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Gorenstein (n, d)-Injective and (n, d)-Flat Modules

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Abstract. Let n, d be non-negative integers. We introduce and study the notions of Gorenstein (n, d)-injective and Gorenstein (n, d)-flat modules by using the notion of special finitely presented, (n, d)-injective and (n, d)-flat modules for any $n \geq d+1$. Then we investigate the properties of these modules in case that every special finitely presented module with finite projective dimension (resp. flat dimension) is 2-presented. Moreover, as applications over special coherent rings, we investigate the relationships between Gorenstein (n, d)-injective modules and Gorenstein (n, d)-flat modules, then we obtain that the classes $\mathcal{GI}_d^n(R)$ and $\mathcal{GF}_d^n(R^{op})$ are covering and preenveloping, where $\mathcal{GI}_d^n(R)$ and $\mathcal{GF}_d^n(R^{op})$ denote the subcategories of Gorenstein (n, d)-injective left R-module and Gorenstein (n, d)-flat right R-modules, respectively.

Keywords: (n, d)-Flat module, (n, d)-Injective module, Gorenstein (n, d)-flat module, Gorenstein (n, d)-injective module, Special finitely presented module.

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1. Introduction

As it is known, the behaviour of Gorenstein modules in algebra is the same as the one of the classical homological notions. Several authors have studied the notions of Gorenstein modules in relative homological algebra on various rings and has become a vigorously active area of research (see, [1, 4, 5, 9, 10, 11, 12, 13, 22]). For example, the notions of Gorenstein injective and flat modules,

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first by Enochs and Jenda in [9, 10] introduced. In 2012, Gao and Wang [11] by using of FP-injective and finitely presented modules introduced a special case of the Gorenstein injective modules and they called Gorenstein FP-injective modules and studied it over coherent rings. All FP-injective and Gorenstein injective modules are in the class of Gorenstein FP-injective modules, but, every Gorenstein FP-injective module need not be FP-injective and Gorenstein injective, see [11, Proposition 2.5 and Theorem 3.8].

Also, in 2004, Zhou [24] using finitely n-presented modules introduced the notion of (n, d)-injective and(n, d)-flat modules, and then homological behavior of these modules on (n, d)-rings and n-coherent rings was investigated, where n, d are non-negative integers. In case $n, d = 0$, an R-module M is $(0, 0)$-injective if and only if M is injective, and if $n = 1, d = 0$, then an R-module M is $(1, 0)$-flat (resp. $(1, 0)$-injective) if and only if M is flat (resp. FP-injective). In general, injective and flat modules are in the class of (n, d)-injective and (n, d)-flat modules, respectively.

In this paper by considering $n \geq d+1$, we study the consequences of extending the notion of (n, d)-injective, Gorenstein FP-injective modules and (n, d)-flat, Gorenstein flat modules to that of Gorenstein (n, d)-injective and Gorenstein (n, d)-flat modules, respectively. Then, by using special finitely presented, (n, d)-injective and (n, d)-flat modules, we introduce a concept of Gorenstein (n, d)-injective and Gorenstein (n, d)-flat modules, and under this definition, Gorenstein (n, d)-injective and Gorenstein (n, d)-flat modules are weaker than the usual Gorenstein FP-injective and Gorenstein flat modules, since for every $m > n$ and $d^{'} \geq d$, $(m, d^{'})$-injective and $(m, d^{'})$-flat modules need not be (n, d)-injective and (n, d)-flat, respectivelt, see Examples 3.3 and 3.4.

Let us describe the contents of the paper in more details.

In Sec. 2, some fundamental concepts and some preliminary results are stated.

