Oscillation behavior for a n-dimensional system of coupled van der Pol oscillators with delays

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Original Research Article

Abstract: In this paper, the oscillatory behavior of the solutions for a n-dimensional system of coupled van der Pol oscillators with delays is investigated. We extend the result in the literature from mathematical point of view. Some sufficient conditions to guarantee the oscillation of the solutions are provided and computer simulations are given to support the present criteria.

Keywords: coupled van der Pol oscillator, delay, instability, oscillation,

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1 Introduction

The van der Pol oscillator which is known to be a primary model for describing self-excited oscillations observed in various biological, physical and engineering systems [1-11]. Recently, Sathiyadevi *et al*. have investigated the following coupled van der Pol oscillators [12]:

$$\begin{cases} x_i' = y_i + \varepsilon_1[\bar{x} - x_i], i = 1, 2, \cdots, n, \\ y_i' = \alpha(1 - x_i^2)y_i - x_i - \varepsilon_2[\bar{y} - y_i]. \end{cases} \tag{1}$$

where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i, \bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i,\ \alpha, \varepsilon_1, \varepsilon_2$ are parameters. For $n = 2$ and $n = 3$, the authors have demonstrated frustration induced transient chaos and multistability, the competing attractive and repulsive coupling plays a crucial role. Bukh and Anishchenko have studied numerically autowave structures in a two-dimensional lattice of van der Pol oscillators with 100×100 elements for the cases of local and nonlocal coupling between the individual oscillators [13]:

$$\begin{cases} x_{i,j}' = y_{i,j} + \frac{\sigma}{B_{i,j}} \sum_{m,n}(x_{m,n} - x_{i,j}), \\ y_{i,j}' = \varepsilon(1 - x_{i,j}^2)y_{i,j} - \omega^2 x_{i,j}. \end{cases} \tag{2}$$

where $m, n \in \{1, 2, \cdots, 100\}$. The classical spiral and target waves which were observed in modeling and analyzing the dynamics of the heart muscle can be realized in the network. Therefore, those obtained results can be useful in cardiology.

Time delayed differential equation is very important tool to study many models such as van der Pol oscillators. More realistic van der pol time delay models should be taken into account the effect of time delays [14-19]. For example, Zhang and Gu have investigated the existence of the Hopf bifurcation for the following system [14]:

$$\begin{cases} x_1'' + \varepsilon(x_1^2 - 1)x_1' + (1+\alpha)x_1 = \alpha y_1'(t - \tau_2), \\ y_1'' + \varepsilon(y_1^2 - 1)y_1' + (1+\alpha)y_1 = \alpha x_1'(t - \tau_1). \end{cases} \tag{3}$$

The stability and direction of the Hopf bifurcation were determined by using the normal form theory and the center manifold theorem. Zhang *et al*. have studied the dynamic behavior for the following symmetric system [15]:

$$\begin{cases} x_1'' + x_1 - \varepsilon(1 - x_1^2)x_1' = k[x_2'(t-\tau) - x_1'(t-\tau)] + k[x_3'(t-\tau) - x_1'(t-\tau)], \\ x_2'' + x_2 - \varepsilon(1 - x_2^2)x_2' = k[x_3'(t-\tau) - x_2'(t-\tau)] + k[x_1'(t-\tau) - x_2'(t-\tau)], \\ x_3'' + x_3 - \varepsilon(1 - x_3^2)x_3' = k[x_1'(t-\tau) - x_3'(t-\tau)] + k[x_2'(t-\tau) - x_3'(t-\tau)]. \end{cases} \tag{4}$$

The dynamics of system (3) was carried out using a symmetric Hopf bifurcation result. Some important results about the spontaneous bifurcations of multiple branches of periodic solutions and their spatio-temporal patterns such as mirror-reflecting waves, standing waves, and discrete waves which describe the oscillatory mode of each oscillator were obtained.

Motivated by the above models, we extend model (1) to the following time delay system:

$$\begin{cases} x_i' = b_i y_i + \sum_{j=1, j\neq i}^{n} k_{ij}[x_j(t-\tau_j) - x_i(t-\tau_i)], \\ y_i' = \alpha_i(1 - x_i^2)y_i - c_i x_i - \sum_{j=1, j\neq i}^{n} r_{ij}[y_j(t-\theta_j) - y_i(t-\theta_i)], i = 1, 2, \cdots, n. \end{cases} \tag{5}$$

where $b_i, c_i, \alpha_i, k_{ij}, r_{ij}$ are parameters, time delays $\tau_j \geq 0, \theta_j \geq 0 (j = 1, 2, \cdots, n)$. Noting that there are $2n$ time delays in system (5), the general bifurcating method is hard to deal with this system. In the present paper, we will use mathematical analysis method to investigate the existence of oscillatory solution for system (5).

2 Preliminaries

The system (5) can be expressed in the following matrix form:

$$z'(t) = Az(t) + Bz(t - \tau) + f(z(t)) \tag{6}$$

where $z = (x_1, y_1, x_2, y_2, \cdots, x_n, y_n)^T$, $z(t-\tau) = [x_1(t-\tau_1), y_1(t-\theta_1), x_2(t-\tau_2), y_2(t-\theta_2), \cdots, x_n(t-\tau_n), y_n(t-\theta_n)]^T$, A and B both are $2n$ by $2n$ matrices, and $f(z(t))$ is a $2n$ by one vector:

$$A = (a_{ij})_{2n\times 2n} = \begin{pmatrix} 0 & b_1 & 0 & 0 & ... & 0 & 0 \\ -c_1 & \alpha_1 & 0 & 0 & ... & 0 & 0 \\ 0 & 0 & 0 & b_2 & ... & 0 & 0 \\ 0 & 0 & -c_2 & \alpha_2 & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & 0 & b_n \\ 0 & 0 & 0 & 0 & ... & -c_n & \alpha_n \end{pmatrix},$$

$$B = (b_{ij})_{2n\times 2n} = \begin{pmatrix} \rho_1 & 0 & k_{12} & 0 & ... & k_{1n} & 0 \\ 0 & \rho_2 & 0 & -r_{12} & ... & 0 & -r_{1n} \\ k_{21} & 0 & \rho_3 & 0 & ... & k_{2n} & 0 \\ 0 & -r_{21} & 0 & \rho_4 & ... & 0 & -r_{2n} \\ ... & ... & ... & ... & ... & ... & ... \\ k_{n,1} & 0 & k_{n,2} & 0 & ... & \rho_{2n-1} & 0 \\ 0 & -r_{n,1} & 0 & -r_{n,2} & ... & 0 & \rho_{2n} \end{pmatrix},$$

