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Some Properties of G^{**}-Autonilpotent Groups

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Abstract. Let G be a group. In this paper, we first introduce a new series on the subgroups generated by G and IA(G) and define G^{**}-autonilpotency in this series. Then we study some properties of these concepts and their relationships.

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1. Introduction

Let G be a group. Let us denote by G', Z(G) and Aut(G), respectively, the commutator subgroup, centre and the full automorphism group. Bachmuth [1] defined an IA-automorphism of group G as

$$IA(G) = \left\{\alpha \in Aut(G) \;\middle|\; g^{-1}\alpha(g) = [g, \alpha] \in G',\ \forall\, g \in G\right\}.$$

Hegarty [3] introduced the autocommutator subgroup as follows:

$$K(G) = \langle [g, \alpha] \mid g \in G,\ \alpha \in Aut(G)\rangle.$$

On similar lines, Ghumde and Ghate [2] introduced the subgroup

$$G^{**} = \langle [g, \alpha] \mid g \in G,\ \alpha \in IA(G)\rangle.$$

For any group G, $G' = G^{**} \leqslant K(G)$.

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First, we study the conditions in which G^{**} is equal to K(G). The following proposition clearly states these conditions in which part (2) is a result of [2] and then all parts are derived from the definitions of G^{**} and $K(G)$.

Proposition 1.1. *For a group G, $G^{**} = K(G)$ if one of the following conditions holds*

1) *G be a complete group, i.e., $G = G'$.*
2) *$[G : G'] = 2$, because then IA(G)=Aut(G).*
3) *Aut(G)=Inn(G).*

For every group G, $G^{**} = \langle 1 \rangle$ if and only if G is a trivial or abelian group. G^{**} is an abelian group if and only if G is a metabelian group.

Lemma 1.2. *Let $G = H_1 \times H_2$, $H_1 \neq \langle 1 \rangle$ and $H_1 \cap G^{**} = \langle 1 \rangle$, then $H_1 \cong C_2$.*

Proof. Suppose by way of contradiction that $|H_1| > 2$, then H_1 has a nontrivial automorphism α and α can be extended to an automorphism β of G where $\beta(h_1) = \alpha(h_1)$, for all $h_1 \in H_1$ and $\beta(h_2) = h_2$, for all $h_2 \in H_2$. If $h_1 \in H_1$ is arbitrary, then

$$[h_1, \beta] = [h_1, \alpha] \in H_1 \cap G^{**} = \langle 1 \rangle.$$

Thus $\alpha(h_1) = h_1$ contradicting the fact that α is nontrivial. □

2. IA-commutator series

Parvaneh and Moghaddam [4] introduced the concept of autonilpotent groups in 2010. They defined the autocommutator subgroup of weight n+1 in the following way:

$$\begin{aligned} K_n(G) &= [K_{n-1}(G), Aut(G)] \\ &= \langle [g, \alpha_1, \alpha_2, \ldots, \alpha_n] \mid g \in G,\ \alpha_1, \alpha_2, \ldots, \alpha_n \in Aut(G) \rangle, \end{aligned}$$

for all $n \geq 1$. Moreover, they obtained a descending chain of autocommutator subgroups of G as follows:

$$\cdots \subseteq K_n(G) \subseteq \cdots \subseteq K_2(G) \subseteq K_1(G) = K(G) \subseteq K_0 = G.$$

Also, they called a group to be autonilpotent of class at most n if there exists a positive integer n such that $K_n(G) = G$.

In this section, we first introduce the IA-commutator series and state the properties of its terms. We define new automorphisms in this series and study their relationships with IA(G), $Aut_c(G)$, Inn(G) and each other.

2.1. **The terms of the IA-commutator series.**

Definition 2.1. We define the IA-commutator series or G^{**} series of G in the following way:

$$\cdots \subseteq G_n^{**} \subseteq \cdots \subseteq G_2^{**} \subseteq G_1^{**} = G^{**} = G' \subseteq G_0^{**} = G \tag{2.1}$$

where n is a positive integer and

$$\begin{aligned} G_n^{**} =& \langle [g, \alpha_1, \ldots, \alpha_n] \mid g \in G,\ \alpha_1, \ldots, \alpha_n \in IA(G) \rangle \\ =& [G_{n-1}^{**}, IA(G)]. \end{aligned}$$

$G_n^{**} \leqslant K(G)$ and if G is an abelian group, then $G_n^{**} = \langle 1 \rangle$, for every positive integer n.

