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Some Results on Common Fixed Point Using Expansion Mapping in C^*-Algebra Valued Metric Spaces

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Abstract. In the present manuscript, for four weakly compatible mappings in pairs, some theorems using expansion mappings have been established in C^*-algebra valued metric space. Also, we proved the theorems using $(E.A.)$ property and (CLR) property. The proved results are the extension and generalization of several results available in metric spaces. To support the findings, suitable examples are also discussed.

Keywords: Common fixed points, $(E.A.)$ property, (CLR) property, Weakly compatible mappings, C^*-algebra valued metric space.

2010 Mathematics subject classification: 47H10, 54H25, 46J10, 46J15.

1. Introduction

Problems originated in various disciplines of mathematical analysis, such as differential equations, integral equations, mathematical economics, game theory etc., are solved using fixed point theory. The study of fixed point theory for self mappings has been fascinating areas of research for the last few decades. The formal theoretic approach of fixed point was originated from Picard's work. However, it was Stefen Banach [17] who underlined the idea into an abstract framework and provided a constructive tool to establish the fixed points of

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mapping in the metric space.

Various researchers discovered the number of fixed point results in the metric space and several generalized metric spaces using the different definitions and approaches (see [2], [5], [7], [8], [10], [13], [14], [16], [20], [21] and references cites therein). Ma et al. [9] introduced a generalized space called C^*-algebra valued metric space by replacing the set of real numbers with the positive elements of unital C^*-algebra. Later on, Ma et al. [15] introduced compatible and weakly compatible mapping in C^*-algebra valued metric space and gave certain fixed point results in this framework.

In the present article, for four weakly compatible mappings in pairs, some theorems using expansion mappings have been established in C^*-algebra valued metric space. Also, we proved the theorem using $(E.A.)$ property and (CLR) property. The proved results are the extension and generalization of several results available in metric spaces. To support the findings, suitable examples are also discussed.

2. Preliminaries

In this section, we gave some notations and definitions mentioned in [9].

A $*$-algebra $\mathbb{A}$ is a complex algebra with linear involution $*$ such that for all $\alpha, \beta \in \mathbb{A}$. $(\alpha, \beta)^* = \beta^* \alpha^*$ and $\alpha^{**} = \alpha$. The pair $(\mathbb{A}, *)$ is called a unital $*$-algebra if it contains the unity element $I_{\mathbb{A}}$. If a unital $*$-algebra satisfying $\|\alpha^*\| = \|\alpha\|$, for all $\alpha \in \mathbb{A}$ is called Banach $*$-algebra. A Banach C*-algebra satisfying $\|\alpha^* \alpha\| = \|\alpha\|^2$, for all $\alpha \in \mathbb{A}$ is called C*-algebra. If $\alpha = \alpha^*$ and $\sigma(\alpha) = \gamma \in \mathcal{R} : \beta I_{\mathbb{A}} - \alpha$ is non-invertible, α is called the positive element of $\mathbb{A}$. If $\alpha \in \mathbb{A}$ is positive, we write it as $\alpha \succ \theta_{\mathbb{A}}$, where $\theta_{\mathbb{A}}$ is zero element of $\mathbb{A}$. The partial ordering on $\mathbb{A}$ can be defined as follows: $\alpha \succ \beta$ if and only if $\alpha - \beta \succ \theta_{\mathbb{A}}$. Throughout the paper, by $\mathbb{A}$, we denote a unital C^*- algebra with the unity element $I_{\mathbb{A}}$.

Definition 2.1. [9] Suppose Γ is a non-empty set. The mapping $d_{\mathbb{A}} : \Gamma \times \Gamma \to \mathbb{A}$ satisfies the following conditions :

(i) $d_{\mathbb{A}}(p, q) \succeq \theta_{\mathbb{A}}$ and $d_{\mathbb{A}}(p, q) = \theta_{\mathbb{A}}$ if and only if $p = q$;
(ii) $d_{\mathbb{A}}(p, q) = d_{\mathbb{A}}(q, p)$;
(iii) $d_{\mathbb{A}}(p, r) \preceq d_{\mathbb{A}}(p, q) + d_{\mathbb{A}}(q, r)$ for all $p, q, r \in \Gamma$.

Then, $d_{\mathbb{A}}$ is called a C^*-algebra valued metric on Γ and $(\Gamma, \mathbb{A}, d_{\mathbb{A}})$ is called a C^*-algebra valued metric space.

Definition 2.2. [9] A sequence $\{\vartheta_n\}$ in $(\Gamma, \mathbb{A}, d_{\mathbb{A}})$ is said to be

(i) convergent with respect to $\mathbb{A}$, if for given $\epsilon > 0$, there exists a positive integer k such that $\|d_{\mathbb{A}}(\vartheta_n, \vartheta)\| < \epsilon$, for all $n > k$;

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(ii) Cauchy sequence with respect to $\mathbb{A}$ if for any $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $\|d_{\mathbb{A}}(\vartheta_n, \vartheta_m)\| < \epsilon$, for all $n, m > k$.

$(\Gamma, \mathbb{A}, d_{\mathbb{A}})$ is called a complete C^*-algebra valued metric space if every Cauchy sequence with respect to $\mathbb{A}$ is convergent.

Definition 2.3. [15] If two self mappings T and S commute at all of their coincidence points, i.e. $TS\rho = ST\rho$ for all $\rho \in \{\rho \in \Gamma : T\rho = S\rho\}$, then T and S are called weakly compatible.

Remark 2.4. [15] In the metric spaces, if the mappings T and S are compatible then they are weakly compatible, where as the converse is not true. The same results holds for the C^*-algebra valued metric spaces.

Aamri and El Moutawakil [17] introduced the concept of $(E.A.)$ property in metric space.

Definition 2.5. [17] Let f and g are two self-mappings of a metric space (Γ, d). Then, the pair (f, g) is said to satisfy $(E.A.)$ property if there exists a sequence $\{\vartheta_n\}$ in Γ such that

$$\lim_{n\to+\infty} f\vartheta_n = \lim_{n\to+\infty} g\vartheta_n = \tau \quad \text{for some} \quad \tau \in \Gamma.$$

The $(E.A.)$ property allows to replace the completeness requirement of the space with a more natural condition of closedness of the range.

Sintunavarat and Kumam introduced [18] the concept of (CLR) property in which there is no requirement of closedness of the space.

Definition 2.6. [18] Let f and g are two self-mappings of a metric space (Γ, d). Then, the pair (f, g) is said to satisfy (CLR_f) property if there exists a sequence $\{\vartheta_n\} \in \Gamma$ such that

$$\lim_{n\to+\infty} f\vartheta_n = \lim_{n\to+\infty} g\vartheta_n = f\tau \quad \text{for some} \quad \tau \in \Gamma.$$

From the above, if $f\Gamma$ is closed subspace of Γ in $(E.A.)$ property then it must satisfy (CLR_f) property.

The (CLR) property was extended by (Wu et al. [19], 2017) to b-metric space for three mappings.

Definition 2.7. [19] Let $A,\ B,\ C : \Gamma \to \Gamma$ be three self-mappings of a b-metric space $(\Gamma, d_{\mathbb{A}})$. The pair $A,\ B$ satisfies the common limit in the range of C property (CLR_C) if there exists a sequence $\{\vartheta_n\} \subseteq \Gamma$ and a point $\vartheta \in \Gamma$ such that $\lim_{n\to+\infty} A\vartheta_n = \lim_{n\to+\infty} B\vartheta_n = C\vartheta \in C\Gamma$.

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3. Main results

3.1. Common fixed point theorem.

Theorem 3.1. *Let $(\Gamma, \mathbb{A}, d_{\mathbb{A}})$ be C^*-algebra valued metric space and A, B, f and g are four self mappings on Γ satisfying the following conditions:*

(*i*) *$A\Gamma \subseteq g\Gamma$ and $B\Gamma \subseteq f\Gamma$;*
(*ii*) *for every $\vartheta, \varsigma \in \Gamma$, $\alpha, \beta, \gamma \in \mathbb{A}^+$ with $\|\alpha\|, \|\beta\|, \|\gamma\| > 1$,*

$$d_{\mathbb{A}}(f\vartheta, g\varsigma) \succeq \alpha d_{\mathbb{A}}(f\vartheta, A\vartheta) d_{\mathbb{A}}(A\vartheta, g\varsigma) + \beta d_{\mathbb{A}}(g\varsigma, B\varsigma) + \gamma d_{\mathbb{A}}(A\vartheta, B\varsigma).$$

If one of $f\Gamma$, $g\Gamma$, $A\Gamma$ and $B\Gamma$ is a complete subspace of Γ, the pairs (A, f) and (B, g) have a coincidence point. Moreover, if the pairs (A, f) and (B, g) are weakly compatible then the mappings A, B, f and g have a common fixed point in Γ.