$$f(z) = [0, -\alpha_1 x_1^2 y_1, 0, -\alpha_2 x_2^2 y_2, \cdots, 0, -\alpha_n x_n^2 y_n]^T.$$

where $\rho_1 = -\sum_{j=2}^{n}(k_{1j}), \rho_2 = \sum_{j=2}^{n}(r_{1j}), \cdots, \rho_{2n-1} = -\sum_{j=1}^{n-1}(k_{nj}), \rho_{2n} = \sum_{j=1}^{n-1}(r_{nj})$. The linearized system of (6) is

$$z'(t) = Az(t) + Bz(t-\tau) \tag{7}$$

Lemma 1 If matrix $S\ (= A + B)$ is a nonsingular matrix for selected parameters, then there exists a unique equilibrium point for system (5) (or (6)).

Proof Assume that $z^* = [x_1^*, y_1^*, x_2^*, y_2^*, \cdots, x_n^*, y_n^*]^T$ is an equilibrium point of system (5), then we have the following algebraic equation

$$\begin{cases} b_1 y_1^* + \sum_{j=2}^{n} k_{1j}(x_j^* - x_1^*) = 0, \\ \alpha_1[1-(x_1^*)^2]y_1^* - c_1 x_1^* - \sum_{j=2}^{n} r_{1j}(y_j^* - y_1^*) = 0, \\ b_2 y_2^* + \sum_{j=1, j\neq 2}^{n} k_{2j}(x_j^* - x_2^*) = 0, \\ \alpha_2[1-(x_2^*)^2]y_2^* - c_2 x_2^* - \sum_{j=1, j\neq 2}^{n} r_{1j}(y_j^* - y_2^*) = 0, \\ .... \\ b_n y_n^* + \sum_{j=1}^{n-1} k_{nj}(x_j^* - x_1^*) - x_n^*) = 0, \\ \alpha_n[1-(x_n^*)^2]y_n^* - c_n x_n^* - \sum_{j=1}^{n-1} r_{nj}(y_j^* - y_n^*) = 0, \end{cases} \tag{8}$$

From (7) we get

$$
\begin{cases}
b_1y_1^* + \sum_{j=2}^n k_{1j}(x_j^* - x_1^*) = 0, \\
\alpha_1 y_1^* - c_1x_1^* - \sum_{j=2}^n r_{1j}(y_j^* - y_1^*) = \alpha_1(x_1^*)^2y_1^*, \\
b_2y_2^* + \sum_{j=1,j\neq 2}^n k_{2j}(x_j^* - x_2^*) = 0, \\
\alpha_2 y_2^* - c_2x_2^* - \sum_{j=1,j\neq 2}^n r_{1j}(y_j^* - y_2^*) = \alpha_2(x_2^*)^2y_2^*, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
b_ny_n^* + \sum_{j=1}^{n-1} k_{nj}(x_j^* - x_n^*) = 0, \\
\alpha_n y_n^* - c_nx_n^* - \sum_{j=1}^{n-1} r_{nj}(y_j^* - y_n^*) = \alpha_n(x_n^*)^2y_n^*.
\end{cases}
\tag{9}
$$

Now we first consider the correspondent homogeneous system of (9):

$$
\begin{cases}
b_1y_1^* + \sum_{j=2}^n k_{1j}(x_j^* - x_1^*) = 0, \\
\alpha_1 y_1^* - c_1x_1^* - \sum_{j=2}^n r_{1j}(y_j^* - y_1^*) = 0, \\
b_2y_2^* + \sum_{j=1,j\neq 2}^n k_{2j}(x_j^* - x_2^*) = 0, \\
\alpha_2 y_2^* - c_2x_2^* - \sum_{j=1,j\neq 2}^n r_{1j}(y_j^* - y_2^*) = 0, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
b_ny_n^* + \sum_{j=1}^{n-1} k_{nj}(x_j^* - x_n^*) = 0, \\
\alpha_n y_n^* - c_nx_n^* - \sum_{j=1}^{n-1} r_{nj}(y_j^* - y_n^*) = 0.
\end{cases}
\tag{10}
$$

The matrix form of system (10) is

$$(A+B)z^* = Sz^* = \mathbf{0} \tag{11}$$

According to the basic linear algebraic knowledge, system (11) (or (10)) has a unique trivial solution since $S = A + B$ is a nonsingular matrix. Namely, system (11)(or (10)) has a unique solution $z^* = (0, 0, \cdots, 0)^T$. From systems (9) and (10) we know that all the $(2i-1)$, $i = 1, 2, \cdots, n$ equations are the same. It is easy to know that system (9) also has a unique trivial solution. The proof is completed.

Lemma 2 All solutions of system (5) (or (6)) are bounded assume that $\alpha_i > 0, i = 1, 2, \cdots, n$.

Proof To prove the boundedness of the solutions in system (5), we construct a Lyapunov function $V(t) = \sum_{i=1}^n \frac{1}{2}[x_i^2(t) + y_i^2(t)]$. Calculating the derivative of $V(t)$ through system (5) we get

$$
\begin{aligned}
V'(t)|_{(5)} &= \sum_{i=1}^n [x_i'(t)x_i(t) + y_i'(t)y_i(t)] \\
&= \sum_{i=1}^n (b_i - c_i)x_iy_i + \sum_{i=1}^n \sum_{j=1,j\neq i}^n k_{ij}[x_j(t-\tau_j) - x_i(t-\tau_i)]x_i + \sum_{i=1}^n \alpha_i y_i^2 \\
&\quad - \sum_{i=1}^n \sum_{j=1,j\neq i}^n r_{ij}[y_j(t-\theta_j) - y_i(t-\theta_i)]y_i - \sum_{i=1}^n \alpha_i x_i^2 y_i^2
\end{aligned}
\tag{12}
$$

Noting that $\alpha_i (i = 1, 2, \cdots, n)$ are positive constants. Obviously, when $x_i \to +\infty, y_i \to \infty (1 \le i \le n)$, $x_i^2 y_i^2$ are higher order infinity than x_i^2, y_i^2, and $x_i x_j$, respectively. Therefore, there exists suitably large $K > 0$ such that $V'(t)|_{(5)} < 0$ as $|x_i| > K, |y_i| > K$. This means that all solutions of system (5) are bounded.