In Sec. 3, we introduce Gorenstein (n, d)-injective and Gorenstein (n, d)-flat modules for an integer $n \geq d + 1$ and then we give some examples and some characterizations of these modules. Among other results, over every arbitrary ring R, we prove that, every special finitely presented module with $\mathrm{pd}_R(K_{d-1}) < \infty$ is 2-presented if and only if for every short exact sequence $0 \to X \to Y \to Z \to 0$ of R-modules, Z is Gorenstein (n, d)-injective if X and Y are Gorenstein (n, d)-injective, then we conclud that an R-module M is Gorenstein (n, d)-injective if and only if every pure submodule and any pure epimorphic image of M are Gorenstein (n, d)-injective. Also, if every special finitely presented module with $\mathrm{fd}_R(K_{d-1}) < \infty$ is 2-presented, it follows that (1) in the exact sequence $0 \to X \to Y \to Z \to 0$ of R-modules, X is Gorenstein (n, d)-flat if Z and Y are Gorenstein (n, d)-flat, (2) an right R-module M is Gorenstein (n, d)-flat if and only if every pure submodule and any pure epimorphic image of M are Gorenstein (n, d)-flat.

In Sec. 4, by using the results of Section 3, it is shown that over special coherent rings, (1) an R-module M is Gorenstein (n, d)-injective (resp. Gorenstein (n, d)-flat) if and only if M^* is Gorenstein (n, d)-flat (resp. Gorenstein (n, d)-injective), where $M^* = \mathrm{Hom}_{\mathbb{Z}}(M, \frac{\mathbb{Q}}{\mathbb{Z}})$ denotes the character module of M, (2) the pairs $(\mathcal{GI}_d^n(R), \mathcal{GF}_d^n(R^{op}))$ and $(\mathcal{GF}_d^n(R^{op}), \mathcal{GI}_d^n(R))$ are duality pair, (3) the class $\mathcal{GI}_d^n(R)$ (resp. $\mathcal{GF}_d^n(R^{op})$) is covering and preenveloping, where $\mathcal{GI}_d^n(R)$ and $\mathcal{GF}_d^n(R^{op})$ are classes of Gorenstein (n, d)-injective and Gorenstein (n, d)-flat R-modules, respectively. We also establish some equivalent characterizations in terms of Gorenstein (n, d)-injective and Gorenstein (n, d)-flat modules.

2. Preliminaries

Throughout this paper R will be an associative ring with identity, and all modules will be unital left R-modules (unless specied otherwise).

In this section, some fundamental concepts and notations are stated.

Let M an R-module. Then

(1) M is said to be *Gorenstein injective* (resp. *Gorenstein flat*) [9, 10] if there is an exact sequence

$$\cdots \longrightarrow I_1 \longrightarrow I_0 \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

of injective (resp. flat) R-modules with $M = \ker(I^0 \to I^1)$ such that Hom_R $(U, -)$ (resp. $U \otimes_R -$) leaves the sequence exact whenever U is an injective R-module.

(2) M is said to be *FP-injective* [19] if $\mathrm{Ext}_R^1(U, M) = 0$ for any finitely presented R-module U.

(3) M is said to be *Gorenstein FP-injective* [11] if there is an exact sequence

$$\mathbf{E} = \cdots \longrightarrow E_1 \longrightarrow E_0 \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots$$

of FP-injective modules with $M = \ker(E^0 \to E^1)$ such that $\mathrm{Hom}_R(P, \mathbf{E})$ is an exact sequence whenever P is a finitely presented module with $\mathrm{pd}_R(P) < \infty$.

(4) An R-module U is called *finitely n-presented* [2, 6, 7] and if there exists an exact sequence $0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to U \to 0$, where each F_i is finitely generated and free.

(5) M is said to be *(n, d)-injective* [24] if $\mathrm{Ext}_R^{d+1}(U, M) = 0$ for any finitely n-presented R-module U. If $n \geq 1$, then a right R-module M is called *(n, d)-flat* [24] if $\mathrm{Tor}_{d+1}^R(M, U) = 0$ for any finitely n-presented R-module U.

(6) Let R be a ring. Then R is said to be *(n, d)-ring* [24] if every finitely n-presented R-module has the projective dimension at most d. A ring R is said to be *weak (n, d)-ring* [24] if every finitely n-presented R-module has the flat dimension at most d.

The following definition is derived from [25, Definition 3.1].