Proof. Let $\vartheta_0 \in \Gamma$ be an arbitrary point. From (i), we can construct a sequence $\{\varsigma_n\}$ in Γ as follow :

$$\varsigma_{2n+1} = A\vartheta_{2n} = g\vartheta_{2n+1} \quad \text{and} \quad \varsigma_{2n+2} = B\vartheta_{2n+1} = f\vartheta_{2n+2}.$$

Define $d_{\mathbb{A}_n} = d_{\mathbb{A}}(\varsigma_n, \varsigma_{n+1})$. Suppose that $d_{\mathbb{A}_{2n}} = \theta_{\mathbb{A}}$ i.e. $d_{\mathbb{A}}(\varsigma_{2n}, \varsigma_{2n+1}) = \theta_{\mathbb{A}}$ for some n. Then, $A\vartheta_{2n} = g\vartheta_{2n+1} = B\vartheta_{2n-1} = f\vartheta_{2n}$. Thus, A and f have coincidence point. Hence, the result.
Now, suppose that $d_{\mathbb{A}_{2n}} \succ \theta_{\mathbb{A}}$ for all $n \in \mathbb{N}$. Then, substitute $\vartheta = \vartheta_{2n}$ and $\varsigma = \vartheta_{2n+1}$ in condition (ii), we have

$$\begin{aligned} d_{\mathbb{A}}(f\vartheta_{2n}, g\vartheta_{2n+1}) \quad \succeq \quad & \alpha d_{\mathbb{A}}(f\vartheta_{2n}, A\vartheta_{2n}) d_{\mathbb{A}}(A\vartheta_{2n}, g\vartheta_{2n+1}) \\ & + \beta d_{\mathbb{A}}(g\vartheta_{2n+1}, B\vartheta_{2n+1}) + \gamma d_{\mathbb{A}}(B\vartheta_{2n+1}, A\vartheta_{2n}), \end{aligned}$$

or

$$\begin{aligned} d_{\mathbb{A}}(\varsigma_{2n}, \varsigma_{2n+1}) \quad \succeq \quad & \alpha d_{\mathbb{A}}(\varsigma_{2n}, \varsigma_{2n+1}) d_{\mathbb{A}}(\varsigma_{2n+1}, \varsigma_{2n+1}) + \beta d_{\mathbb{A}}(\varsigma_{2n+1}, \varsigma_{2n+2}) \\ & + \gamma d_{\mathbb{A}}(\varsigma_{2n+1}, \varsigma_{2n+2}), \end{aligned}$$

or

$$d_{\mathbb{A}}(\varsigma_{2n}, \varsigma_{2n+1}) \succeq \beta d_{\mathbb{A}}(\varsigma_{2n+1}, \varsigma_{2n+2}) + \gamma d_{\mathbb{A}}(\varsigma_{2n+1}, \varsigma_{2n+2}).$$

Thus, we have

$$d_{\mathbb{A}_{2n+1}} \preceq \frac{1}{\beta + \gamma} d_{\mathbb{A}_{2n}} = h d_{\mathbb{A}_{2n}},$$

where $\|h\| < 1$ as $\|\beta + \gamma\| > 1$.

On the same argument, we can conclude that $d_{\mathbb{A}_{2n}} \preceq h d_{\mathbb{A}_{2n-1}}$, $d_{\mathbb{A}_{2n-1}} \preceq h d_{\mathbb{A}_{2n-2}}$ and so on. In general, we have

$$d_{\mathbb{A}_n} \preceq h d_{\mathbb{A}_{n-1}} \quad \text{for all} \quad n \in \mathbb{N},$$

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i.e

$$\begin{aligned} d_{\mathbb{A}}(\varsigma_n, \varsigma_{n+1}) &\preceq h d_{\mathbb{A}}(\varsigma_{n-1}, \varsigma_n) \\ &\preceq h^2 d_{\mathbb{A}}(\varsigma_{n-2}, \varsigma_{n-1}) \\ &\quad \cdots \\ &\preceq h^n d_{\mathbb{A}}(\varsigma_0, \varsigma_1). \end{aligned}$$

For any $p \in \mathbb{N}$ and by using triangle inequality, we get

$$\begin{aligned} d_{\mathbb{A}}(\varsigma_{n+p}, \varsigma_n) &\preceq d_{\mathbb{A}}(\varsigma_{n+p}, \varsigma_{n+p-1}) + d_{\mathbb{A}}(\varsigma_{n+p-1}, \varsigma_{n+p-2}) + \cdots + d_{\mathbb{A}}(\varsigma_{n+1}, \varsigma_n) \\ &\preceq \sum_{m=n}^{n+p-1} h^m d_{\mathbb{A}}(\varsigma_0, \varsigma_1) \\ &\preceq \sum_{m=n}^{n+p-1} (\kappa h^{\frac{m}{2}})^* \kappa h^{\frac{m}{2}} \\ &\preceq \sum_{m=n}^{n+p-1} |(\kappa h^m)|^2 \\ &\preceq \sum_{m=n}^{n+p-1} \|(\kappa h^m)^2\| I_{\mathbb{A}} \\ &\preceq \|\kappa\|^2 I_{\mathbb{A}} \sum_{m=n}^{n+p-1} (h^m)^2 \to 0 \quad \text{as} \quad n \to +\infty, \end{aligned}$$

where $|\kappa|^2 = d_{\mathbb{A}}(\varsigma_0, \varsigma_1)$ for some $\kappa \in \mathbb{A}^+$ and $I_{\mathbb{A}}$ is the unity element in $\mathbb{A}$. Hence, $\{\varsigma_n\}$ is a Cauchy sequence. Since, $f\Gamma$ is complete subspace of Γ. Therefore, $\{\varsigma_n\}$ is contained in $f\Gamma$ and has a limit in $f\Gamma$, μ (say). Let $\nu \in f^{-1}\mu$, then $f\nu = \mu$. Next, we shall show that $A\nu = \mu$. Assume that, $A\nu \neq \mu$. Substituting $\vartheta = \nu$ and $\varsigma = \vartheta_{n-1}$ in condition (ii), we get

$$\begin{aligned} d_{\mathbb{A}}(f\nu, g\vartheta_{n-1}) &\succeq \alpha d_{\mathbb{A}}(f\nu, A\nu) d_{\mathbb{A}}(A\nu, g\vartheta_{n-1}) + \beta d_{\mathbb{A}}(g\vartheta_{n-1}, B\vartheta_{n-1}) \\ &\quad + \gamma d_{\mathbb{A}}(A\nu, B\vartheta_{n-1}), \end{aligned}$$

or

$$\begin{aligned} d_{\mathbb{A}}(f\nu, \varsigma_n) &\succeq \alpha d_{\mathbb{A}}(f\nu, A\nu) d_{\mathbb{A}}(A\nu, \varsigma_n) + \beta d_{\mathbb{A}}(\varsigma_{n-1}, \varsigma_n) \\ &\quad + \gamma d_{\mathbb{A}}(\varsigma_{n-1}, A\nu). \end{aligned}$$

Taking limit as $n \to +\infty$ on both sides, we get

$$d_{\mathbb{A}}(\mu, \mu) \succeq \alpha d_{\mathbb{A}}(\mu, A\nu) d_{\mathbb{A}}(A\nu, \mu) + \beta d_{\mathbb{A}}(\mu, \mu) + \gamma d_{\mathbb{A}}(\mu, A\nu),$$

or

$$\alpha d_{\mathbb{A}}(\mu, A\nu) d_{\mathbb{A}}(A\nu, \mu) + \gamma d_{\mathbb{A}}(\mu, A\nu) \preceq \theta_{\mathbb{A}}.$$

Taking norm on both side, we get $\|d_{\mathbb{A}}(A\nu, \mu)\| \|\alpha d_{\mathbb{A}}(A\nu, \mu) + \gamma\| \leq 0$. This implies $\|d_{\mathbb{A}}(A\nu, \mu)\| = 0$; hence $A\nu = \mu$.
Thus, we have $f\nu = \mu = A\nu$ where ν is the coincidence point of the pair(A, f).

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Since, $A\Gamma \subseteq g\Gamma$, $A\nu = \mu$ implies $\mu \in g\Gamma$. Let $\varpi \in g^{-1}\mu$, then $g\varpi = \mu$. Using same argument as above, we can easily verify that $B\varpi = g\varpi = \mu$, i.e ϖ is the coincidence point of the pair (B, g). On the similar lines, the result holds if $g\Gamma$ is complete instead of $f\Gamma$. Now, if $B\Gamma$ is complete, then by condition (i), $\mu \in B\Gamma \subseteq f\Gamma$ and the result holds. Similarly, if $A\Gamma$ is complete $\mu \in A\Gamma \subseteq g\Gamma$, the result holds.

Next if, the pair (A, f) and (B, g) are weakly compatible. Then,

$$\mu = A\nu = f\nu = g\varpi = B\varpi,$$

then

$$\begin{aligned} g\mu &= gB\varpi = Bg\varpi = B\mu, \\ f\mu &= fA\nu = Af\nu = A\mu. \end{aligned}$$

We claim that $B\mu = \mu$. If possible, let $B\mu \neq \mu$.

$$\begin{aligned} d_{\mathbb{A}}(\mu, B\mu) = d_{\mathbb{A}}(f\nu, g\mu) &\succeq \alpha d_{\mathbb{A}}(f\nu, A\nu) d_{\mathbb{A}}(A\nu, g\mu) + \beta d_{\mathbb{A}}(g\mu, B\mu) \\ &\quad + \gamma d_{\mathbb{A}}(B\mu, A\nu) \\ &= \gamma d_{\mathbb{A}}(\mu, B\mu) \\ \theta_{\mathbb{A}} &\succeq (\gamma - I_{\mathbb{A}}) d_{\mathbb{A}}(\mu, B\mu). \end{aligned}$$

Taking norm on both side, we get

$$\|(\gamma - 1)\| \|d_{\mathbb{A}}(\mu, B\mu)\| \leq 0,$$

implies $\|d_{\mathbb{A}}(\mu, B\mu)\| = 0$. Hence, $B\mu = \mu$. On the similar lines, we can show that $A\mu = \mu$. Thus, we get $A\mu = f\mu = g\mu = B\mu = \mu$. Hence, μ is the common fixed point of A, B, f and g.