3 The existence of oscillatory solutions

Theorem 1 Assume that zero is the unique equilibrium point of system (5) (or(6)) for selecting parameter values. Let $\alpha_1, \alpha_2, \cdots, \alpha_{2n}$ and $\beta_1, \beta_2, \cdots, \beta_{2n}$ be characteristic values of matrix A and B, respectively. If the real part of each α_i and $\beta_i (i = 1, 2, \cdots, 2n)$ is negative, then the trivial solution of system (5) is stable. If there exists some $Re(\alpha_k) > 0$, with $Re(\alpha_k) > |Re(\beta_k)| + |Im(\beta_k)|$ or $Re(\alpha_k) < 0$ with $|Re(\alpha_k)| < Re(\beta_k)$, then the unique equilibrium point of system (5) is unstable. System (5) generates an oscillatory solution.

Proof According to the time delay basic differential equation theory, if the real part of each α_i and β_i $(i = 1, 2, \cdots, 2n)$ is negative, then the trivial solution of system (7) is stable. Noting that the nonlinear term $f(z)$ is a higher order infinitesimal as $x_i \to 0$ and $y_i \to 0$. Therefore, the stability of the trivial solution of system (7) ensures the stability of the trivial solution of system (5) (or (6)). Obviously, if the trivial solution of system (7) is unstable, then the trivial solution of system (5)(or (6)) is also unstable. Therefore, in order to discuss the instability of the trivial solution of system (5) (or (6)) we only need to deal with the instability of the trivial solution of system (7). Consider an auxiliary system of (7) as follows:

$$z'(t) = Az(t) + Bz(t - \tau_*) \tag{13}$$

where $z(t - \tau_*) = [x_1(t - \tau_*), y_1(t - \tau_*), x_2(t - \tau_*), y_2(t - \tau_*), \cdots, x_n(t - \tau_*), y_n(t - \tau_*)]^T$, $\tau_* = \min_{1 \le i \le n}\{\tau_i, \theta_i\}$. If the trivial solution of system (13) is unstable, then the trivial solution of system (7) is unstable according to the property of delayed differential equation [24]. So we only need to show the instability of the trivial solution of system (13). Since α_i and β_i $(i = 1, 2, \cdots, 2n)$ are characteristic values of matrix A and B, respectively, then the characteristic equation corresponding to system (13) is the following:

$$\Pi_{i=1}^{2n}(\lambda - \alpha_i - \beta_i e^{-\lambda \tau_*}) = 0 \tag{14}$$

So, we are led to an investigation of the nature of the roots for some $k, k \in \{1, 2, \cdots, 2n\}$:

$$\lambda - \alpha_k - \beta_k e^{-\lambda \tau_*} = 0 \tag{15}$$

System (15) is a transcendental equation which is hard to find all solutions for the equation. However, we show that there exists a positive real part eigenvalue of equation (15) under the assumption of Theorem 1. If $Re(\alpha_k) > 0$, with $Re(\alpha_k) > |Re(\beta_k)|+|Im(\beta_k)|$ hold, let $\lambda = \sigma+i\gamma, \alpha_k = \alpha_{k1}+i\alpha_{k2}, \beta_k = \beta_{k1} + i\beta_{k2}$, where $\sigma = Re(\lambda), \gamma = Im(\lambda), \alpha_{k1} = Re(\alpha_k), \alpha_{k2} = Im(\alpha_k), \beta_{k1} = Re(\beta_k), \beta_{k2} = Im(\beta_k)$. Separating the real and imaginary parts from equation (15), we have

$$\sigma = \alpha_{k1} + \beta_{k1}e^{-\sigma\tau_*}\cos(\gamma\tau^*) - \beta_{k2}e^{-\sigma\tau_*}\sin(\gamma\tau_*). \tag{16}$$

$$\gamma = \alpha_{k2} - \beta_{k1}e^{-\sigma\tau_*}\sin(\gamma\tau_*) + \beta_{k2}e^{-\sigma\tau_*}\cos(\gamma\tau_*). \tag{17}$$

We show that equation (15) has a positive real part root. Let

$$\Psi(\sigma) = \sigma - \alpha_{k1} - \beta_{k1}e^{-\sigma\tau_*}\cos(\gamma\tau_*) + \beta_{k2}e^{-\sigma\tau_*}\sin(\gamma\tau_*). \tag{18}$$

Obviously, $\Psi(\sigma)$ is a continuous function of σ. Noting that $\alpha_{k1} > |\beta_{k1}| + |\beta_{k2}|$. Then $\Psi(0) = -\alpha_{k1} - \beta_{k1}\cos(\gamma\tau_*) + \beta_{k2}\sin(\gamma\tau_*) \leq -\alpha_{k1} + |\beta_{k1}| + |\beta_{k2}| < 0$. Since $\lim_{\sigma\to+\infty} e^{-\sigma\tau_*} = 0$, so there exists a suitably large σ say $\sigma_1(> 0)$ such that $\Psi(\sigma_1) = \sigma_1 - \alpha_{k1} - \beta_{k1}e^{-\sigma_1\tau_*}\cos(\gamma\tau_*) + \beta_{k2}e^{-\sigma_1\tau_*}\sin(\gamma\tau_*) > 0$. By means of the Intermediate Value Theorem, there exists a $\sigma_0 \in (0, \sigma_1)$ such that $\Psi(\sigma_0) = \sigma_0 - \alpha_{k1} - \beta_{k1}e^{-\sigma_0\tau_*}\cos(\gamma\tau_*) + \beta_{k2}e^{-\sigma_0\tau_*}\sin(\gamma\tau_*) = 0$. This means that the characteristic value λ has a positive real part. Thus, the trivial solution of system (13) is unstable, implying that the trivial solution of system (7) (thus system (5)) is unstable. Since all solutions of system (5) are bounded, the instability of the trivial solution and the boundedness of the solutions will force system (5) to generate an oscillatory solution [25, 26]. For the case of $Re(\alpha_k) < 0$ with $|Re(\alpha_k)| < Re(\beta_k)$, equation (15) will have a positive real part characteristic root. The proof is similar and we omit it.