Proposition 2.2. *Let G be a group, then for any positive integer n, G_n^{**} is a characteristic subgroup of G.*

Proof. Clearly, $G_n^{**} \leqslant G$. Now, let $\beta \in Aut(G)$ and $[g, \alpha_1, \ldots, \alpha_n] \in G_n^{**}$, for every $g \in G$ and $\alpha_1, \ldots, \alpha_n \in IA(G)$. Then, one can write

$$\begin{aligned} \beta([g, \alpha_1, \ldots, \alpha_n]) &= \beta\big(g^{-1}\alpha_1 \cdots \alpha_n(g)\big) \\ &= \beta(g^{-1})\beta\big(\alpha_1 \cdots \alpha_n(g)\big) \\ &= \big(\beta(g)^{-1}\big)\beta\big(\alpha_1 \cdots \alpha_n \beta^{-1}\beta(g)\big) \\ &= \big(\beta(g)^{-1}\big)\beta\alpha_1 \cdots \alpha_n \beta^{-1}\big(\beta(g)\big) \\ &= [\underbrace{\beta(g)}_{\in G}, \underbrace{\beta\alpha_1 \cdots \alpha_n \beta^{-1}}_{\in IA(G) \trianglelefteq Aut(G)}] \in G_{n+2}^{**} \leqslant G_n^{**}. \end{aligned}$$

□

Lemma 2.3. *a) Let H and K be two arbitrary groups, then for any positive integer n,*

$$H_n^{**} \times K_n^{**} \subseteq (H \times K)_n^{**}.$$

b) If H and K be finite groups such that $(|H|, |K|) = 1$, Then for any positive integer n,

$$H_n^{**} \times K_n^{**} = (H \times K)_n^{**}.$$

Proof. a) Because $IA(H \times K) = IA(H) \times IA(K)$ and

$$H^{**} \times K^{**} = H' \times K' = (H \times K)' = (H \times K)^{**},$$

by induction on n, it is easy to check that

$$([h, \alpha_1, \ldots, \alpha_n], [k, \beta_1, \ldots, \beta_n]) = [(h, k), \alpha_1 \times \beta_1, \ldots, \alpha_n \times \beta_n],$$

for $\alpha_i \in IA(H)$, $\beta_i \in IA(K)$, $h \in H$ and $k \in K$.
b) We just have to prove that

$$(H \times K)_n^{**} \subseteq H_n^{**} \times K_n^{**}.$$

It is easily seen that $\sigma|_H \in IA(H)$ and $\sigma|_K \in IA(K)$, for all $\sigma \in IA(H \times K)$. Now by induction on n, we have

$$[(h, k), \sigma_1, \ldots, \sigma_n] = ([h, \sigma_1|_H, \ldots, \sigma_n|_H], [k, \sigma_1|_K, \ldots, \sigma_n|_K]),$$

for all $h \in H$, $k \in K$ and $\sigma_1, \ldots, \sigma_n \in IA(H \times K)$. This implies the result. □

Lemma 2.4. *If H is a characteristic subgroup of index two of a given group G, then G_n^{**} is contained in H, for any positive integer n.*

Proof. It follows from lemma 2.4 [4]. □

2.2. **The automorphisms on the terms of the IA-commutator series.** In this section, after some new definitions, we give our main results about the automorphisms of the IA-commutator series.

Definition 2.5. The kernel of the natural homomorphism from Aut(G) to $Aut(G/G_j^{**})$ is named the group of G_j^{**}-automorphism and is denoted by $Aut_{G_j^{**}}$.

According to the above definition, a G_j^{**}-automorphism group acts as the identity on G modulo G_j^{**}, Thus:

$$Aut_{G_j^{**}}(G) = \{\alpha \in Aut(G) \mid g^{-1}\alpha(g) \in G_j^{**},\ \forall\, g \in G\} \trianglelefteq Aut(G).$$

Also, we have $Aut_{G^{**}}(G) = IA(G)$ and $Aut_{G_j^{**}}(G) \leqslant IA(G)$, for every $j \geq 2$.

Notation 2.6. We use the notation $Aut_{AG_j^{**}}(G) = Aut_c(G) \cap Aut_{G_j^{**}}(G)$. Another definition of $Aut_{AG_j^{**}}(G)$ is given by

$$Aut_{AG_j^{**}}(G) = \{\alpha \in Aut(G) \mid g^{-1}\alpha(g) \in Z(G) \cap G_j^{**},\ \forall\, g \in G\}.$$

Proposition 2.7. *Let G be any group,*

a) *$\varphi \in Aut_{G_j^{**}}(G)$ if and only if $[\alpha, \varphi] \in Aut_{G_j^{**}}(G)$, for every $\alpha \in Aut(G)$.*
b) *$Aut_{AG_j^{**}}(G)$ is a normal subgroup of $Aut_c(G)$ and we have*

$$\frac{Aut_c(G)}{Aut_{AG_j^{**}}(G)} \cong \frac{Aut_c(G)Aut_{G_j^{**}}(G)}{Aut_{G_j^{**}}(G)}.$$