Uniqueness: To prove uniqueness, let χ be another common fixed point different from μ i.e. $\chi \neq \mu$ of A, B, f and g. Then,

$$\begin{aligned} d_{\mathbb{A}}(\chi, \mu) = d_{\mathbb{A}}(f\chi, g\mu) &\succeq \alpha d_{\mathbb{A}}(f\chi, A\chi) d_{\mathbb{A}}(A\chi, g\mu) + \beta d_{\mathbb{A}}(g\mu, B\mu) \\ &\quad + \gamma d_{\mathbb{A}}(B\mu, A\chi) \\ &\succeq \alpha d_{\mathbb{A}}(\chi, \chi) d_{\mathbb{A}}(\chi, \mu) + d_{\mathbb{A}}(\mu, \mu) + \gamma d_{\mathbb{A}}(\mu, \chi) \\ &\succeq \gamma d_{\mathbb{A}}(\mu, \chi) \\ \theta_{\mathbb{A}} &\succeq (\gamma - I_{\mathbb{A}}) d_{\mathbb{A}}(\mu, \chi). \end{aligned}$$

Taking norm on both side, we get

$$\|(\gamma - 1)\| \|d_{\mathbb{A}}(\mu, \chi)\| \leq 0,$$

implies $\|d_{\mathbb{A}}(\mu, \chi)\| = 0$. Hence, $\chi = \mu$. □

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3.2. **Common fixed point theorem using (E.A.) property.**

Theorem 3.2. *Let* $(\Gamma, \mathbb{A}, d_{\mathbb{A}})$ *be* C^**-algebra valued metric space and* A, B, f *and* g *are four self mappings on* Γ *satisfying the following conditions:*

(i) $A\Gamma \subseteq g\Gamma$ *and* $B\Gamma \subseteq f\Gamma$*;*

(ii) *for every* $\vartheta, \varsigma \in \Gamma$, $\alpha, \beta, \gamma \in \mathbb{A}^+$ *with* $\|\alpha\|, \|\beta\|, \|\gamma\| > 1$,

$$d_{\mathbb{A}}(f\vartheta, g\varsigma) \succeq \alpha d_{\mathbb{A}}(f\vartheta, A\vartheta) d_{\mathbb{A}}(A\vartheta, g\varsigma) + \beta d_{\mathbb{A}}(g\varsigma, B\varsigma) + \gamma d_{\mathbb{A}}(A\vartheta, B\varsigma);$$

(iii) *the pair* (A, f) *and* (B, g) *are weakly compatible;*

(iv) *one of the pair* (A, f) *or* (B, g) *satisfy* $(E.A.)$ *property.*

If the range of one of the mapping $f\Gamma$ *or* $g\Gamma$ *is a closed subspace of* Γ*, then the mappings* A, B, f *and* g *have a unique common fixed point in* Γ*.*

Proof. Firstly, we assume that the pair (B, g) satisfies $(E.A.)$ property. Then, by definition (2.5), there exists a sequence $\{\vartheta_n\}$ in Γ such that $\lim_{n\to+\infty} B(\vartheta_n) = \lim_{n\to+\infty} g(\vartheta_n) = \tau$ for some $\tau \in \Gamma$.

Further, $B\Gamma \subseteq f\Gamma$, there exists a sequence $\{\varsigma_n\}$ in Γ such that $B(\vartheta_n) = f(\varsigma_n)$. Hence, $\lim_{n\to+\infty} f(\varsigma_n) = \tau$. We claim that $\lim_{n\to+\infty} A(\varsigma_n) = \tau$. Let if possible $\lim_{n\to+\infty} A(\vartheta_n) = \tau_1 \neq \tau$. Then, substituting $\vartheta = \varsigma_n$ and $\varsigma = \vartheta_n$ in condition (ii), we have

$$d_{\mathbb{A}}(f\varsigma_n, g\vartheta_n) \succeq \alpha d_{\mathbb{A}}(f\varsigma_n, A\varsigma_n) d_{\mathbb{A}}(A\varsigma_n, g\vartheta_n) + \beta d_{\mathbb{A}}(g\vartheta_n, B\vartheta_n) + \gamma d_{\mathbb{A}}(A\varsigma_n, B\vartheta_n).$$

Taking norm and limit as $n \to +\infty$ on both side, we get

$$\begin{aligned} d_{\mathbb{A}}(\tau, \tau) &\succeq \alpha d_{\mathbb{A}}(\tau, \tau_1) d_{\mathbb{A}}(\tau_1, \tau) + \beta d_{\mathbb{A}}(\tau, \tau) + \gamma d_{\mathbb{A}}(\tau_1, \tau) \\ 0 &\geq \|d_{\mathbb{A}}(\tau, \tau_1)\| \|\alpha d_{\mathbb{A}}(\tau, \tau_1) + \gamma\|. \end{aligned}$$

This implies $\|d_{\mathbb{A}}(\tau_1, \tau)\| = 0$; then $\tau_1 = \tau$ i.e. $\lim_{n\to+\infty} A(\varsigma_n) = \lim_{n\to+\infty} B(\vartheta_n) = \tau$. Now, we suppose that $f\Gamma$ is closed subspace of Γ and $f\mu = \tau$ for some $\mu \in \Gamma$. Subsequently, we get

$$\lim_{n\to+\infty} A(\varsigma_n) = \lim_{n\to+\infty} B(\vartheta_n) = \lim_{n\to+\infty} g(\vartheta_n) = \lim_{n\to+\infty} f(\varsigma_n) = \tau = f\mu.$$

We claim that $A\mu = f\mu$. Substitute $\vartheta = \mu$ and $\varsigma = \vartheta_n$ in condition (ii), we have

$$d_{\mathbb{A}}(f\mu, g\vartheta_n) \succeq \alpha d_{\mathbb{A}}(f\mu, A\mu) d_{\mathbb{A}}(A\mu, g\vartheta_n) + \beta d_{\mathbb{A}}(g\vartheta_n, B\vartheta_n) + \gamma d_{\mathbb{A}}(A\mu, B\vartheta_n).$$

Taking norm and limit as $n \to +\infty$ on both side, we get

$$\begin{aligned} d_{\mathbb{A}}(\tau, \tau) &\succeq \alpha d_{\mathbb{A}}(\tau, A\mu) d_{\mathbb{A}}(A\mu, \tau) + \beta d_{\mathbb{A}}(\tau, \tau) + \gamma d_{\mathbb{A}}(A\mu, \tau) \\ 0 &\geq \|d_{\mathbb{A}}(A\mu, \tau)\| \|\alpha d_{\mathbb{A}}(\tau, A\mu) + \gamma\|. \end{aligned}$$

This implies $\|d_{\mathbb{A}}(A\mu, \tau)\| = 0$ i.e. $A\mu = \tau$

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Now the weak compatibility of the pair (A, f) implies that $Af\mu = fA\mu$ or $A\tau = f\tau$.

Since $A\Gamma \subseteq g\Gamma$, there exits $\nu \in \Gamma$ such that $A\mu = g\nu = f\mu = \tau$. Now, we prove that ν is coincidence point of pair (B, g), i.e $B\nu = g\nu = \tau$. Substituting $\vartheta = \mu$ and $\varsigma = \nu$ in condition (ii), we get

$$\begin{aligned} d_{\mathbb{A}}(f\mu, g\nu) &\succeq \alpha d_{\mathbb{A}}(f\mu, A\mu) d_{\mathbb{A}}(A\mu, g\nu) + \beta d_{\mathbb{A}}(g\nu, B\nu) + \gamma d_{\mathbb{A}}(A\mu, B\nu) \\ \theta_{\mathbb{A}} &\succeq (\beta + \gamma) d_{\mathbb{A}}(\tau, B\nu). \end{aligned}$$

Taking norm on both side, we get

$$(\|\beta\| + \|\gamma\|)\|d_{\mathbb{A}}(\tau, B\nu)\| \leq 0,$$

implies $\|d_{\mathbb{A}}(\tau, B\nu)\| \leq 0$. Hence, $\tau = B\nu$. Thus, $B\nu = g\nu = \tau$ and ν is a coincidence point of B and g.

Further, the weak compatibility of pair (B, g) implies that $Bg\nu = gB\nu$, or $B\tau = g\tau$. Therefore, τ is a common coincidence point of A, B, f and g.

Now, we prove that τ is a common fixed point of A, B, f and g. Substituting $\vartheta = \mu$ and $\varsigma = \tau$ in condition (ii), we get

$$\begin{aligned} d_{\mathbb{A}}(A\mu, B\tau) &= d_{\mathbb{A}}(f\mu, g\tau) \\ &\succeq \alpha d_{\mathbb{A}}(f\mu, A\mu) d_{\mathbb{A}}(A\mu, g\tau) + \beta d_{\mathbb{A}}(g\tau, B\tau) + \gamma d_{\mathbb{A}}(A\mu, B\tau) \\ &= \gamma d_{\mathbb{A}}(\tau, B\tau) \\ \theta_{\mathbb{A}} &\succeq (\gamma - I_{\mathbb{A}}) d_{\mathbb{A}}(\tau, B\tau). \end{aligned}$$

Taking norm on both side, we get

$$\|(\gamma - 1)\|\|d_{\mathbb{A}}(\tau, B\tau)\| \leq 0,$$

implies $\|d_{\mathbb{A}}(\tau, B\tau)\| = 0$. Hence, we have $B\tau = \tau$. Thus, $A\tau = B\tau = f\tau = g\tau = \tau$.

On the similar lines, the result holds if we assume that $g\Gamma$ is closed subspace of Γ. Similarly, $(E.A.)$ property of the pair (A, f) will give a similar result.