For simplify, setting $\| B \|= \max_{1\leq j\leq 2n}\sum_{i=1}^{2n}|b_{ij}|, p = \max_{1\leq i\leq n}\{-c_i, b_i + \alpha_i\}$, then we have

Theorem 2 Assume that the conditions of Lemma 1 and Lemma 2 hold. If the following inequality holds:

$$7|p|\tau_*^2 \| B \|> 4e^{|p|\tau_*}. \tag{19}$$

Then the trivial solution of system (13) (thus the system (7)) is unstable, implying that system (5) has a periodic solution.

Proof To prove the instability of the trivial solution of system (13), let $w(t) = \sum_{i=1}^{n}(|x_i(t)| + |y_i(t)|)$. Therefore, $w(t) > 0$ and

$$w'(t) \leq pw(t)+ \| B \| w(t - \tau_*). \tag{20}$$

Specifically, consider equation

$$v'(t) = pv(t)+ \| B \| v(t - \tau_*). \tag{21}$$

If the trivial solution of equation (21) is stable, then the characteristic equation associated with (21) given by

$$\lambda = p + \parallel B \parallel e^{-\lambda\tau_*} \tag{22}$$

must have a real negative root say λ_*, and we have from (22)

$$|\lambda_*| > \parallel B \parallel e^{|\lambda_*|\tau_*} - |p| \tag{23}$$

One can prove that $e^x \geq \frac{7}{4}x^2 (x > 0)$. So we have

$$1 \geq \frac{\parallel B \parallel e^{|\lambda_*|\tau_*}}{|\lambda_*| + |p|} = \frac{\tau_* \parallel B \parallel e^{(|\lambda_*|+|p|)\tau_*}}{(|\lambda_*| + |p|)\tau_* \cdot e^{|p|\tau_*}} = \frac{7\tau_*^2 \parallel B \parallel (|\lambda_*| + |p|)}{4e^{|p|\tau_*}} > \frac{7|p|\tau_*^2 \parallel B \parallel}{4e^{|p|\tau_*}} \tag{24}$$

A contradiction with equation (19), implying that the trivial solution of equation (22) (thus (21)) is unstable. This suggests that the trivial solution of system (20) is unstable. Thus, for all $\{\tau_i, \theta_i\} \geq \tau_*, i = 1, 2, \cdots, n$, the trivial solution of system (13) is unstable, implying that the trivial solution in system (5) is unstable. Similar to theorem 1, system (5) generates an oscillatory solution. The proof is completed.

4 Simulation results

We first consider $n = 4$ in system (5) as the following:

$$\begin{cases} x_1' = b_1 y_1 + \sum_{j=2}^{4} k_{1j}[x_j(t-\tau_j) - x_1(t-\tau_1)], \\ y_1' = \alpha_1(1-x_1^2)y_1 - c_1 x_1 - \sum_{j=2}^{4} r_{1j}[y_j(t-\theta_j) - y_1(t-\theta_1)], \\ x_2' = b_2 y_2 + \sum_{j=1,j\neq 2}^{4} k_{2j}[x_j(t-\tau_j) - x_2(t-\tau_2)], \\ y_2' = \alpha_2(1-x_2^2)y_2 - c_2 x_2 - \sum_{j=1,j\neq 2}^{4} r_{2j}[y_j(t-\theta_j) - y_2(t-\theta_2)], \\ x_3' = b_3 y_3 + \sum_{j=1,j\neq 3}^{4} k_{3j}[x_j(t-\tau_j) - x_3(t-\tau_3)], \\ y_3' = \alpha_3(1-x_3^2)y_3 - c_3 x_3 - \sum_{j=1,j\neq 3}^{4} r_{3j}[y_j(t-\theta_j) - y_3(t-\theta_3)], \\ x_4' = b_4 y_4 + \sum_{j=1}^{3} k_{3j}[x_j(t-\tau_j) - x_4(t-\tau_4)], \\ y_4' = \alpha_4(1-x_4^2)y_4 - c_4 x_4 - \sum_{j=1}^{3} r_{4j}[y_j(t-\theta_j) - y_4(t-\theta_4)]. \end{cases} \tag{25}$$

First the parameters are selected as follows: $b_1 = 1.45, b_2 = 1.48, b_3 = 1.46, b_4 = 1.44, c_1 = 1.75, c_2 = 1.78, c_3 = 1.74, c_4 = 1.76, \alpha_1 = 0.94, \alpha_2 = 0.98, \alpha_3 = 0.96, \alpha_4 = 0.95; k_{12} = 0.125, k_{13} = 0.116, k_{14} = 0.114, k_{21} = 0.115, k_{23} = 0.120, k_{24} = 0.125, k_{31} = 0.124, k_{32} = 0.122, k_{34} = 0.125, k_{41} = 0.112, k_{42} = 0.116, k_{43} = 0.118, r_{12} = 0.64, r_{13} = 0.82, r_{14} = 0.76, r_{21} = 0.65, r_{23} = 0.72, r_{24} = 0.75, r_{31} = 0.64, r_{32} = 0.55, r_{34} = 0.68, r_{41} = 0.62, r_{42} = 0.55, r_{43} = 0.64$. The time delays $\tau_1 = 0.87, \tau_2 =$

$0.90, \tau_3 = 0.88, \tau_4 = 0.86, \theta_1 = 0.86, \theta_2 = 0.92, \theta_3 = 0.94, \theta_4 = 0.92$. Then the characteristic values of matrix A in system (25) are $0.4700 \pm 1.5220i, 0.4900 \pm 1.5474i, 0.4800 \pm 1.5199i, 0.4750 \pm 1.5195i$, the characteristic values of matrix B in system (25) are $2.8627, 2.6731, 2.5161, 0.0137, 0.0037, -0.4542,$ $-0.4748, -0.4928$. Since all characteristic value of matrix B are real numbers, so $Re(\beta_k) = \beta_k$, and $Re(\alpha_5) = 0.4800 > 0.0037 = \beta_5$, the condition of Theorem 1 are satisfied. There exists an oscillatory solution for system (25) (see Fig.1). When we change time delays as $\tau_1 = 1.45, \tau_2 = 1.48, \tau_3 = 1.40, \tau_4 = 1.42, \theta_1 = 1.46, \theta_2 = 1.42, \theta_3 = 1.38, \theta_4 = 1.36$. The other parameters are the same as Fig.1, we see that the oscillatory behavior of the solutions is still maintained (see Fig.2).

