

Some Properties of G^{**} -Autonilpotent Groups

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ABSTRACT. Let G be a group. In this paper, we first introduce a new series on the subgroups generated by G and $IA(G)$ and define G^{**} -autonilpotency in this series. Then we study some properties of these concepts and their relationships.

Keywords: IA-group, IA-commutator series, Autonilpotent groups, G^{**} -autonilpotent groups.

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1. INTRODUCTION

Let G be a group. Let us denote by G' , $Z(G)$ and $Aut(G)$, respectively, the commutator subgroup, centre and the full automorphism group. Bachmuth [1] defined an IA-automorphism of group G as

$$IA(G) = \{ \alpha \in Aut(G) \mid g^{-1}\alpha(g) = [g, \alpha] \in G', \forall g \in G \}.$$

Hegarty [3] introduced the autocommutator subgroup as follows:

$$K(G) = \langle [g, \alpha] \mid g \in G, \alpha \in Aut(G) \rangle.$$

On similar lines, Ghumde and Ghate [2] introduced the subgroup

$$G^{**} = \langle [g, \alpha] \mid g \in G, \alpha \in IA(G) \rangle.$$

For any group G , $G' = G^{**} \leq K(G)$.

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First, we study the conditions in which G^{**} is equal to $K(G)$. The following proposition clearly states these conditions in which part (2) is a result of [2] and then all parts are derived from the definitions of G^{**} and $K(G)$.

Proposition 1.1. *For a group G , $G^{**} = K(G)$ if one of the following conditions holds*

- 1) G be a complete group, i.e., $G = G'$.
- 2) $[G : G'] = 2$, because then $IA(G) = \text{Aut}(G)$.
- 3) $\text{Aut}(G) = \text{Inn}(G)$.

For every group G , $G^{**} = \langle 1 \rangle$ if and only if G is a trivial or abelian group. G^{**} is an abelian group if and only if G is a metabelian group.

Lemma 1.2. *Let $G = H_1 \times H_2$, $H_1 \neq \langle 1 \rangle$ and $H_1 \cap G^{**} = \langle 1 \rangle$, then $H_1 \cong C_2$.*

Proof. Suppose by way of contradiction that $|H_1| > 2$, then H_1 has a nontrivial automorphism α and α can be extended to an automorphism β of G where $\beta(h_1) = \alpha(h_1)$, for all $h_1 \in H_1$ and $\beta(h_2) = h_2$, for all $h_2 \in H_2$. If $h_1 \in H_1$ is arbitrary, then

$$[h_1, \beta] = [h_1, \alpha] \in H_1 \cap G^{**} = \langle 1 \rangle.$$

Thus $\alpha(h_1) = h_1$ contradicting the fact that α is nontrivial. $\square$

2. IA-COMMUTATOR SERIES

Parvaneh and Moghaddam [4] introduced the concept of autonilpotent groups in 2010. They defined the autocommutator subgroup of weight $n+1$ in the following way:

$$\begin{aligned} K_n(G) &= [K_{n-1}(G), \text{Aut}(G)] \\ &= \langle [g, \alpha_1, \alpha_2, \dots, \alpha_n] \mid g \in G, \alpha_1, \alpha_2, \dots, \alpha_n \in \text{Aut}(G) \rangle, \end{aligned}$$

for all $n \geq 1$. Moreover, they obtained a descending chain of autocommutator subgroups of G as follows:

$$\dots \subseteq K_n(G) \subseteq \dots \subseteq K_2(G) \subseteq K_1(G) = K(G) \subseteq K_0 = G.$$

Also, they called a group to be autonilpotent of class at most n if there exists a positive integer n such that $K_n(G) = G$.

In this section, we first introduce the IA-commutator series and state the properties of its terms. We define new automorphisms in this series and study their relationships with $IA(G)$, $\text{Aut}_c(G)$, $\text{Inn}(G)$ and each other.