Uniqueness: To prove the uniqueness, let ϖ is another common fixed point of A, B, f and g. Then, substituting $\vartheta = \varpi$ and $\varsigma = \tau$ in condition (ii), we get

$$\begin{aligned} d_{\mathbb{A}}(\varpi, \tau) &= d_{\mathbb{A}}(f\varpi, g\tau) \\ &\succeq \alpha d_{\mathbb{A}}(f\varpi, A\varpi) d_{\mathbb{A}}(A\varpi, g\tau) + \beta d_{\mathbb{A}}(g\tau, B\tau) + \gamma d_{\mathbb{A}}(A\varpi, B\tau) \\ &= \gamma d_{\mathbb{A}}(\varpi, \tau) \\ \theta_{\mathbb{A}} &\succeq (\gamma - I_{\mathbb{A}}) d_{\mathbb{A}}(\varpi, \tau). \end{aligned}$$

Taking norm on both side, we get

$$\|(\gamma - 1)\|\|d_{\mathbb{A}}(\varpi, \tau)\| \leq 0,$$

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implies $\|d_{\mathbb{A}}(\varpi,\tau)\| = 0$. Hence, τ is the unique common fixed point of A, B, f and g. □

3.3. **Common fixed point theorem using (CLR) property.**

Theorem 3.3. *Let $(\Gamma, \mathbb{A}, d_{\mathbb{A}})$ be C^*-algebra valued metric space and A, B, f and g are four self mappings on Γ satisfying the following conditions:*

(i) *$A\Gamma \subseteq g\Gamma$ and $B\Gamma \subseteq f\Gamma$;*

(ii) *for every $\vartheta, y \in \Gamma$, $\alpha, \beta, \gamma \in \mathbb{A}^+$ with $\|\alpha\|, \|\beta\|, \|\gamma\| > 1$,*

$$d_{\mathbb{A}}(f\vartheta, g\varsigma) \succeq \alpha d_{\mathbb{A}}(f\vartheta, A\vartheta) d_{\mathbb{A}}(A\vartheta, g\varsigma) + \beta d_{\mathbb{A}}(g\varsigma, B\varsigma) + \gamma d_{\mathbb{A}}(A\vartheta, B\varsigma);$$

(iii) *the pair (A, f) and (B, g) are weakly compatible;*

(iv) *one of the pair satisfy (CLR) property.*

Then, the mappings A, B, f and g have a unique common fixed point in Γ.

Proof. Firstly, we suppose that the pair (B, g) satisfies (CLR_B) property. By definition (2.6) there exist a sequence $\{\vartheta_n\}$ in Γ such that

$$\lim_{n\to+\infty} B(\vartheta_n) = \lim_{n\to+\infty} g(\vartheta_n) = B\vartheta = \tau,$$

for some $\vartheta \in \Gamma$.

Since, $B\Gamma \subseteq f\Gamma$, we have $B\vartheta = f\mu$, for some $\mu \in \Gamma$. We claim that $A\mu = f\mu = \tau$ (say). Substituting $\vartheta = \mu$ and $\varsigma = \vartheta_n$ in condition (ii), we get

$$d_{\mathbb{A}}(f\mu, g\vartheta_n) \succeq \alpha d_{\mathbb{A}}(f\mu, A\mu) d_{\mathbb{A}}(A\mu, g\vartheta_n) + \beta d_{\mathbb{A}}(g\vartheta_n, B\vartheta_n) + \gamma d_{\mathbb{A}}(A\mu, B\vartheta_n).$$

Taking limit as $n \to +\infty$ and norm on both side

$$\begin{aligned} d_{\mathbb{A}}(\tau,\tau) &\succeq \alpha d_{\mathbb{A}}(\tau, A\mu) d_{\mathbb{A}}(A\mu, \tau) + \beta d_{\mathbb{A}}(\tau,\tau) + \gamma d_{\mathbb{A}}(A\mu, \tau) \\ 0 &\geq \|d_{\mathbb{A}}(\tau, A\mu)\| \|\alpha d_{\mathbb{A}}(A\mu, \tau) + \gamma\|. \end{aligned}$$

This implies $\|d_{\mathbb{A}}(A\mu, \tau)\| = 0$ i.e. $A\mu = \tau$.

Since, $A\Gamma \subseteq g\Gamma$, there exists $\nu \in \Gamma$ such that $g\nu = A\mu = f\mu = \tau$.

Now, we prove that $g\nu = B\nu = \tau$. i.e ν is the coincidence point of (B, g). Substituting $\vartheta = \mu$ and $\varsigma = \nu$ in condition (ii), we get

$$\begin{aligned} d_{\mathbb{A}}(f\mu, g\nu) &\succeq \alpha d_{\mathbb{A}}(f\mu, A\mu) d_{\mathbb{A}}(A\mu, g\nu) + \beta d_{\mathbb{A}}(g\nu, B\nu) + \gamma d_{\mathbb{A}}(A\mu, B\nu) \\ \theta_{\mathbb{A}} &\succeq (\beta+\gamma) d_{\mathbb{A}}(\tau, B\nu). \end{aligned}$$

Taking norm on both side, we get

$$\|(\beta+\gamma)\| \|d_{\mathbb{A}}(\tau, B\nu)\| \leq 0,$$

implies $\|d_{\mathbb{A}}(\tau, B\nu)\| = 0$, we get $B\nu = \tau$ i.e $B\nu = g\nu = \tau$ and ν is a coincidence point of B and g.

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Further, the weak compatibility of pair (B, g) implies that $Bg\nu = gB\nu$, or $B\tau = g\tau$. Therefore, τ is a common coincidence point of A, B, f and g.

Now, we prove that τ is common fixed point of A, B, f and g. Substituting $\vartheta = \mu$ and $\varsigma = \tau$ in condition (ii), we get

$$\begin{aligned} d_{\mathbb{A}}(A\mu, B\tau) = d_{\mathbb{A}}(f\mu, g\tau) &\succeq \alpha d_{\mathbb{A}}(f\mu, A\mu) d_{\mathbb{A}}(A\mu, g\tau) + \beta d_{\mathbb{A}}(g\tau, B\tau) \\ &\quad + \gamma d_{\mathbb{A}}(A\mu, B\tau) \\ d_{\mathbb{A}}(\tau, B\tau) &\succeq \gamma d_{\mathbb{A}}(\tau, B\tau) \\ \theta_{\mathbb{A}} &\succeq (\gamma - I_{\mathbb{A}}) d_{\mathbb{A}}(\tau, B\tau). \end{aligned}$$

Taking norm on both side, we get

$$\|(\gamma - 1)\| \|d_{\mathbb{A}}(\tau, B\tau)\| \leq 0,$$

implies $\|d_{\mathbb{A}}(\tau, B\tau)\| = 0$. Hence, $B\tau = \tau$. Thus, $A\tau = B\tau = f\tau = g\tau = \tau$ i.e τ is a common fixed point of A, B, f and g.

Uniqueness: To prove that the uniqueness, let ϖ is another common fixed point of A, B, f and g. Then, substituting $\vartheta = \varpi$ and $\varsigma = \tau$ in condition (ii), we get

$$\begin{aligned} d_{\mathbb{A}}(\varpi, \tau) = d_{\mathbb{A}}(f\varpi, g\tau) &\succeq \alpha d_{\mathbb{A}}(f\varpi, A\varpi) d_{\mathbb{A}}(A\varpi, g\tau) + \beta d_{\mathbb{A}}(g\tau, B\tau) \\ &\quad + \gamma d_{\mathbb{A}}(A\varpi, B\tau) \\ &\succeq \gamma d_{\mathbb{A}}(\varpi, \tau) \\ \theta_{\mathbb{A}} &\succeq (\gamma - I_{\mathbb{A}}) d_{\mathbb{A}}(\varpi, \tau). \end{aligned}$$

Taking norm on both side, we get

$$\|(\gamma - 1)\| \|d_{\mathbb{A}}(\varpi, \tau)\| = 0,$$

implies $\|d_{\mathbb{A}}(\varpi, \tau)\| = 0$. Hence, τ is a unique common fixed point of A, B, f and g. □

Theorem 3.4. *Let $(\Gamma, \mathbb{A}, d_{\mathbb{A}})$ be C^*-algebra valued metric space and A, B, f and g are four self mappings on Γ satisfying the following conditions:*

(i) *$A\Gamma \subseteq S\Gamma$ and $B\Gamma \subseteq T\Gamma$;*

(ii) *for every $\vartheta, \varsigma \in \Gamma$, $\alpha, \beta, \gamma \in \mathbb{A}^+$ with $\|\alpha\|, \|\beta\|, \|\gamma\| > 1$,*

$$d_{\mathbb{A}}(T\vartheta, S\varsigma) \succeq \alpha d_{\mathbb{A}}(T\vartheta, A\vartheta) d_{\mathbb{A}}(A\vartheta, S\varsigma) + \beta d_{\mathbb{A}}(S\varsigma, B\varsigma) + \gamma d_{\mathbb{A}}(A\vartheta, B\varsigma);$$

(iii) *the pair (A, T) and (B, S) are weakly compatible;*

(iv) *A, T satisfies (CLR_S) and B, S satisfies (CLR_T) property.*

Then, the mappings A, B, f and g have a unique common fixed point in Γ.