Then we consider $n = 9$ in system (5) as the following:

$$\begin{cases} x_1' = b_1 y_1 + \sum_{j=2}^{9} k_{1j}[x_j(t-\tau_j) - x_1(t-\tau_1)], \\ y_1' = \alpha_1(1-x_1^2)y_1 - c_1 x_1 - \sum_{j=2}^{9} r_{1j}[y_j(t-\theta_j) - y_1(t-\theta_1)], \\ x_2' = b_2 y_2 + \sum_{j=1,j\neq 2}^{9} k_{2j}[x_j(t-\tau_j) - x_2(t-\tau_2)], \\ y_2' = \alpha_2(1-x_2^2)y_2 - c_2 x_2 - \sum_{j=1,j\neq 2}^{9} r_{2j}[y_j(t-\theta_j) - y_2(t-\theta_2)], \\ x_3' = b_3 y_3 + \sum_{j=1,j\neq 3}^{9} k_{3j}[x_j(t-\tau_j) - x_3(t-\tau_3)], \\ y_3' = \alpha_3(1-x_3^2)y_3 - c_3 x_3 - \sum_{j=1,j\neq 3}^{9} r_{3j}[y_j(t-\theta_j) - y_3(t-\theta_3)], \\ x_4' = b_4 y_4 + \sum_{j=1,j\neq 4}^{9} k_{4j}[x_j(t-\tau_j) - x_4(t-\tau_4)],, \\ y_4' = \alpha_4(1-x_4^2)y_4 - c_4 x_4 - \sum_{j=1,j\neq 4}^{9} r_{4j}[y_j(t-\theta_j) - y_4(t-\theta_4)], \\ x_5' = b_5 y_5 + \sum_{j=1,j\neq 5}^{9} k_{5j}[x_j(t-\tau_j) - x_5(t-\tau_5)], \\ y_5' = \alpha_5(1-x_5^2)y_5 - c_5 x_5 - \sum_{j=1,j\neq 5}^{9} r_{5j}[y_j(t-\theta_j) - y_5(t-\theta_5)], \\ x_6' = b_6 y_6 + \sum_{j=1,j\neq 6}^{9} k_{6j}[x_j(t-\tau_j) - x_6(t-\tau_6)], \\ y_6' = \alpha_6(1-x_6^2)y_6 - c_6 x_6 - \sum_{j=1,j\neq 6}^{9} r_{6j}[y_j(t-\theta_j) - y_6(t-\theta_6)], \\ x_7' = b_7 y_7 + \sum_{j=1,j\neq 7}^{9} k_{7j}[x_j(t-\tau_j) - x_7(t-\tau_7)],, \\ y_7' = \alpha_7(1-x_7^2)y_7 - c_7 x_7 - \sum_{j=1,j\neq 7}^{9} r_{7j}[y_j(t-\theta_j) - y_7(t-\theta_7)], \\ x_8' = b_8 y_8 + \sum_{j=1,j\neq 8}^{9} k_{8j}[x_j(t-\tau_j) - x_8(t-\tau_8)], \\ y_8' = \alpha_8(1-x_8^2)y_8 - c_8 x_8 - \sum_{j=1,j\neq 8}^{9} r_{8j}[y_j(t-\theta_j) - y_8(t-\theta_8], \\ x_9' = b_9 y_9 + \sum_{j=1}^{8} k_{9j}[x_j(t-\tau_j) - x_9(t-\tau_9)], \\ y_9' = \alpha_9(1-x_9^2)y_9 - c_9 x_9 - \sum_{j=1}^{8} r_{9j}[y_j(t-\theta_j) - y_9(t-\theta_9)]. \end{cases} \tag{26}$$

We select the parameters as $b_1 = 1.55, b_2 = 1.58, b_3 = 1.56, b_4 = 1.52, b_5 = 1.55, b_6 = 1.58, b_7 = 1.56, b_8 = 1.52, b_8 = 1.55, c_1 = 1.45, c_2 = 1.42, c_3 = 1.50, c_4 = 1.48, c_5 = 1.45, c_6 = 1.44, c_7 = 1.46, c_8 = 1.42, c_9 = 1.48, \alpha_1 = 0.92, \alpha_2 = 0.90, \alpha_3 = 0.98, \alpha_4 = 0.97, \alpha_5 = 0.98, \alpha_6 = 0.92, \alpha_7 = 0.92, \alpha_8 = 0.95, \alpha_9 = 0.88; k_{12} = 0.110, k_{13} = 0.111, k_{14} = 0.112, k_{15} = 0.113, k_{16} = 0.114, k_{17} = 0.115, k_{18} = 0.116, k_{19} = 0.118, k_{21} = 0.68, k_{23} = 0.121, k_{24} = 0.122, k_{25} = 0.123, k_{26} = 0.124, k_{27} = 0.125, k_{28} = 0.126, k_{29} = 0.128, k_{31} = 0.67, k_{32} = 0.131, k_{34} = 0.132, k_{35} = 0.133, k_{36} = 0.134, k_{37} = 0.135, k_{38} =$