2.1. The terms of the IA-commutator series.

Definition 2.1. We define the IA-commutator series or G^{**} series of G in the following way:

$$\dots \subseteq G_n^{**} \subseteq \dots \subseteq G_2^{**} \subseteq G_1^{**} = G^{**} = G' \subseteq G_0^{**} = G \quad (2.1)$$

where n is a positive integer and

$$\begin{aligned} G_n^{**} &= \langle [g, \alpha_1, \dots, \alpha_n] \mid g \in G, \alpha_1, \dots, \alpha_n \in IA(G) \rangle \\ &= [G_{n-1}^{**}, IA(G)]. \end{aligned}$$

$G_n^{**} \leq K(G)$ and if G is an abelian group, then $G_n^{**} = \langle 1 \rangle$, for every positive integer n .

Proposition 2.2. *Let G be a group, then for any positive integer n , G_n^{**} is a characteristic subgroup of G .*

Proof. Clearly, $G_n^{**} \leq G$. Now, let $\beta \in \text{Aut}(G)$ and $[g, \alpha_1, \dots, \alpha_n] \in G_n^{**}$, for every $g \in G$ and $\alpha_1, \dots, \alpha_n \in IA(G)$. Then, one can write

$$\begin{aligned} \beta([g, \alpha_1, \dots, \alpha_n]) &= \beta(g^{-1} \alpha_1 \cdots \alpha_n(g)) \\ &= \beta(g^{-1}) \beta(\alpha_1 \cdots \alpha_n(g)) \\ &= (\beta(g)^{-1}) \beta(\alpha_1 \cdots \alpha_n \beta^{-1} \beta(g)) \\ &= (\beta(g)^{-1}) \beta \alpha_1 \cdots \alpha_n \beta^{-1} (\beta(g)) \\ &= \underbrace{[\beta(g)]}_{\in G}, \underbrace{\beta \alpha_1 \cdots \alpha_n \beta^{-1}}_{\in IA(G) \trianglelefteq \text{Aut}(G)} \in G_{n+2}^{**} \leq G_n^{**}. \end{aligned}$$

□

Lemma 2.3. a) *Let H and K be two arbitrary groups, then for any positive integer n ,*

$$H_n^{**} \times K_n^{**} \subseteq (H \times K)_n^{**}.$$

b) *If H and K be finite groups such that $(|H|, |K|) = 1$, Then for any positive integer n ,*

$$H_n^{**} \times K_n^{**} = (H \times K)_n^{**}.$$

Proof. a) Because $IA(H \times K) = IA(H) \times IA(K)$ and

$$H^{**} \times K^{**} = H' \times K' = (H \times K)' = (H \times K)^{**},$$

by induction on n , it is easy to check that

$$([h, \alpha_1, \dots, \alpha_n], [k, \beta_1, \dots, \beta_n]) = [(h, k), \alpha_1 \times \beta_1, \dots, \alpha_n \times \beta_n],$$

for $\alpha_i \in IA(H)$, $\beta_i \in IA(K)$, $h \in H$ and $k \in K$.

b) We just have to prove that

$$(H \times K)_n^{**} \subseteq H_n^{**} \times K_n^{**}.$$

It is easily seen that $\sigma|_H \in IA(H)$ and $\sigma|_K \in IA(K)$, for all $\sigma \in IA(H \times K)$. Now by induction on n , we have

$$[(h, k), \sigma_1, \dots, \sigma_n] = ([h, \sigma_1|_H, \dots, \sigma_n|_H], [k, \sigma_1|_K, \dots, \sigma_n|_K]),$$

for all $h \in H$, $k \in K$ and $\sigma_1, \dots, \sigma_n \in IA(H \times K)$. This implies the result. □

Lemma 2.4. *If H is a characteristic subgroup of index two of a given group G , then G_n^{**} is contained in H , for any positive integer n .*

Proof. It follows from lemma 2.4 [4]. $\square$

2.2. The automorphisms on the terms of the IA-commutator series.

In this section, after some new definitions, we give our main results about the automorphisms of the IA-commutator series.