Proof. a) It is obvious by the normality of $Aut_{G_j^{**}}(G)$.

b) First, we prove that $G_j^{**} \overset{ch}{\leqslant} G$. Because G_j^{**} is a generated subgroup, so clearly $G_j^{**} \leqslant G$. Let $\psi \in Aut(G)$ and $[g, \alpha_1, \ldots, \alpha_j] \in G_j^{**}$. Then, one can write

$$\begin{aligned}
\psi([g, \alpha_1, \ldots, \alpha_j]) &= \psi\big(g^{-1}\alpha_1 \cdots \alpha_j(g)\big) \\
&= \psi(g)^{-1}\psi\big(\alpha_1 \cdots \alpha_j(g)\big) \\
&= \big(\psi(g)\big)^{-1}\psi\alpha_1 \cdots \alpha_j\big(\psi^{-1}\psi(g)\big) \\
&= \big(\psi(g)\big)^{-1}(\psi\alpha_1 \cdots \alpha_j\psi^{-1})\big(\psi(g)\big) \\
&= [\underbrace{\psi(g)}_{\in G}, \underbrace{\psi\alpha_1 \cdots \alpha_j\psi^{-1}}_{\in IA(G)}] \in G_j^{**}.
\end{aligned}$$

Therefore, $\psi([g, \alpha_1, \ldots, \alpha_j]) \in G_j^{**}$.

Now, let $\sigma \in Aut_c(G)$ and $\beta \in Aut_{AG_j^{**}}(G)$. We show that $\sigma^{-1}\beta\sigma \in Aut_{AG_j^{**}}(G)$. For every $g \in G$, we have

$$\begin{aligned} g^{-1}\big(\sigma^{-1}\beta\sigma\big)(g) &= g^{-1}\sigma^{-1}\Big(\sigma(g)\big(\sigma(g)\big)^{-1}\beta\big(\sigma(g)\big)\Big) \\ &= g^{-1}g\sigma^{-1}\Big(\underbrace{\big(\sigma(g)\big)^{-1}\beta\big(\sigma(g)\big)}_{\in Z(G)\cap G_j^{**}}\Big). \end{aligned}$$

Because the intersection of two characteristic subgroups is a characteristic subgroup, the first part is proved.

For the second part, the result follows from the definition of $Aut_{AG_j^{**}}(G)$ and the third isomorphism theorem. $\square$

Corollary 2.8. *For any group G, $[Aut(G), Aut_{G_j^{**}}(G)] \leqslant Aut_{G_j^{**}}(G)$.*

Theorem 2.9. *Let G be a group. If $Aut_c\big(G/G_j^{**}\big) = Inn\big(G/G_j^{**}\big)$, then*

$$Aut_c(G) \leqslant Inn(G)Aut_{G_j^{**}}(G).$$

Proof. Let $\alpha \in Aut_c(G)$. By hypothesis, $Aut_c\big(G/G_j^{**}\big) = Inn\big(G/G_j^{**}\big)$, so there exists $g \in G$ such that $\alpha(x)G_j^{**} = x^g G_j^{**}$, for all $x \in G$. Hence,

$$\begin{aligned} & x^{-g}\alpha(x) = \Big(x^{-1}\big(\alpha(x)\big)^{g^{-1}}\Big)^g \in G_j^{**} \\ \Longrightarrow\quad & x^{-1}\big(\alpha(x)\big)^{g^{-1}} \in G_j^{**} \\ \Longrightarrow\quad & x^{-1}g\big(\alpha(x)\big)g^{-1} \in G_j^{**} \\ \Longrightarrow\quad & x^{-1}\varphi_g^{-1}\alpha(x) \in G_j^{**} \end{aligned}$$

where $\varphi_g \in Inn(G)$.

Consequently, $\varphi_g^{-1}\alpha \in Aut_{G_j^{**}}(G)$, i.e., $\alpha = \varphi_g\varphi_g^{-1}\alpha \in Inn(G)Aut_{G_j^{**}}(G)$. $\square$

In the special case j=1, we have the following result.

Corollary 2.10. *For any group G, If $Aut_c\big(G/G'\big) = Inn\big(G/G'\big)$, then*

$$Aut_c(G) \leqslant Inn(G)IA(G).$$

3. G^{**}-AUTONILPOTENT GROUPS

In the previous section, we saw that Parvaneh and Moghaddam [4] introduced autonilpotent groups. In this section, we define G^{**}-autonilpotent groups and study their properties.