Definition 2.1. Let R be a ring, U a finitely n-presented R-module and let n, d be non-negative integers. If $n \geq d+1$ and $0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to U \to 0$ is a finitely generated projective left resolution of U, then

(1) The module $K_{d-1} = \mathrm{Ker}(F_{d-1} \to F_{d-2})$ is called *special finitely presented.*

(2) The sequence $0 \to K_d \to F_d \to K_{d-1} \to 0$ of R-modules will be called a *special short exact sequence.*

(3) Let R be a ring. Then R is said to be *special coherent* if every special finitely presented R-module is 2-presented.

(4) A short exact sequence $0 \to A \to B \to C \to 0$ of R-modules is called special pure if the induced sequence

$$0 \longrightarrow \mathrm{Hom}_R(K_{d-1}, A) \longrightarrow \mathrm{Hom}_R(K_{d-1}, B) \longrightarrow \mathrm{Hom}_R(K_{d-1}, C) \longrightarrow 0$$

is exact for every special finitely presented module K_{d-1}. In this case A is said to be special pure in B.

(5) $\mathrm{Ext}_R^{d+1}(U, -) \cong \mathrm{Ext}_R^1(K_{d-1}, -)$ and $\mathrm{Tor}_{d+1}^R(-, U) \cong \mathrm{Tor}_1^R(-, K_{d-1})$, where U is a finitely n-presented module with an associated special finitely presented module K_{d-1}. This fact will be used throughout the paper.

(6) We recall that the (n, d)-injective dimension of an R-module M, denoted by (n, d)-$id_R M$, is defined to be the least integer k such that $\mathrm{Ext}_R^{k+1}(K_{d-1}, M) = 0$ for any special finitely presented R-module K_{d-1} with respect to every finitely n-presented R-module U. Also, $l.(n, d)$-$dim(R) = sup\{(n, d)$-$id_R M \mid M$ *is an* R-*module*$\}$.

Proposition 2.2. *Let every special finitely presented module K_{d-1} be finitely 2-presented. Then (n, d)-$id_R M \leq m$ if and only if there is an exact sequence $0 \to M \to C^0 \to C^1 \to \cdots \to C^m \to 0$, where every C^i is (n, d)-injective.*

Proof. Is clear, since similar to the proof of Proposition 3.7(1), $\mathrm{Ext}_R^{i+1}(K_{d-1}, C^m) \cong \mathrm{Ext}_R^{m+i+1}(K_{d-1}, M)$ for any $i \geq 0$. □

3. Gorenstein (n, d)-injective and gorenstein (n, d)-flat modules

In this section, we introduce and study Gorenstein (n, d)-Injective modules which are defined as follows:

Definition 3.1. Let R be a ring and n, d non-negative integers with $n \geq d+1$. Then, an R-module M is called *Gorenstein (n, d)-injective* if there exists an exact sequence of (n, d)-injective R-modules of this form:

$$\mathcal{I}_{\mathcal{D}}^{\mathcal{N}} = \cdots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow C^0 \longrightarrow C^1 \longrightarrow \cdots$$

with $M = \mathrm{Ker}(C^0 \to C^1)$ such that $\mathrm{Hom}_R(K_{d-1}, \mathcal{I}_{\mathcal{D}}^{\mathcal{N}})$ is an exact sequence whenever K_{d-1} is a special finitely presented R-module with $\mathrm{pd}_R(K_{d-1}) < \infty$.

A right R-module N is called Gorenstein (n, d)-flat if there exists the following exact sequence of (n, d)-flat right R-modules of this form:

$$\mathcal{F}_{\mathcal{D}}^{\mathcal{N}} = \cdots \longrightarrow D_1 \longrightarrow D_0 \longrightarrow D^0 \longrightarrow D^1 \longrightarrow \cdots$$

with $N = \ker(D^0 \to D^1)$ such that $\mathcal{F}_{\mathcal{D}}^{\mathcal{N}} \otimes_R K_{d-1}$ is an exact sequence whenever K_{d-1} is a special finitely presented R-module with $\mathrm{fd}_R(K_{d-1}) < \infty$.