$0.136, k_{39} = 0.138, k_{41} = 0.66, k_{42} = 0.141, k_{43} = 0.142, k_{45} = 0.143, k_{46} = 0.144, k_{47} = 0.145, k_{48} = 0.146, k_{49} = 0.148, k_{51} = 0.65, k_{52} = 0.121, k_{53} = 0.122, k_{54} = 0.123, k_{56} = 0.124, k_{57} = 0.125, k_{58} = 0.156, k_{59} = 0.158, k_{61} = 0.66, k_{62} = 0.113, k_{63} = 0.114, k_{64} = 0.115, k_{65} = 0.116, k_{67} = 0.117, k_{68} = 0.118, k_{69} = 0.119, k_{71} = 0.68, k_{72} = 0.133, k_{73} = 0.134, k_{74} = 0.135, k_{75} = 0.136, k_{76} = 0.137, k_{78} = 0.138, k_{79} = 0.139, k_{81} = 0.69, k_{82} = 0.093, k_{83} = 0.094, k_{84} = 0.095, k_{85} = 0.096, k_{86} = 0.097, k_{87} = 0.098, k_{89} = 0.099, k_{91} = 0.67, k_{92} = 0.123, k_{93} = 0.124, k_{94} = 0.125, k_{95} = 0.126, k_{96} = 0.127, k_{97} = 0.128, k_{98} = 0.130; r_{12} = -0.61, r_{13} = 0.62, r_{14} = -0.63, r_{15} = 0.64, r_{16} = -0.65, r_{17} = 0.66, r_{18} = -0.67, r_{19} = 0.68, r_{21} = -0.51, r_{23} = 0.52, r_{24} = -0.53, r_{25} = 0.54, r_{26} = -0.55, r_{27} = 0.56, r_{28} = -0.57, r_{29} = 0.58, r_{31} = -0.71, r_{32} = 0.72, r_{34} = -0.73, r_{35} = 0.74, r_{36} = -0.75, r_{37} = 0.76, r_{38} = -0.77, r_{39} = 0.78, r_{41} = -0.64, r_{42} = 0.65, r_{43} = -0.66, r_{45} = 0.67, r_{46} = -0.68, r_{47} = 0.69, r_{48} = -0.70, r_{49} = 0.72, r_{51} = -0.72, r_{52} = 0.73, r_{53} = -0.74, r_{54} = 0.75, r_{56} = -0.76, r_{57} = 0.77, r_{58} = -0.78, r_{59} = 0.79, r_{61} = -0.63, r_{62} = 0.64, r_{63} = -0.65, r_{64} = 0.66, r_{65} = -0.67, r_{67} = 0.68, r_{68} = -0.69, r_{69} = 0.70, r_{71} = -0.52, r_{72} = 0.53, r_{73} = -0.54, r_{74} = 0.55, r_{75} = -0.56, r_{76} = 0.57, r_{78} = -0.58, r_{79} = 0.59, r_{81} = -0.42, r_{82} = 0.43, r_{83} = -0.44, r_{84} = 0.45, r_{85} = -0.46, r_{86} = 0.47, r_{87} = -0.48, r_{89} = 0.49, r_{91} = -0.65, r_{92} = 0.66, r_{93} = -0.57, r_{94} = 0.68, r_{95} = -0.69, r_{96} = 0.70, r_{97} = -0.71, r_{98} = 0.72$. We see that $p = 2.54$, and $\| B \| = 4.448$ in system (26). When time delays are selected as $\tau_1 = 0.45, \tau_2 = 0.48, \tau_3 = 0.40, \tau_4 = 0.42, \tau_5 = 0.45, \tau_6 = 0.48, \tau_7 = 0.40, \tau_8 = 0.54, \tau_9 = 0.45, \theta_1 = 0.46, \theta_2 = 0.42, \theta_3 = 0.38, \theta_4 = 0.36, \theta_5 = 0.46, \theta_6 = 0.42, \theta_7 = 0.52, \theta_8 = 0.48, \theta_9 = 0.48$. Then $\tau_* = 0.36$, and $7|p|\tau_*^2 \| B \| = 10.2495 > 9.9869 = 4e^{|p|\tau_*}$. The condition of Theorem 2 are satisfied. There exists an oscillatory solution for system (26) (see Fig.3). When we change time delays as $\tau_1 = 0.60, \tau_2 = 0.62, \tau_3 = 0.61, \tau_4 = 0.61, \tau_5 = 0.60, \tau_6 = 0.61, \tau_7 = 0.60, \tau_8 = 0.61, \tau_9 = 0.60, \theta_1 = 0.61, \theta_2 = 0.60, \theta_3 = 0.62, \theta_4 = 0.62, \theta_5 = 0.62, \theta_6 = 0.61, \theta_7 = 0.60, \theta_8 = 0.62, \theta_9 = 0.61$. The other parameters are the same as Fig.3, we see that the oscillatory behavior of the solutions is still kept (see Fig.4). It is pointed out that the synchronization of the solutions is appeared (see Fig.5).

5 Conclusion

In this paper, we have discussed the oscillatory behavior of the solutions for a n-dimensional system of coupled van der Pol model with delays. Based on mathematical analysis method, we provided some sufficient conditions to guarantee the oscillation of the solutions. Some simulations are provided to indicate the effectness of the criteria. Also the synchronized behavior of the solutions is appeared.

Competing Interests

Author has declared that no competing interests exist.

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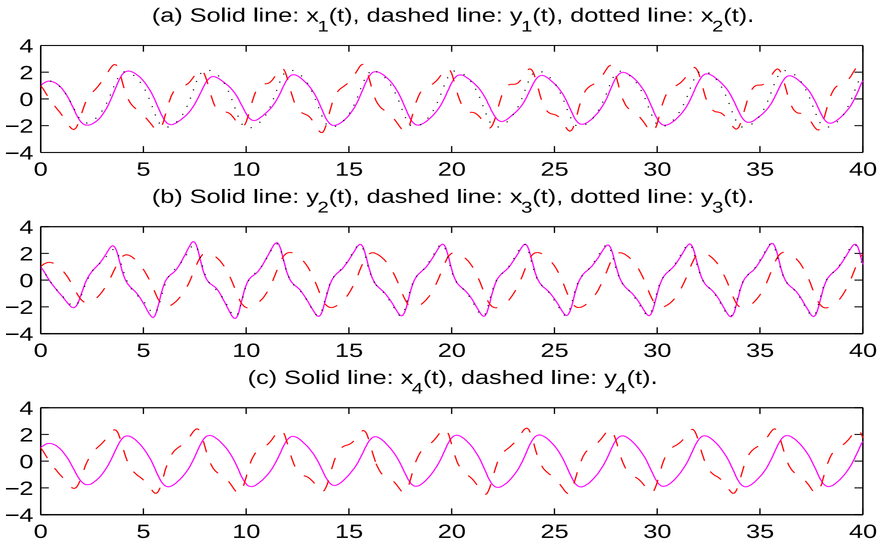

Fig.1 Oscillation of the solutions, delays: 0.87, 0.86, 0.90, 0.92, 0.88, 0.94, 0.86, 0.92.