Proof. As A, T satisfies (CLR_S) and B, S satisfies (CLR_T) property, there exist two sequences $\{\vartheta_n\}$ and $\{\varsigma_n\}$ in Γ and $\varrho, \upsilon \in \Gamma$ such that $\lim_{n\to+\infty} A\vartheta_n =$

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$\lim_{n\to+\infty} T\vartheta_n = S\varrho \in S\Gamma$ and $\lim_{n\to+\infty} B\varsigma_n = \lim_{n\to+\infty} S\varsigma_n = T\upsilon \in T\Gamma$. Substituting $\vartheta = \vartheta_n$ and $\varsigma = \varsigma_n$ in condition (ii) and taking limit as $n \to +\infty$, we get

$$\begin{aligned} d_{\mathbb{A}}(T\vartheta_n, S\varsigma_n) &\succeq \alpha d_{\mathbb{A}}(T\vartheta_n, A\vartheta_n) d_{\mathbb{A}}(A\vartheta_n, S\varsigma_n) + \beta d_{\mathbb{A}}(S\varsigma_n, B\varsigma_n) \\ &\quad + \gamma d_{\mathbb{A}}(A\vartheta_n, B\varsigma_n) \\ d_{\mathbb{A}}(S\varrho, T\upsilon) &\succeq \alpha d_{\mathbb{A}}(S\varrho, S\varrho) d_{\mathbb{A}}(S\varrho, T\upsilon) + \beta d_{\mathbb{A}}(T\upsilon, T\upsilon) + \gamma d_{\mathbb{A}}(S\varrho, T\upsilon) \\ &= \gamma d_{\mathbb{A}}(S\varrho, T\upsilon) \\ \theta_{\mathbb{A}} &\succeq (\gamma - I_{\mathbb{A}}) d_{\mathbb{A}}(S\varrho, T\upsilon). \end{aligned}$$

Taking norm on both side, we get

$$\|(\gamma - I_{\mathbb{A}})\| \|d_{\mathbb{A}}(S\varrho, T\upsilon)\| \leq 0,$$

implies $\|d_{\mathbb{A}}(S\varrho, T\upsilon)\| = 0$. Hence, $T\upsilon = S\varrho$.
Now, substituting $\vartheta_n = \vartheta$ and $\varsigma = \alpha$ in condition (ii) with limit as $n \to +\infty$, we get

$$\begin{aligned} d_{\mathbb{A}}(T\vartheta_n, S\varrho) &\succeq \alpha d_{\mathbb{A}}(T\vartheta_n, A\vartheta_n) d_{\mathbb{A}}(A\vartheta_n, S\varrho) + \beta d_{\mathbb{A}}(S\varrho, B\varrho) + \gamma d_{\mathbb{A}}(A\vartheta_n, B\varrho) \\ d_{\mathbb{A}}(S\varrho, S\varrho) &\succeq \alpha d_{\mathbb{A}}(S\varrho, S\varrho) d_{\mathbb{A}}(S\varrho, S\varrho) + \beta d_{\mathbb{A}}(S\varrho, B\varrho) + \gamma d_{\mathbb{A}}(S\varrho, B\varrho) \\ &\succeq (\beta + \gamma) d_{\mathbb{A}}(S\varrho, B\varrho). \end{aligned}$$

Taking norm on both side, we get $\|d_{\mathbb{A}}(S\varrho, B\varrho)\| \leq 0$. Hence, $B\varrho = S\varrho$.

Similarly, one can proved that $A\upsilon = T\upsilon$. Hence, we get $A\upsilon = T\upsilon = B\varrho = S\varrho = \zeta$.

Since, A, T are weakly compatible mapping so

$$A\zeta = AT\upsilon = TA\upsilon = T\zeta.$$

Substituting $\vartheta = \zeta$ and $\varsigma = \varrho$ in condition (ii), we get

$$\begin{aligned} d_{\mathbb{A}}(T\zeta, S\varrho) &\succeq \alpha d_{\mathbb{A}}(T\zeta, A\zeta) d_{\mathbb{A}}(A\zeta, S\varrho) + \beta d_{\mathbb{A}}(S\varrho, B\varrho) + \gamma d_{\mathbb{A}}(A\zeta, B\varrho) \\ d_{\mathbb{A}}(T\zeta, \zeta) &\succeq \gamma d_{\mathbb{A}}(T\zeta, \zeta) \\ \theta_{\mathbb{A}} &\succeq (\gamma - I_{\mathbb{A}}) d_{\mathbb{A}}(T\zeta, \zeta). \end{aligned}$$

Taking norm on both side, we get

$$\|(\gamma - I_{\mathbb{A}})\| \|d_{\mathbb{A}}(T\zeta, \zeta)\| \leq 0,$$

implies $\|d_{\mathbb{A}}(T\zeta, \zeta)\| = 0$. Hence, $T\zeta = \zeta = A\zeta$.
Similarly, $S\zeta = B\zeta = \zeta$.

Uniqueness: To prove the uniqueness, let η be another common fixed of

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A, B, T and S. Substituting $\vartheta = \zeta$ and $\varsigma = \eta$ in condition (ii), we get

$$\begin{aligned} d_{\mathbb{A}}(\zeta, \eta) &= d_{\mathbb{A}}(T\zeta, S\eta) \\ &\succeq \alpha d_{\mathbb{A}}(T\zeta, A\zeta) d_{\mathbb{A}}(A\zeta, S\eta) + \beta d_{\mathbb{A}}(S\eta, B\eta) + \gamma d_{\mathbb{A}}(A\zeta, B\eta) \\ &= \gamma d_{\mathbb{A}}(\zeta, \eta) \\ \theta_{\mathbb{A}} &\succeq (\gamma - I_{\mathbb{A}}) d_{\mathbb{A}}(\zeta, \eta). \end{aligned}$$

Taking norm on both side, we get

$$\|(\gamma - I_{\mathbb{A}})\| \|d_{\mathbb{A}}(\zeta, \eta)\| \leq 0,$$

implies $\|d_{\mathbb{A}}(\zeta, \eta)\| = 0$. Hence, $\zeta = \eta$. □

Theorem 3.5. *Let $(\Gamma, \mathbb{A}, d_{\mathbb{A}})$ be C^*-algebra valued metric space and A, B, f and g are four self mappings on Γ satisfying the following conditions:*

(i) *$A\Gamma \subseteq S\Gamma$ and $B\Gamma \subseteq T\Gamma$;*

(i) *for every $\vartheta, \varsigma \in \Gamma$, $\alpha, \gamma \in \mathbb{A}^+$ with $\|\alpha\|, \|\gamma\| > 1$,*

$$d_{\mathbb{A}}(T\vartheta, S\varsigma) \succeq \alpha \frac{d_{\mathbb{A}}(T\vartheta, A\vartheta) d_{\mathbb{A}}(A\vartheta, S\varsigma) + d_{\mathbb{A}}(B\varsigma, S\varsigma) d_{\mathbb{A}}(S\varsigma, T\vartheta)}{d_{\mathbb{A}}(A\vartheta, B\varsigma)} + \gamma d_{\mathbb{A}}(A\vartheta, B\varsigma);$$

(iii) *the pair (A, T) and (B, S) are weakly compatible;*

(iv) *A, T satisfies (CLR_S) and B, S satisfies (CLR_T) property.*

Then, the mappings A, B, f and g have a unique common fixed point in Γ.

Proof. As A, T satisfies (CLR_S) and B, S satisfies (CLR_T) property, there exist two sequences $\{\vartheta_n\}$ and $\{\varsigma_n\}$ in Γ and $\varrho, \upsilon \in \Gamma$ such that $\lim_{n\to+\infty} A\vartheta_n = \lim_{n\to+\infty} T\vartheta_n = S\varrho \in S\Gamma$ and $\lim_{n\to+\infty} B\varsigma_n = \lim_{n\to+\infty} S\varsigma_n = T\upsilon \in T\Gamma$. Substituting $\vartheta = \vartheta_n$ and $\varsigma = \varsigma_n$ in condition (ii) and taking limit as $n \to +\infty$, we get

$$\begin{aligned} d_{\mathbb{A}}(T\vartheta_n, S\varsigma_n) &\succeq \alpha \frac{d_{\mathbb{A}}(T\vartheta_n, A\vartheta_n) d_{\mathbb{A}}(A\vartheta_n, S\varsigma_n) + d_{\mathbb{A}}(B\varsigma_n, S\varsigma_n) d_{\mathbb{A}}(S\varsigma_n, T\vartheta_n)}{d_{\mathbb{A}}(A\vartheta_n, B\varsigma_n)} \\ &\quad + \gamma d_{\mathbb{A}}(A\vartheta_n, B\varsigma_n) \\ d_{\mathbb{A}}(S\varrho, T\upsilon) &\succeq \alpha \frac{d_{\mathbb{A}}(S\varrho, S\varrho) d_{\mathbb{A}}(S\varrho, T\upsilon) + d_{\mathbb{A}}(T\upsilon, T\upsilon) d_{\mathbb{A}}(T\upsilon, S\varrho)}{d_{\mathbb{A}}(S\varrho, T\upsilon)} \\ &\quad + \gamma d_{\mathbb{A}}(S\varrho, T\upsilon) \\ &\succeq \gamma d_{\mathbb{A}}(S\varrho, T\upsilon) \\ \theta_{\mathbb{A}} &\succeq (\gamma - 1) d_{\mathbb{A}}(S\varrho, T\upsilon). \end{aligned}$$

Since, $\|(\gamma - 1)\| \neq 0$; hence the result holds for $\|d_{\mathbb{A}}(S\varrho, T\upsilon)\| = 0$ implies $S\varrho = T\upsilon$.

Now, substituting $\vartheta_n = \vartheta$ and $\varsigma = \alpha$ in condition (ii) with limit as $n \to +\infty$,

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we get

$$\begin{aligned} d_{\mathbb{A}}(T\vartheta_n, S\varrho) &\succeq \alpha \frac{d_{\mathbb{A}}(T\vartheta_n, A\vartheta_n) d_{\mathbb{A}}(A\vartheta_n, S\varrho) + d_{\mathbb{A}}(B\varrho, S\varrho) d_{\mathbb{A}}(S\varrho, T\vartheta_n)}{d_{\mathbb{A}}(A\vartheta_n, B\varrho)} \\ &\quad + \gamma d_{\mathbb{A}}(A\vartheta_n, B\varrho) \\ d_{\mathbb{A}}(S\varrho, S\varrho) &\succeq \alpha \frac{d_{\mathbb{A}}(S\varrho, S\varrho) d_{\mathbb{A}}(S\varrho, S\varrho) + d_{\mathbb{A}}(B\varrho, S\varrho) d_{\mathbb{A}}(S\varrho, S\varrho)}{d_{\mathbb{A}}(S\varrho, B\varrho)} \\ &\quad + \gamma d_{\mathbb{A}}(S\varrho, B\varrho) \\ \theta_{\mathbb{A}} &\succeq \gamma d_{\mathbb{A}}(S\varrho, B\varrho). \end{aligned}$$

Hence, $\|d_{\mathbb{A}}(S\varrho, B\varrho)\| = 0$ implies $S\varrho = B\varrho$.