Definition 2.5. The kernel of the natural homomorphism from $\text{Aut}(G)$ to $\text{Aut}(G/G_j^{**})$ is named the group of G_j^{**} -automorphism and is denoted by $\text{Aut}_{G_j^{**}}$.

According to the above definition, a G_j^{**} -automorphism group acts as the identity on G modulo G_j^{**} , Thus:

$$\text{Aut}_{G_j^{**}}(G) = \{\alpha \in \text{Aut}(G) \mid g^{-1}\alpha(g) \in G_j^{**}, \forall g \in G\} \trianglelefteq \text{Aut}(G).$$

Also, we have $\text{Aut}_{G^{**}}(G) = \text{IA}(G)$ and $\text{Aut}_{G_j^{**}}(G) \leq \text{IA}(G)$, for every $j \geq 2$.

Notation 2.6. We use the notation $\text{Aut}_{AG_j^{**}}(G) = \text{Aut}_c(G) \cap \text{Aut}_{G_j^{**}}(G)$. Another definition of $\text{Aut}_{AG_j^{**}}(G)$ is given by

$$\text{Aut}_{AG_j^{**}}(G) = \{\alpha \in \text{Aut}(G) \mid g^{-1}\alpha(g) \in Z(G) \cap G_j^{**}, \forall g \in G\}.$$

Proposition 2.7. *Let G be any group,*

- a) $\varphi \in \text{Aut}_{G_j^{**}}(G)$ if and only if $[\alpha, \varphi] \in \text{Aut}_{G_j^{**}}(G)$, for every $\alpha \in \text{Aut}(G)$.
- b) $\text{Aut}_{AG_j^{**}}(G)$ is a normal subgroup of $\text{Aut}_c(G)$ and we have

$$\frac{\text{Aut}_c(G)}{\text{Aut}_{AG_j^{**}}(G)} \cong \frac{\text{Aut}_c(G)\text{Aut}_{G_j^{**}}(G)}{\text{Aut}_{G_j^{**}}(G)}.$$

Proof. a) It is obvious by the normality of $\text{Aut}_{G_j^{**}}(G)$.

b) First, we prove that $G_j^{**} \stackrel{ch}{\leq} G$. Because G_j^{**} is a generated subgroup, so clearly $G_j^{**} \leq G$. Let $\psi \in \text{Aut}(G)$ and $[g, \alpha_1, \dots, \alpha_j] \in G_j^{**}$. Then, one can write

$$\begin{aligned} \psi([g, \alpha_1, \dots, \alpha_j]) &= \psi(g^{-1}\alpha_1 \cdots \alpha_j(g)) \\ &= \psi(g)^{-1}\psi(\alpha_1 \cdots \alpha_j(g)) \\ &= (\psi(g))^{-1}\psi\alpha_1 \cdots \alpha_j(\psi^{-1}\psi(g)) \\ &= (\psi(g))^{-1}(\psi\alpha_1 \cdots \alpha_j\psi^{-1})(\psi(g)) \\ &= \underbrace{[\psi(g), \psi\alpha_1 \cdots \alpha_j\psi^{-1}]}_{\in G} \in G_j^{**}. \end{aligned}$$

Therefore, $\psi([g, \alpha_1, \dots, \alpha_j]) \in G_j^{**}$.

Now, let $\sigma \in \text{Aut}_c(G)$ and $\beta \in \text{Aut}_{AG_j^{**}}(G)$. We show that $\sigma^{-1}\beta\sigma \in \text{Aut}_{AG_j^{**}}(G)$. For every $g \in G$, we have

$$\begin{aligned} g^{-1}(\sigma^{-1}\beta\sigma)(g) &= g^{-1}\sigma^{-1}\left(\sigma(g)(\sigma(g))^{-1}\beta(\sigma(g))\right) \\ &= g^{-1}g\sigma^{-1}\left(\underbrace{(\sigma(g))^{-1}\beta(\sigma(g))}_{\in Z(G) \cap G_j^{**}}\right). \end{aligned}$$

Because the intersection of two characteristic subgroups is a characteristic subgroup, the first part is proved.