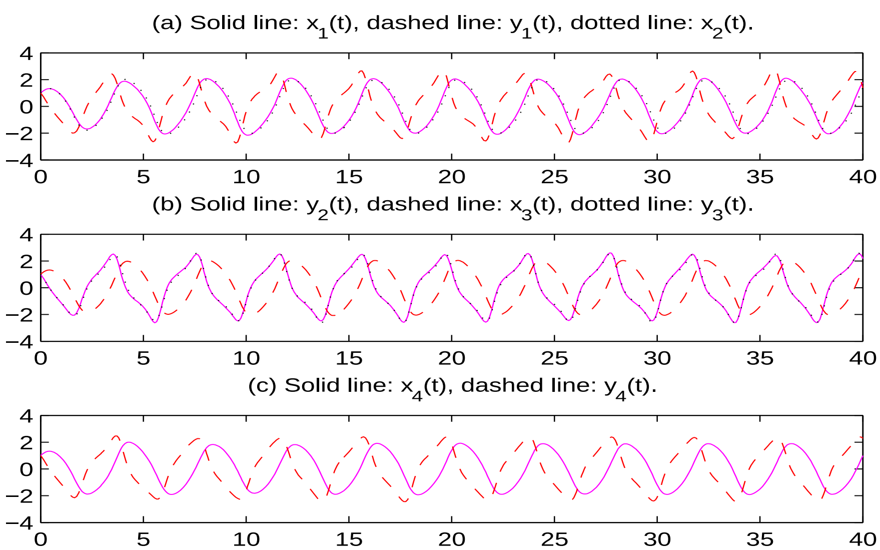

Fig.2 Oscillation of the solutions, delays: 1.45, 1.46, 1.48, 1.42, 1.40, 1.38, 1.42, 1.36.

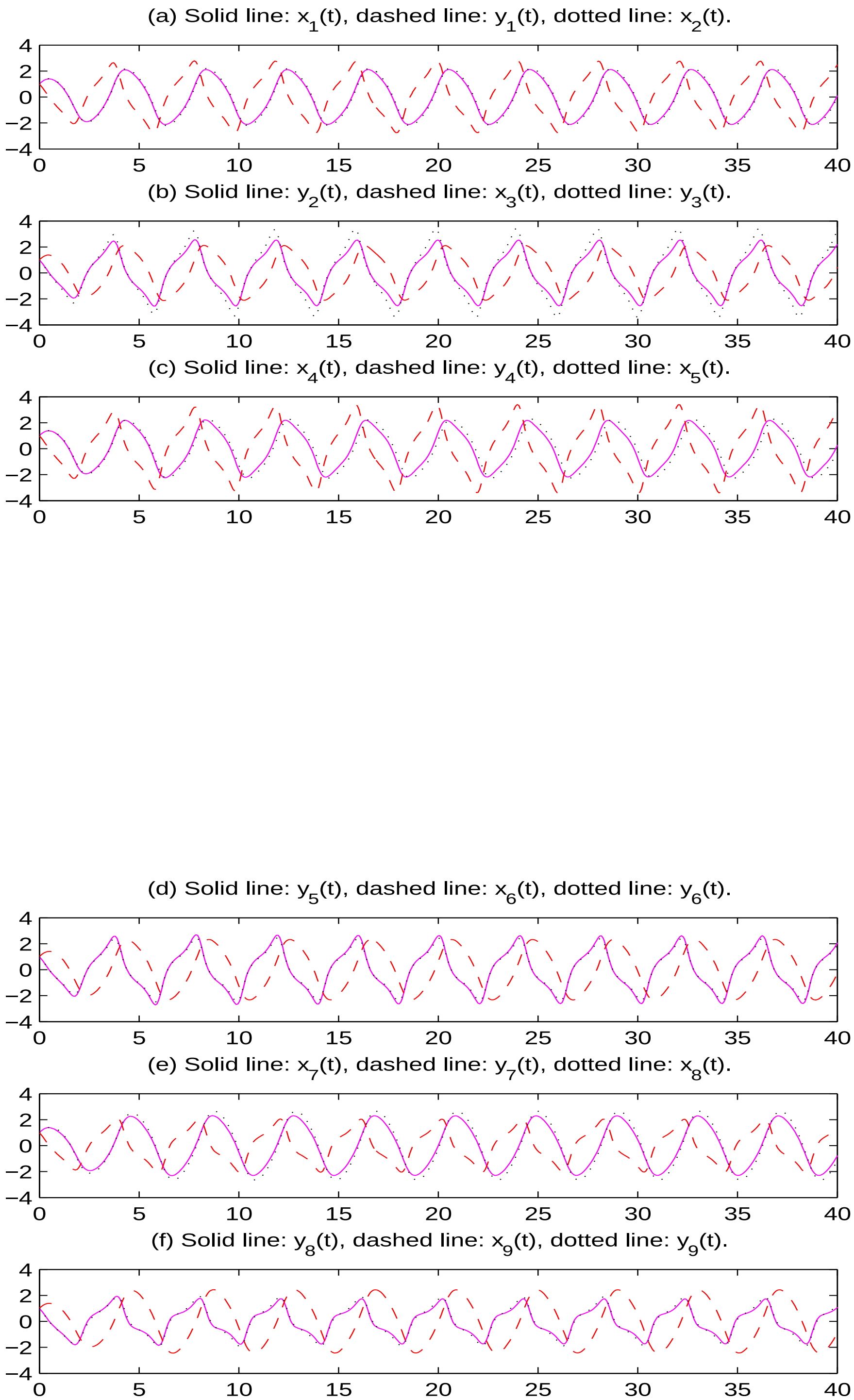

Fig.3 Oscillation of the solutions, minimum delay: 0.36.