Similarly, one can proved that $A\upsilon = T\upsilon$. Hence, we get $A\upsilon = T\upsilon = B\varrho = S\varrho = \zeta$. Since A, T are weakly compatible mapping so

$$A\zeta = AT\upsilon = TA\upsilon = T\zeta.$$

Putting $\vartheta = \zeta$ and $\varsigma = \alpha$ in condition (ii), we get

$$\begin{aligned} d_{\mathbb{A}}(T\zeta, S\varrho) &\succeq \alpha \frac{d_{\mathbb{A}}(T\zeta, A\zeta) d_{\mathbb{A}}(A\zeta, S\varrho) + d_{\mathbb{A}}(B\varrho, S\varrho) d_{\mathbb{A}}(S\varrho, T\zeta)}{d_{\mathbb{A}}(A\zeta, B\varrho)} \\ &\quad + \gamma d_{\mathbb{A}}(A\zeta, B\varrho) \\ d_{\mathbb{A}}(T\zeta, \zeta) &\succeq \alpha \frac{d_{\mathbb{A}}(T\zeta, T\zeta) d_{\mathbb{A}}(T\zeta, \zeta) + d_{\mathbb{A}}(\zeta, \zeta) d_{\mathbb{A}}(\zeta, T\zeta)}{d_{\mathbb{A}}(T\zeta, \zeta)} + \gamma d_{\mathbb{A}}(T\zeta, \zeta) \\ \theta_{\mathbb{A}} &\succeq (\gamma - 1) d_{\mathbb{A}}(T\zeta, \zeta). \end{aligned}$$

Since, $\|(\gamma - 1)\| \neq 0$; hence the result holds for $\|d_{\mathbb{A}}(T\zeta, \zeta)\| = 0$ implies $T\zeta = \zeta$. Hence, $T\zeta = \zeta = A\zeta$

Similarly, $S\zeta = B\zeta = \zeta$. Hence ζ is common fixed point of A, B, T and S.

Uniqueness: To prove the uniqueness, let η be another common fixed of A, B, T and S. Substituting $\vartheta = \zeta$ and $\varsigma = \eta$ in condition (ii), we get

$$\begin{aligned} d_{\mathbb{A}}(\zeta, \eta) &= d_{\mathbb{A}}(T\zeta, S\eta) \\ &\succeq \alpha \frac{d_{\mathbb{A}}(T\zeta, A\zeta) d_{\mathbb{A}}(A\zeta, S\eta) + d_{\mathbb{A}}(B\eta, S\eta) d_{\mathbb{A}}(S\eta, T\zeta)}{d_{\mathbb{A}}(A\zeta, B\eta)} + \gamma d_{\mathbb{A}}(A\zeta, B\eta) \\ \theta_{\mathbb{A}} &\succeq (\gamma - 1) d_{\mathbb{A}}(\zeta, \eta). \end{aligned}$$

Since, $\|(\gamma - 1)\| \neq 0$; hence results holds for $\|d_{\mathbb{A}}(\zeta, \eta)\| = 0$ implies $\zeta = \eta$. Hence, $\zeta = \eta$. □

Theorem 3.6. *Let $(\Gamma, \mathbb{A}, d_{\mathbb{A}})$ be C^*-algebra valued metric space and A, B, f and g are four self mappings on Γ satisfying the following conditions :*

(i) $A\Gamma \subseteq S\Gamma$ *and* $B\Gamma \subseteq T\Gamma$*;*

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(ii) *for every* $\vartheta, \varsigma \in \Gamma$, $\alpha, \beta, \gamma, \delta \in \mathbb{A}^+$ *with* $\|\alpha\|, \|\beta\|, \|\gamma\|, \|\delta\| > 1$,

$$\begin{aligned} d_{\mathbb{A}}(T\vartheta, S\varsigma) &\succeq \alpha\frac{d^2(T\vartheta, A\vartheta) + d^2(B\varsigma, S\varsigma)}{d_{\mathbb{A}}(A\vartheta, S\varsigma) + d_{\mathbb{A}}(B\varsigma, T\vartheta)} + \beta\frac{d^2(S\varsigma, A\vartheta) + d^2(B\varsigma, T\vartheta)}{d_{\mathbb{A}}(A\vartheta, S\varsigma) + d_{\mathbb{A}}(B\varsigma, T\vartheta)} \\ &\quad +\gamma(d_{\mathbb{A}}(A\vartheta, T\vartheta) + d_{\mathbb{A}}(S\varsigma, B\varsigma)) + \delta d_{\mathbb{A}}(A\vartheta, B\varsigma); \end{aligned}$$

(iii) *the pair* (A, T) *and* (B, S) *are weakly compatible;*

(iv) A, T *satisfies* (CLR_S) *and* B, S *satisfies* (CLR_T) *property.*

Then, the mappings A, B, f *and* g *have a unique common fixed point in* Γ.

Proof. As A, T satisfies (CLR_S) and B, S satisfies (CLR_T) property, there exist two sequences $\{\vartheta_n\}$ and $\{\varsigma_n\}$ in Γ and $\varrho, \upsilon \in \Gamma$ such that $\lim_{n\to+\infty} A\vartheta_n = \lim_{n\to+\infty} T\vartheta_n = S\varrho \in S\Gamma$ and $\lim_{n\to+\infty} B\varsigma_n = \lim_{n\to+\infty} S\varsigma_n = T\upsilon \in T\Gamma$. Substituting $\vartheta = \vartheta_n$ and $\varsigma = \varsigma_n$ in condition (ii) and taking limit as $n \to +\infty$, we get

$$\begin{aligned} d_{\mathbb{A}}(T\vartheta_n, S\varsigma_n) &\succeq \alpha\frac{d^2(T\vartheta_n, A\vartheta_n) + d^2(B\varsigma_n, S\varsigma_n)}{d_{\mathbb{A}}(A\vartheta_n, S\varsigma_n) + d_{\mathbb{A}}(B\varsigma_n, T\vartheta_n)} \\ &\quad +\beta\frac{d^2(S\varsigma_n, A\vartheta_n) + d^2(B\varsigma_n, T\vartheta_n)}{d_{\mathbb{A}}(A\vartheta_n, S\varsigma_n) + d_{\mathbb{A}}(B\varsigma_n, T\vartheta_n)} \\ &\quad +\gamma(d_{\mathbb{A}}(A\vartheta_n, T\vartheta_n) + d_{\mathbb{A}}(S\varsigma_n, B\varsigma_n)) + \delta d_{\mathbb{A}}(A\vartheta_n, B\varsigma_n) \\ d_{\mathbb{A}}(S\varrho, T\upsilon) &\succeq \alpha\frac{d^2(S\varrho, S\varrho) + d^2(T\upsilon, T\upsilon)}{d_{\mathbb{A}}(S\varrho, T\upsilon) + d_{\mathbb{A}}(T\upsilon, S\varrho)} + \beta\frac{d^2(T\upsilon, S\varrho) + d^2(T\upsilon, S\varrho)}{d_{\mathbb{A}}(S\varrho, T\upsilon) + d_{\mathbb{A}}(T\upsilon, S\varrho)} \\ &\quad +\gamma(d_{\mathbb{A}}(S\varrho, S\varrho) + d_{\mathbb{A}}(T\upsilon, T\upsilon)) + \delta d_{\mathbb{A}}(S\varrho, T\upsilon) \\ &= \beta d_{\mathbb{A}}(S\varrho, T\upsilon) + \delta d_{\mathbb{A}}(S\varrho, T\upsilon) \\ \theta_{\mathbb{A}} &\succeq (\beta + \delta - 1)d_{\mathbb{A}}(S\varrho, T\upsilon). \end{aligned}$$

Since, $\|(\beta + \delta - 1)\| \neq 0$; hence the results holds for $\|d_{\mathbb{A}}(S\varrho, T\upsilon)\| = 0$ implies $S\varrho = T\upsilon$.