For the second part, the result follows from the definition of $\text{Aut}_{AG_j^{**}}(G)$ and the third isomorphism theorem. $\square$

Corollary 2.8. *For any group G , $[\text{Aut}(G), \text{Aut}_{G_j^{**}}(G)] \leq \text{Aut}_{G_j^{**}}(G)$.*

Theorem 2.9. *Let G be a group. If $\text{Aut}_c(G/G_j^{**}) = \text{Inn}(G/G_j^{**})$, then*

$$\text{Aut}_c(G) \leq \text{Inn}(G)\text{Aut}_{G_j^{**}}(G).$$

Proof. Let $\alpha \in \text{Aut}_c(G)$. By hypothesis, $\text{Aut}_c(G/G_j^{**}) = \text{Inn}(G/G_j^{**})$, so there exists $g \in G$ such that $\alpha(x)G_j^{**} = x^gG_j^{**}$, for all $x \in G$. Hence,

$$\begin{aligned} x^{-g}\alpha(x) &= \left(x^{-1}(\alpha(x))^{g^{-1}}\right)^g \in G_j^{**} \\ \implies x^{-1}(\alpha(x))^{g^{-1}} &\in G_j^{**} \\ \implies x^{-1}g(\alpha(x))g^{-1} &\in G_j^{**} \\ \implies x^{-1}\varphi_g^{-1}\alpha(x) &\in G_j^{**} \end{aligned}$$

where $\varphi_g \in \text{Inn}(G)$.

Consequently, $\varphi_g^{-1}\alpha \in \text{Aut}_{G_j^{**}}(G)$, i.e., $\alpha = \varphi_g\varphi_g^{-1}\alpha \in \text{Inn}(G)\text{Aut}_{G_j^{**}}(G)$. $\square$

In the special case $j=1$, we have the following result.

Corollary 2.10. *For any group G , If $\text{Aut}_c(G/G') = \text{Inn}(G/G')$, then*

$$\text{Aut}_c(G) \leq \text{Inn}(G)IA(G).$$

3. G^{**} -AUTONILPOTENT GROUPS

In the previous section, we saw that Parvaneh and Moghaddam [4] introduced autonilpotent groups. In this section, we define G^{**} -autonilpotent groups and study their properties.

Definition 3.1. A group G is called a G^{**} -autonilpotent group of class at most n if the series (2.1) ends in the identity subgroup after a finite number of steps.

Remark 3.2. The autonilpotent groups are G^{**} -autonilpotent, but the converse is not true in general. As an example, all of the abelian groups are G^{**} -autonilpotent, but

$$G = \bigoplus_{i=1}^l \mathbb{Z}_{2^{m_i}}, \quad l > 1, \quad m_1 = m_2 \geq \dots \geq m_l$$

is not autonilpotent, because $K_n(G) = G$.

Proposition 3.3. If H is not H^{**} -autonilpotent or K is not K^{**} -autonilpotent group, then $H \times K$ is not $(H \times K)^{**}$ -autonilpotent too.

Proof. It is clear by lemma 2.3. $\square$

Corollary 3.4. If $H_1, H_2, \dots, H_l$ are H_i^{**} -autonilpotent groups with coprime orders, then $H_1 \times H_2 \times \dots \times H_l$ is also $(H_1 \times H_2 \times \dots \times H_l)^{**}$ -autonilpotent.

Proof. The result is followed by induction on l . $\square$

Theorem 3.5. For a characteristic subgroup H of a given group G , if H and G/H are H^{**} -autonilpotent and $(G/H)^{**}$ -autonilpotent, respectively, then G is also G^{**} -autonilpotent.

