L, then $A \cong (A + I)/I \leq L/I$ and so $dimA = 1$ and L is supersolvable, a contradiction.
Therefore, we suppose that A is a unique minimal ideal of L. If $A \leq \phi(L)$, then $L/\phi(L)$ is supersolvable and so L is supersolvable by [1, Theorem 7], a contradiction. If $A \nleq \phi(L)$, then there is a maximal subalgebra M of L such that $L = A \dot{+} M$. Now, if C is a maximal nilpotent subalgebra of L with $A \leq C$, then we consider two cases:
1. $A = C$: In this case, A is a maximal nilpotent subalgebra of L and so by the assumption, every maximal subalgebra of A is a weak ϕ-ideal or weak $c^{\#}$-ideal of L. Hence $dimA = 1$, thanks to Lemma 2.2.
2. $A < C$: In this case, we have $C = A + (C \cap M)$. Now, let B be a maximal subalgebra of C that contains $C \cap M$. Then if B is a weak $c^{\#}$-ideal of L, then there is a subideal K of L such that $L = B + K$ and $B \cap K$ is a CAP-subalgebra of L. Therefore $B \cap K$ covers or avoids $A/\{0\}$. If $(B \cap K) + A = B \cap K$, then $A \leq B \cap K \leq B$ and so $C \leq B$, a contradiction.
If $B \cap K \cap A = \{0\}$, then by noting to [2, Lemma 4.2], if $A \subseteq K$, then $B \cap A = \{0\}$ and $C = B \dot{+} A$. Thus $C/B \cong A$ and consequently $dimA = 1$

and L is supersolvable, a contradiction. If $[L, A] = 0$, then clearly $dimA = 1$, a contradiction.

Now, if $A < C$ and B is a maximal subalgebra of C that contains A and if we suppose that B is a weak ϕ-ideal of L, then there is a subideal K of L with $L = B + K$ and $B \cap K \leq \phi(B)$. Since A is a minimal ideal of L, so $A \subseteq K$ or $[L, A] = 0$, by [2, Lemma 4.2]. If $A \subseteq K$, then $A \subseteq B \cap K \leq \phi(B)$. Thus we have

$$B = B \cap L = B \cap (A + M) = A + (B \cap M) = B \cap M.$$

Therefore $A \subseteq B \subseteq M$, a contradiction. If $[L, A] = 0$, then $dimA = 1$ and L is supersolvable, a contradiction. □

Theorem 3.4. *Let L be a Lie algebra over a field of characteristic zero.*
(i) If every maximal subalgebra of $N(L)$ is a weak ϕ-ideal of L, then L is solvable.
(ii) If every maximal subalgebra of $N(L)$ is a weak $c^{\#}$-ideal of L, then L is supersolvable.

Proof. (i) We suppose that L be a minimal counterexample. If $\phi(L) \neq 0$, then we can easily show that L is solvable, a contradiction. If $\phi(L) = 0$, then $N(L) = A_1 \oplus A_2 \oplus ... \oplus A_n$ such that A_i's are minimal abelian ideals of L, by [7, Theorem 3]. If A_i^* is a maximal subalgebra of A_i, then $M = A_i^* + (A_1 \oplus ... \oplus \hat{A}_i \oplus ... \oplus A_n)$ is a maximal subalgebra of $N(L)$. So by the assumption, there is a subideal C of L such that $L = M + C$ and $M \cap C \leq \phi(M) = 0$. Therefore M is a weak c-ideal of L. Consequently L is solvable, by [2, Theorem 3.6].

(ii) Again, we suppose that L be a minimal counterexample. If $\phi(L) \neq 0$, then we can easily show that L is supersolvable, a contradiction. Now, we suppose thet $\phi(L) = 0$. Then $N(L) = A_1 \oplus A_2 \oplus ... \oplus A_n$ such that A_i's are minimal abelian ideals of L, by [7, Theorem 3]. If A_i^* is a maximal subalgebra of A_i, then $M = A_i^* + (A_1 \oplus ... \oplus \hat{A}_i \oplus ... \oplus A_n)$ is a maximal subalgebra of $N(L)$. So by the assumption, there is a subideal K of L such that $L = M + K$ and $M \cap K$ is a CAP-subalgebra of L. Hence, $M \cap K$ covers or avoids $\dfrac{A_i}{\{0\}}$. If $M \cap K + A_i = M \cap K$, then $A_i \subseteq M \cap K \subseteq M$, a contradiction. If $M \cap K \cap A_i = 0$, then $A_i^* \cap K = 0$. Since A_i is a minimal ideal of L, either $A_i \subseteq K$ or $[L, A_i] = 0$, by [2, Lemma 4.2]. If $A_i \subseteq K$, then $A_i^* = 0$ and so $\dim A_i = 1$. But, if $[L, A_i] = 0$, then $[L, A_i^*] = 0$. Thus, A_i^* is an ideal of L and so $A_i^* = 0$. Therefore, $\dim A_i = 1$.

Now, we put $K_i = A_1 \oplus A_2 \oplus \cdots \oplus A_i$, $i = 1, \cdots, n$. It follows that $N(L)$ has a series $0 \leq K_1 \leq \cdots \leq K_n = N(L)$ such that $\dim(K_i/K_{i-1}) = 1$, for $i = 1, \cdots, n$. Therefore, L is supersolvable, by [6, Corollary 2.9], a contradiction. □

Finally, we bring an example that shows that the converse of the first part of the above theorem is not valid.

Example 3.5. Consider the Lie algebra $L = A^{a,b,c}_{5,7} = span\{e_1, e_2, e_3, e_4, e_5\}$, with nonzero multiplications $[e_1, e_5] = e_1$, $[e_2, e_5] = ae_2$, $[e_3, e_5] = be_3$ and $[e_4, e_5] = ce_4$, where $abc \neq 0$, $-1 \leq c \leq b \leq a \leq 1$ (See [5]). There is a maximal subalgebra $M = span\{e_1, e_2, e_3\}$ of $N(L) = span\{e_1, e_2, e_3, e_4\}$ such that M is not a weak ϕ-ideal, while L is a solvable Lie algebra.

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