Now, substituting $\vartheta_n = \vartheta$ and $\varsigma = \varrho$ in condition (ii) with limit as $n \to +\infty$, we get

$$\begin{aligned} d_{\mathbb{A}}(T\vartheta_n, S\varrho) &\succeq \alpha\frac{d^2(T\vartheta_n, A\vartheta_n) + d^2(B\varrho, S\varrho)}{d_{\mathbb{A}}(A\vartheta_n, S\varrho) + d_{\mathbb{A}}(B\varrho, T\vartheta_n)} \\ &\quad +\beta\frac{d^2(S\varrho, A\vartheta_n) + d^2(B\varrho, T\vartheta_n)}{d_{\mathbb{A}}(A\vartheta_n, S\varrho) + d_{\mathbb{A}}(B\varrho, T\vartheta_n)} \\ &\quad +\gamma(d_{\mathbb{A}}(A\vartheta_n, T\vartheta_n) + d_{\mathbb{A}}(S\varrho, B\varrho)) + \delta d_{\mathbb{A}}(A\vartheta_n, B\varrho) \\ d_{\mathbb{A}}(S\varrho, S\varrho) &\succeq \alpha\frac{d^2(S\varrho, S\varrho) + d^2(B\varrho, S\varrho)}{d_{\mathbb{A}}(S\varrho, S\varrho) + d_{\mathbb{A}}(B\varrho, S\varrho)} + \beta\frac{d^2(S\varrho, S\varrho) + d^2(B\varrho, S\varrho)}{d_{\mathbb{A}}(S\varrho, S\varrho) + d_{\mathbb{A}}(B\varrho, S\varrho)} \\ &\quad +\gamma(d_{\mathbb{A}}(S\varrho, S\varrho) + d_{\mathbb{A}}(B\varrho, S\varrho)) + \delta d_{\mathbb{A}}(B\varrho, S\varrho) \\ &= \alpha d_{\mathbb{A}}(B\varrho, S\varrho) + \beta d_{\mathbb{A}}(S\varrho, T\upsilon) + \gamma d_{\mathbb{A}}(B\varrho, S\varrho) + \delta d_{\mathbb{A}}(S\varrho, T\upsilon) \\ \theta_{\mathbb{A}} &\succeq (\alpha + \beta + \gamma + \delta)d_{\mathbb{A}}(B\varrho, S\varrho). \end{aligned}$$

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Hence, $\|d_{\mathbb{A}}(S\varrho, B\varrho)\| = 0$ implies $S\varrho = B\varrho$.

Similarly, it may be proved that $Av = Tv$. Hence, we get $Av = Tv = B\varrho = S\varrho = \zeta$. Since A, T are weakly compatible mapping so

$$A\zeta = ATv = TAv = T\zeta.$$

Substituting $\vartheta = \zeta$ and $\varsigma = \varrho$ in condition (ii), we get

$$\begin{aligned} d_{\mathbb{A}}(T\zeta, S\varrho) &\succeq \alpha\frac{d^2(T\zeta, A\zeta) + d^2(B\varrho, S\varrho)}{d_{\mathbb{A}}(A\zeta, S\varrho) + d_{\mathbb{A}}(B\varrho, T\zeta)} + \beta\frac{d^2(S\varrho, A\zeta) + d^2(B\varrho, T\zeta)}{d_{\mathbb{A}}(A\zeta, S\varrho) + d_{\mathbb{A}}(B\varrho, T\zeta)} \\ &\quad +\gamma(d_{\mathbb{A}}(A\zeta, T\zeta) + d_{\mathbb{A}}(S\varrho, B\varrho)) + \delta d_{\mathbb{A}}(A\zeta, B\varrho) \\ d_{\mathbb{A}}(T\zeta, \zeta) &\succeq \beta d_{\mathbb{A}}(T\zeta, \zeta) + \delta d_{\mathbb{A}}(T\zeta, \zeta) \\ \theta_{\mathbb{A}} &\succeq (\beta + \delta - 1)d_{\mathbb{A}}(T\zeta, \zeta). \end{aligned}$$

Since, $\|(\beta + \delta - 1)\| \neq 0$; hence the results holds for $\|d_{\mathbb{A}}(\zeta, T\zeta)\| = 0$ implies $\zeta = T\zeta$. Hence, $T\zeta = \zeta = A\zeta$

Similarly, $S\zeta = B\zeta = \zeta$. Hence, ζ is common fixed point of A, B, T and S.

Uniqueness: To prove the uniqueness, let η be another common fixed of A, B, T and S. Substituting $\vartheta = \zeta$ and $\varsigma = \eta$ in condition (ii), we get

$$\begin{aligned} d_{\mathbb{A}}(\zeta, \eta) = d_{\mathbb{A}}(T\zeta, S\eta) &\succeq \alpha\frac{d^2(T\zeta, A\zeta) + d^2(B\eta, S\eta)}{d_{\mathbb{A}}(A\zeta, S\eta) + d_{\mathbb{A}}(B\eta, T\zeta)} \\ &\quad +\beta\frac{d^2(S\eta, A\zeta) + d^2(B\eta, T\zeta)}{d_{\mathbb{A}}(A\zeta, S\eta) + d_{\mathbb{A}}(B\eta, T\zeta)} \\ &\quad +\gamma(d_{\mathbb{A}}(A\zeta, T\zeta) + d_{\mathbb{A}}(S\eta, B\eta)) + \delta d_{\mathbb{A}}(A\zeta, B\eta) \\ d_{\mathbb{A}}(\zeta, \eta) &\succeq \beta d_{\mathbb{A}}(\zeta, \eta) + \delta d_{\mathbb{A}}(\zeta, \eta) \\ \theta_{\mathbb{A}} &\succeq (\beta + \delta - 1)d_{\mathbb{A}}(\zeta, \eta). \end{aligned}$$

Since, $\|(\beta + \delta - 1)\| \neq 0$; hence the results for $\|d_{\mathbb{A}}(\zeta, \eta)\| = 0$ implies $\zeta = \eta$. $\square$

EXAMPLE 3.7. Let $\Gamma = [1, 5]$ and $\mathbb{A} = \mathbb{C}$. Define $d_{\mathbb{A}} : \Gamma \times \Gamma \to \mathbb{A}$ by $d_{\mathbb{A}}(\vartheta, \varsigma) = |\vartheta - \varsigma|$. Then, $(\Gamma, \mathbb{A}, d_{\mathbb{A}})$ is C^*-algebra valued metric space.
Define four self maps A, B, f and g on Γ by

$$A(\vartheta) = \begin{cases} 5 & \text{if } \vartheta < 3 \\ 3 & \text{if } \vartheta \geq 3 \end{cases}, \qquad B(\vartheta) = \begin{cases} \frac{\vartheta+3}{2} & \text{if } \vartheta \leq 3 \\ 3 & \text{if } \vartheta > 3 \end{cases}$$

$$g(\vartheta) = \begin{cases} 9 - 2\vartheta & \text{if } \vartheta \leq 3 \\ 3 & \text{if } \vartheta > 3 \end{cases}, \qquad f(\vartheta) = \begin{cases} 21 & \text{if } \vartheta < 3 \\ 3(4 - \vartheta) & \text{if } \vartheta \geq 3. \end{cases}$$

Clearly, $A\Gamma \subset g\Gamma$ and $B\Gamma \subset f\Gamma$ for all $\vartheta \in \Gamma$.

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To prove that (A,f) satisfy (CLR_g) and (B,g) satisfy (CLR_f) property, consider sequences $\{\vartheta_n\}$ and $\{\varsigma_n\}$ defined by $\varsigma_n = 3 - \dfrac{n}{n^3-2n}$ and $\vartheta_n = 3 + \dfrac{n}{n^3-2n}$.

We have $\lim\limits_{n\to+\infty} A\vartheta_n = \lim\limits_{n\to+\infty} f\vartheta_n = 3 = g3 \in g\Gamma$ and $\lim\limits_{n\to+\infty} B\varsigma_n = \lim\limits_{n\to+\infty} g\varsigma_n = 3 = f3 \in f\Gamma$.

It is easily proved that the pairs (A,g) and (B,f) are weakly compatible.

Following cases arises :

Case (i): Let $\vartheta,\varsigma < 3$, let $\varsigma = 2$ and $\alpha=\beta=\gamma=\frac{6}{5}$

$$A\vartheta = 5, \quad f\vartheta = 21, \quad B\varsigma = \frac{5}{2} \quad \text{and} \quad g\varsigma = 5.$$

$$\begin{aligned} d_{\mathbb{A}}(A\vartheta, B\varsigma) &= \frac{5}{2}, \quad d_{\mathbb{A}}(f\vartheta, A\vartheta) = 16, \quad d_{\mathbb{A}}(f\vartheta, g\varsigma) = 16, \\ d_{\mathbb{A}}(g\varsigma, B\varsigma) &= \frac{5}{2} \quad \text{and} \quad d_{\mathbb{A}}(g\varsigma, A\vartheta) = 0. \end{aligned}$$

Therefore,

$$\begin{aligned} &\alpha d_{\mathbb{A}}(f\vartheta, A\vartheta) d_{\mathbb{A}}(A\vartheta, g\varsigma) + \beta d_{\mathbb{A}}(g\varsigma, B\varsigma) + \gamma d_{\mathbb{A}}(A\vartheta, B\varsigma) \\ &= \alpha 16 * 0 + \beta\frac{5}{2} + \gamma\frac{5}{2} \\ &\preceq 16 = d_{\mathbb{A}}(f\vartheta, g\varsigma). \end{aligned}$$

Thus,

$$d_{\mathbb{A}}(f\vartheta, g\varsigma) \succeq \alpha d_{\mathbb{A}}(f\vartheta, A\vartheta) d_{\mathbb{A}}(A\vartheta, g\varsigma) + \beta d_{\mathbb{A}}(g\varsigma, B\varsigma) + \gamma d_{\mathbb{A}}(A\vartheta, B\varsigma)$$

for all $\vartheta,\varsigma < 3$.
Case (ii): Let $\vartheta,\varsigma = 3$ and $\alpha=\beta=\gamma=\frac{6}{5}$

$$A\vartheta = 3, \quad f\vartheta = 3, \quad B\varsigma = 3 \quad \text{and} \quad g\varsigma = 3.$$

$$\begin{aligned} d_{\mathbb{A}}(A\vartheta, B\varsigma) &= 0, \quad d_{\mathbb{A}}(f\vartheta, A\vartheta) = 0, \quad d_{\mathbb{A}}(f\vartheta, g\varsigma) = 0, \\ d_{\mathbb{A}}(g\varsigma, B\varsigma) &= 0 \quad \text{and} \quad d_{\mathbb{A}}(g\varsigma, A\vartheta) = 0. \end{aligned}$$