Proof. Assume that there exist positive integers i and j such that

$$H_i^{**} = \langle 1 \rangle, \quad \left(\frac{G}{H} \right)_j^{**} = \langle 1 \rangle.$$

It is clear that $G^{**}H/H = (G/H)^{**}$ and by induction on j , one gets

$$\frac{G_j^{**}H}{H} \subseteq \left(\frac{G}{H} \right)_j^{**} = 1_{\frac{G}{H}}$$

whence $G_j^{**} \subseteq H$. Let $[g, \alpha]$ be an arbitrary generator of $G_{j+1}^{**} = [G_j^{**}, IA(G)]$. It is easy to see that $[g, \alpha|_H] \in [H, IA(H)]$, thus $G_{j+1}^{**} \subseteq H'$. By induction on i , we have $G_{i+j}^{**} \subseteq H_i^{**} = \langle 1 \rangle$. Therefore, G is a G^{**} -autonilpotent group of class at most $i+j$. $\square$

Theorem 3.6. Let G be a group and H be a proper characteristic subgroup of G with G/H is $(G/H)^{**}$ -autonilpotent of class c . If $H \cap G_c^{**} = \langle 1 \rangle$, then G is G^{**} -autonilpotent.

Proof. With the same argument as the proof of theorem 3.5, one can easily check that $G_c^{**}H/H \subseteq (G/H)^{**}$. Because $H \cap G_c^{**} = \langle 1 \rangle$, we have

$$\frac{G_c^{**}}{H \cap G_c^{**}} \cong \frac{G_c^{**}H}{H} \subseteq \left(\frac{G}{H} \right)_c^{**} = 1_{\frac{G}{H}}.$$

Thus, $G_c^{**} = \langle 1 \rangle$ which gives the G^{**} -autonilpotency of G . $\square$

Theorem 3.7. *Let H be a characteristic subgroup of a G^{**} -autonilpotent group G of class c . If $G = HG^{**}$, then $G=H$.*

Proof. By hypothesis,

$$G^{**} = [G, IA(G)] = [HG^{**}, IA(G)].$$

We prove that

$$[HG^{**}, IA(G)] \leq [H, IA(G)]^{G^{**}} [G^{**}, IA(G)].$$

Let $h \in H$, $g \in G^{**}$ and $\alpha \in IA(G)$, then $[hg, \alpha] \in [HG^{**}, IA(G)]$ and

$$\begin{aligned} [hg, \alpha] &= (hg)^{-1} \alpha (hg) \\ &= g^{-1} h^{-1} \alpha (h) \alpha (g) \\ &= g^{-1} h^{-1} \alpha (h) g g^{-1} \alpha (g) \\ &= [h, \alpha]^g [g, \alpha] \in [H, IA(G)]^{G^{**}} [G^{**}, IA(G)]. \end{aligned}$$

Because H is a characteristic subgroup, we have

$$\begin{aligned} G^{**} &= [HG^{**}, IA(G)] \\ &\leq [H, IA(G)]^{G^{**}} [G^{**}, IA(G)] \\ &\leq HG_2^{**}. \end{aligned}$$

Thus,

$$G = HG^{**} \leq HG_2^{**}.$$

So, $G = HG_2^{**}$. Now, by induction, we have $G = HG_c^{**}$, and $G=H$ since G is a G^{**} -autonilpotent group of class c . $\square$

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REFERENCES

1. S. Bachmuth, Automorphisms of Free Metabelian Groups, *Trans. Amer. Math. Soc.*, **118**(1965), (1963), 93-104.
2. R. G. Ghumde, S. H. Ghatge, IA-automorphisms of p-groups, Finite Polycyclic Groups and other Results, *Matematicki Vesnik*, **67**, (2015), 194-200.
3. P. V. Hegarty, The Absolute Centre of a Group, *J. Algebra*, **169**, (1994), 929-935.
4. F. Parvaneh, M. R. R. Moghaddam, Some Properties of Autosoluble Groups, *J. Math. Ext.*, **5**(1), (2010), 13-19.