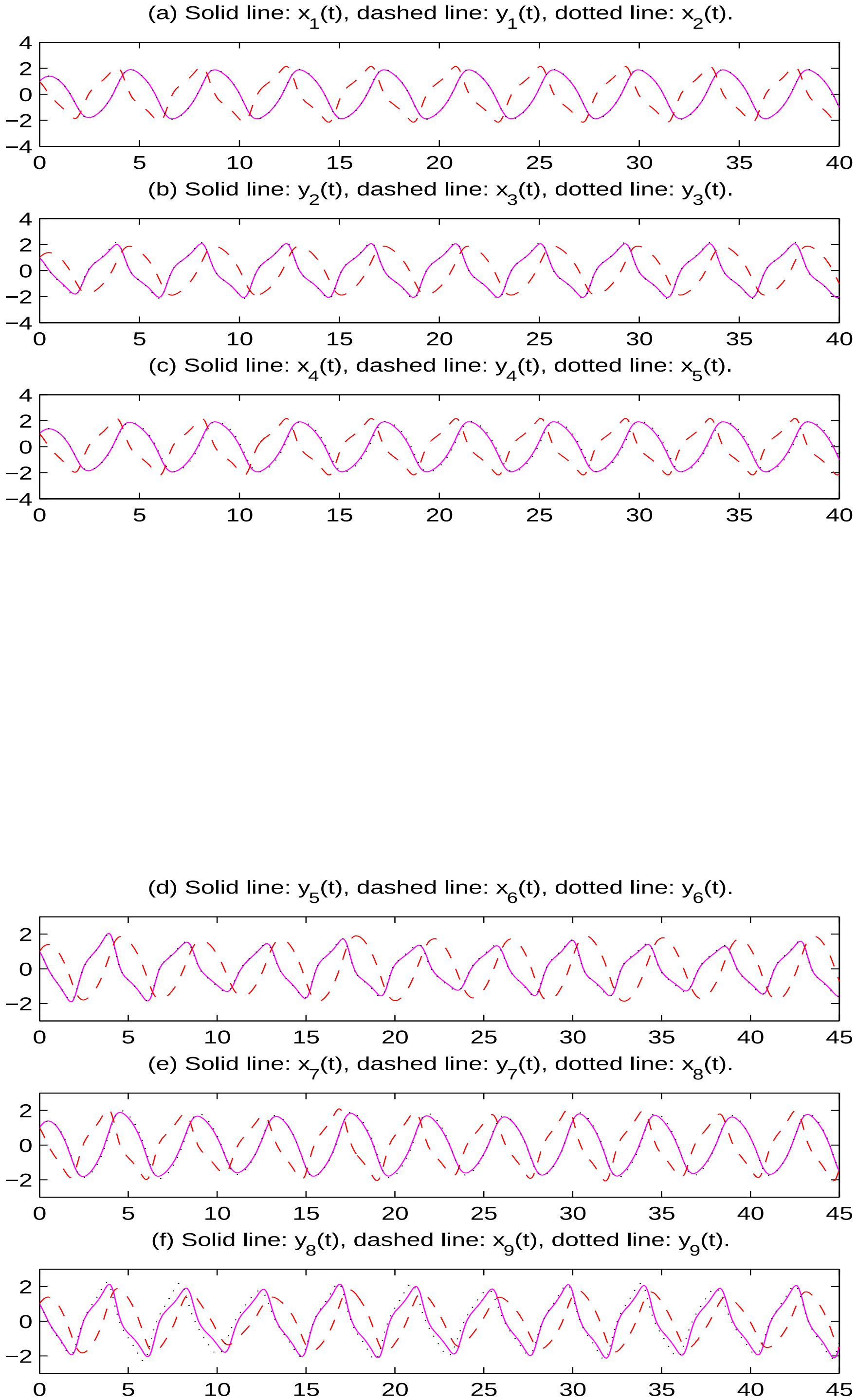

Fig.4 Oscillation of the solutions, minimum delay: 0.60.

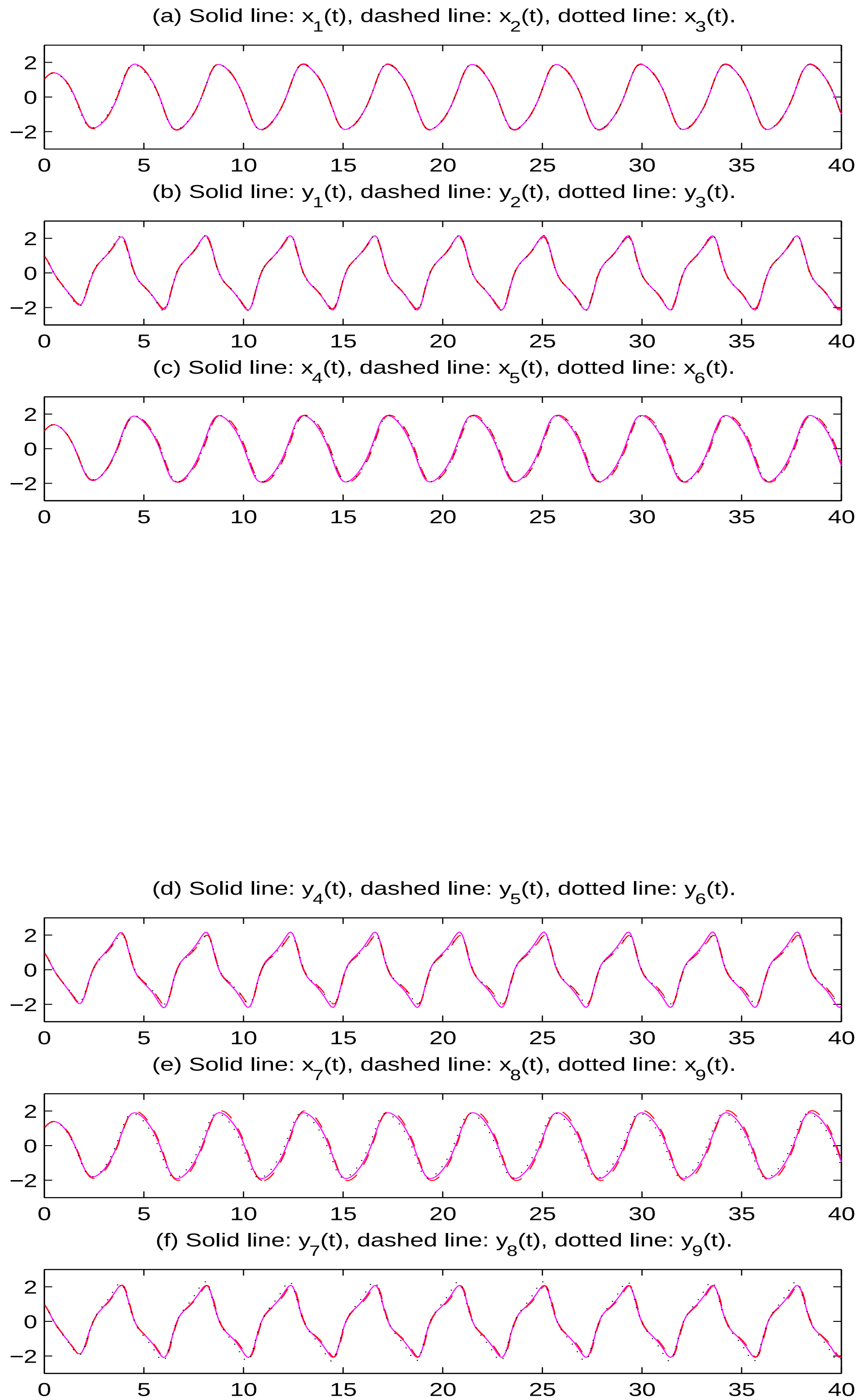

Fig.5 Synchronization of the solutions, minimum delay: 0.60.