Therefore,

$$\alpha d_{\mathbb{A}}(f\vartheta, A\vartheta) d_{\mathbb{A}}(A\vartheta, g\varsigma) + \beta d_{\mathbb{A}}(g\varsigma, B\varsigma) + \gamma d_{\mathbb{A}}(A\vartheta, B\varsigma) = 0 = d_{\mathbb{A}}(f\vartheta, g\varsigma).$$

Thus,

$$d_{\mathbb{A}}(f\vartheta, g\varsigma) \succeq \alpha d_{\mathbb{A}}(f\vartheta, A\vartheta) d_{\mathbb{A}}(A\vartheta, g\varsigma) + \beta d_{\mathbb{A}}(g\varsigma, B\varsigma) + \gamma d_{\mathbb{A}}(A\vartheta, B\varsigma)$$

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for all $\vartheta, \varsigma = 3$.
Case (iii): Let $\vartheta, \varsigma > 3$, let $\vartheta = 4$ and $\alpha = \beta = \gamma = \frac{6}{5}$

$$A\vartheta = 3, \quad f\vartheta = 0, \quad B\varsigma = 3 \quad \text{and} \quad g\varsigma = 3.$$

$$\begin{aligned} d_{\mathbb{A}}(A\vartheta, B\varsigma) &= 0, \quad d_{\mathbb{A}}(f\vartheta, A\vartheta) = 3, \quad d_{\mathbb{A}}(f\vartheta, g\varsigma) = 3, \\ d_{\mathbb{A}}(g\varsigma, B\varsigma) &= 0 \quad \text{and} \quad d_{\mathbb{A}}(g\varsigma, A\vartheta) = 0. \end{aligned}$$

Therefore,

$$\begin{aligned} \alpha d_{\mathbb{A}}(f\vartheta, A\vartheta) d_{\mathbb{A}}(A\vartheta, g\varsigma) + \beta d_{\mathbb{A}}(g\varsigma, B\varsigma) + \gamma d_{\mathbb{A}}(A\vartheta, B\varsigma) &= \alpha 3 * 0 + \beta 0 + \gamma 0 \\ &\preceq 3 = d_{\mathbb{A}}(f\vartheta, g\varsigma). \end{aligned}$$

Thus,

$$d_{\mathbb{A}}(f\vartheta, g\varsigma) \succeq \alpha d_{\mathbb{A}}(f\vartheta, A\vartheta) d_{\mathbb{A}}(A\vartheta, g\varsigma) + \beta d_{\mathbb{A}}(g\varsigma, B\varsigma) + \gamma d_{\mathbb{A}}(A\vartheta, B\varsigma) \text{ for all } \vartheta, \varsigma > 3.$$

Hence, by the Theorem 3.4 the mappings A, B, f and g have unique common fixed point. Indeed, 3 is common unique fixed point.

Example 3.8. Let $\Gamma = [1, 6]$ and $\mathbb{A} = \mathbb{C}$. Define $d_{\mathbb{A}} : \Gamma \times \Gamma \to \mathbb{A}$ by $d_{\mathbb{A}}(\vartheta, \varsigma) = |\vartheta - \varsigma|$. Then, $(\Gamma, \mathbb{A}, d_{\mathbb{A}})$ is C^*-algebra-valued metric space.
Define four self maps A, B, f and g on Γ by

$$A(\vartheta) = \begin{cases} 4 & \text{if } \vartheta \le 4 \\ 11 & \text{if } \vartheta > 4 \end{cases}, \qquad B(\vartheta) = \begin{cases} 4 & \text{if } \vartheta \le 4 \\ \vartheta + 3 & \text{if } \vartheta > 4 \end{cases}$$

$$g(\vartheta) = \begin{cases} 4 & \text{if } \vartheta < 4 \\ 7\vartheta - 24 & \text{if } \vartheta \ge 4 \end{cases}, \qquad f(\vartheta) = \begin{cases} 16 - 5\vartheta & \text{if } \vartheta \le 4 \\ 0 & \text{if } \vartheta > 4. \end{cases}$$

Clearly, $A\Gamma \subset g\Gamma$ and $B\Gamma \subset f\Gamma$ for all $\vartheta \in \Gamma$.

To prove that (A, f) satisfy (CLR_A) property, consider sequences$\{\vartheta_n\}$ defined by $\vartheta_n = 4 - \dfrac{1}{n^2 - 2n + 3}$.

We have $\lim\limits_{n \to +\infty} A\vartheta_n = \lim\limits_{n \to +\infty} f\vartheta_n = 4 = A(4)$.

It is easily proved that A, g and B, f are weakly compatible.

Following cases arises :

Case (i): Let $\vartheta, \varsigma < 4$, let $\vartheta = 3$ and $\alpha = \beta = \gamma = \frac{6}{5}$

$$A\vartheta = 4, \quad f\vartheta = 1, \quad B\varsigma = 4 \quad \text{and} \quad g\varsigma = 4.$$

$$\begin{aligned} d_{\mathbb{A}}(A\vartheta, B\varsigma) &= 0, \quad d_{\mathbb{A}}(f\vartheta, A\vartheta) = 3, \quad d_{\mathbb{A}}(f\vartheta, g\varsigma) = 3, \\ d_{\mathbb{A}}(g\varsigma, B\varsigma) &= 0 \quad \text{and} \quad d_{\mathbb{A}}(g\varsigma, A\vartheta) = 0. \end{aligned}$$

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Therefore,

$$\begin{aligned}&\alpha d_{\mathbb{A}}(f\vartheta, A\vartheta)d_{\mathbb{A}}(A\vartheta, g\varsigma) + \beta d_{\mathbb{A}}(g\varsigma, B\varsigma) + \gamma d_{\mathbb{A}}(A\vartheta, B\varsigma)\\ &= \alpha 16 * 0 + \beta * 0 + \gamma * 0\\ &\preceq 3 = d_{\mathbb{A}}(f\vartheta, g\varsigma).\end{aligned}$$

Thus,

$d_{\mathbb{A}}(f\vartheta, g\varsigma) \succeq \alpha d_{\mathbb{A}}(f\vartheta, A\vartheta)d_{\mathbb{A}}(A\vartheta, g\varsigma)+\beta d_{\mathbb{A}}(g\varsigma, B\varsigma)+\gamma d_{\mathbb{A}}(A\vartheta, B\varsigma)$ for all $\vartheta, y < 4$.

Case (ii): Let $\vartheta, \varsigma = 4$ and $\alpha = \beta = \gamma = \frac{6}{5}$

$$A\vartheta = 4, \quad f\vartheta = 4, \quad B\varsigma = 4 \quad \text{and} \quad g\varsigma = 4.$$

$$\begin{aligned} d_{\mathbb{A}}(A\vartheta, B\varsigma) &= 0, \quad d_{\mathbb{A}}(f\vartheta, A\vartheta) = 0, \quad d_{\mathbb{A}}(f\vartheta, g\varsigma) = 0,\\ d_{\mathbb{A}}(g\varsigma, B\varsigma) &= 0 \quad \text{and} \quad d_{\mathbb{A}}(g\varsigma, A\vartheta) = 0.\end{aligned}$$

Therefore,

$$\alpha d_{\mathbb{A}}(f\vartheta, A\vartheta)d_{\mathbb{A}}(A\vartheta, g\varsigma) + \beta d_{\mathbb{A}}(g\varsigma, B\varsigma) + \gamma d_{\mathbb{A}}(A\vartheta, B\varsigma) = 0 = d_{\mathbb{A}}(f\vartheta, g\varsigma).$$

Thus,

$$d_{\mathbb{A}}(f\vartheta, g\varsigma) \succeq \alpha d_{\mathbb{A}}(f\vartheta, A\vartheta)d_{\mathbb{A}}(A\vartheta, g\varsigma) + \beta d_{\mathbb{A}}(g\varsigma, B\varsigma) + \gamma d_{\mathbb{A}}(A\vartheta, B\varsigma)$$

for all $\vartheta, \varsigma = 4$.
Case (iii): Let $\vartheta, \varsigma > 4$, let $\varsigma = 5$ and $\alpha = \beta = \gamma = \frac{6}{5}$

$$A\vartheta = 11, \quad f\vartheta = 0, \quad B\varsigma = 8 \quad \text{and} \quad g\varsigma = 11.$$

$$\begin{aligned} d_{\mathbb{A}}(A\vartheta, B\varsigma) &= 3, \quad d_{\mathbb{A}}(f\vartheta, A\vartheta) = 11, \quad d_{\mathbb{A}}(f\vartheta, g\varsigma) = 11,\\ d_{\mathbb{A}}(g\varsigma, B\varsigma) &= 3 \quad \text{and} \quad d_{\mathbb{A}}(g\varsigma, A\vartheta) = 0.\end{aligned}$$

Therefore,

$$\begin{aligned}&\alpha d_{\mathbb{A}}(f\vartheta, A\vartheta)d_{\mathbb{A}}(A\vartheta, g\varsigma) + \beta d_{\mathbb{A}}(g\varsigma, B\varsigma) + \gamma d_{\mathbb{A}}(A\vartheta, B\varsigma)\\ &= \alpha 11 * 0 + \beta * 3 + \gamma * 3\\ &\preceq 11 = d_{\mathbb{A}}(f\vartheta, g\varsigma).\end{aligned}$$

Thus,

$$d_{\mathbb{A}}(f\vartheta, g\varsigma) \succeq \alpha d_{\mathbb{A}}(f\vartheta, A\vartheta)d_{\mathbb{A}}(A\vartheta, g\varsigma) + \beta d_{\mathbb{A}}(g\varsigma, B\varsigma) + \gamma d_{\mathbb{A}}(A\vartheta, B\varsigma)$$

for all $\vartheta, \varsigma > 4$. Hence, by the Theorem 3.3 the mappings A, B, f and g have unique common fixed point. Indeed, 4 is common unique fixed point.

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