Definition 3.1. A group G is called a G^{**}-autonilpotent group of class at most n if the series (2.1) ends in the identity subgroup after a finite number of steps.

Remark 3.2. The autonilpotent groups are G^{**}-autonilpotent, but the converse is not true in general. As an example, all of the abelian groups are G^{**}-autonilpotent, but

$$G = \bigoplus_{i=1}^{l} \mathbb{Z}_{2^{m_i}}, \qquad l > 1, \;\; m_1 = m_2 \geq \cdots \geq m_l$$

is not autonilpotent, because $K_n(G) = G$.

Proposition 3.3. *If H is not H^{**}-autonilpotent or K is not K^{**}-autonilpotent group, then $H \times K$ is not $(H \times K)^{**}$-autonilpotent too.*

Proof. It is clear by lemma 2.3. □

Corollary 3.4. *If H_1, H_2, ..., H_l are H_i^{**}-autonilpotent groups with coprime orders, then $H_1 \times H_2 \times \cdots \times H_l$ is also $(H_1 \times H_2 \times \cdots \times H_l)^{**}$-autonilpotent.*

Proof. The result is followed by induction on l. □

Theorem 3.5. *For a characteristic subgroup H of a given group G, if H and G/H are H^{**}-autonilpotent and $(G/H)^{**}$-autonilpotent, respectively, then G is also G^{**}-autonilpotent.*

Proof. Assume that there exist positive integers i and j such that

$$H_i^{**} = \langle 1 \rangle, \qquad \left(\frac{G}{H}\right)_j^{**} = \langle 1 \rangle.$$

It is clear that $G^{**}H/H = (G/H)^{**}$ and by induction on j, one gets

$$\frac{G_j^{**}H}{H} \subseteq \left(\frac{G}{H}\right)_j^{**} = 1_{\frac{G}{H}}$$

whence $G_j^{**} \subseteq H$. Let $[g, \alpha]$ be an arbitrary generator of $G_{j+1}^{**} = [G_j^{**}, IA(G)]$. It is easy to see that $[g, \alpha|_H] \in [H, IA(H)]$, thus $G_{j+1}^{**} \subseteq H'$. By induction on i, we have $G_{i+j}^{**} \subseteq H_i^{**} = \langle 1 \rangle$. Therefore, G is a G^{**}-autonilpotent group of class at most i+j. □

Theorem 3.6. *Let G be a group and H be a proper characteristic subgroup of G with G/H is $(G/H)^{**}$-autonilpotent of class c. If $H \cap G_c^{**} = \langle 1 \rangle$, then G is G^{**}-autonilpotent.*

Proof. With the same argument as the proof of theorem 3.5, one can easily check that $G_c^{**}H/H \subseteq (G/H)_c^{**}$. Because $H \cap G_c^{**} = \langle 1 \rangle$, we have

$$\frac{G_c^{**}}{H \cap G_c^{**}} \cong \frac{G_c^{**}H}{H} \subseteq \left(\frac{G}{H}\right)_c^{**} = 1_{\frac{G}{H}}.$$

Thus, $G_c^{**} = \langle 1 \rangle$ which gives the G^{**}-autonilpotency of G. □

Theorem 3.7. *Let H be a characteristic subgroup of a G^{**}-autonilpotent group G of class c. If $G = HG^{**}$, then G=H.*

Proof. By hypothesis,

$$G^{**} = [G, IA(G)] = [HG^{**}, IA(G)].$$

We prove that

$$[HG^{**}, IA(G)] \leqslant [H, IA(G)]^{G^{**}}[G^{**}, IA(G)].$$

Let $h \in H$, $g \in G^{**}$ and $\alpha \in IA(G)$, then $[hg, \alpha] \in [HG^{**}, IA(G)]$ and

$$\begin{aligned}[hg, \alpha] &= (hg)^{-1}\alpha(hg)\\ &= g^{-1}h^{-1}\alpha(h)\alpha(g)\\ &= g^{-1}h^{-1}\alpha(h)gg^{-1}\alpha(g)\\ &= [h, \alpha]^g[g, \alpha] \in [H, IA(G)]^{G^{**}}[G^{**}, IA(G)].\end{aligned}$$

Because H is a characteristic subgroup, we have

$$\begin{aligned}G^{**} =&[HG^{**}, IA(G)]\\ \leqslant&[H, IA(G)]^{G^{**}}[G^{**}, IA(G)]\\ \leqslant&HG_2^{**}.\end{aligned}$$

Thus,

$$G = HG^{**} \leqslant HG_2^{**}.$$

So, $G = HG_2^{**}$. Now, by induction, we have $G = HG_c^{**}$, and G=H since G is a G^{**}-autonilpotent group of class c. □

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