With a simple observation we have:
Gorenstein injective $\Longrightarrow$ Gorenstein FP-injective $\Longleftrightarrow$ Gorenstein $(1, 0)$-injective $\Longrightarrow$ Gorenstein (n, d)-injective, and
Gorenstein flat $\Longrightarrow$ Gorenstein $(1, 0)$-flat $\Longrightarrow$ Gorenstein (n, d)-flat.

Remark 3.2. Let $n, n^{'}, m$ and $d, d^{'}$ be non-negative integers. Then

(1) Every injective R-module is (n, d)-injective;
(2) Every flat right R-module is (n, d)-flat;
(3) Every (n, d)-injective R-module is Gorenstein (n, d)-injective;
(4) Every (n, d)-flat right R-module is Gorenstein (n, d)-flat;
(5) Every Gorenstein (n, d)-injective R-module is Gorenstein (m, d)-injective for any $n \leq m$. But, every Gorenstein (m, d)-injective module need not be Gorenstein (n, d)-injective for any $m > n$, see Example 3.3;
(6) Every Gorenstein (n, d)-flat right R-module is Gorenstein (m, d)-flat for any $n \leq m$. But, every Gorenstein (m, d)-flat right module need not be Gorenstein (n, d)-flat for any $m > n$, see Example 3.3;
(7) If $n \leq n^{'}$ and $d \leq d^{'}$, then every Gorenstein (n, d)-injective (Gorenstein (n, d)-flat) R-module is Gorenstein $(n^{'}, d^{'})$-injective (Gorenstein $(n^{'}, d^{'})$-flat) . But, not contrary, see Example 3.4;
(8) In Definition 3.1, $\mathrm{Ker}(C_i \to C_{i-1})$ and $\mathrm{Ker}(C^i \to C^{i+1})$ are Gorenstein (n, d)-injective, also, $\mathrm{Ker}(D_i \to D_{i-1})$ and $\mathrm{Ker}(D^i \to D^{i+1})$ are Gorenstein (n, d)-flat for any $i \geq 1$.

A ring R is called n-regular [23] if it is a $(n, 0)$-ring. In case $n = 1$, R is called regular ring.

Example 3.3. Let K be a field, E a nonzero K-vector space and $R = K \ltimes E$ be a trivial extension of K by E. If $\dim_K E = 1$, then by Remark 3.2, every R-module is Gorenstein (n, d)-injective, see [3, Corollary 2.2]. If E is a K-vector space with infinite rank, then by [17, Theorem 3.4], every finitely 2-presented R-module is projective. So, R is $(2, 0)$-ring and then by [24, Proposition 2.6](1), every R-module is $(2, 0)$-injective and hence, every R-module is Gorenstein $(2, 0)$-injective. Also, R is weak $(2, 0)$-ring, so by [24, Proposition 2.6](2) and Remark 3.2, every R-module is Gorenstein $(2, 0)$-flat, too. But, if every R-module is Gorenstein $(1, 0)$-injective (Gorenstein $(1, 0)$-flat), then R is $(1, 0)$-ring (or regular), contradiction.

Example 3.4. Let $R = k[X]$, where k is a field. Then, we claim that every R-module is Gorenstein $(2,1)$-injective (Gorenstein $(2,1)$-flat). Since $l.(1,0)$-$dim(R) \leq 1$, for every R-module M by Proposition 3.8, there exists an exact sequence $0 \to M \to C^0 \to C^1 \to 0$, where C^0 and C^1 are $(1,0)$-injective. Then for every finitely 2-presented R-module U, $0 = \mathrm{Ext}^1_R(U, C^1) \cong \mathrm{Ext}^2_R(U, M)$. So $\mathrm{Ext}^2_R(U, M) = 0$, consequently M is $(2,1)$-injective and then by Remark 3.2, M is Gorenstein $(2,1)$-injective. Also, it follows that N^* is a Gorenstein $(2,1)$-injective R-module for every R-module N. Then by Theorem 4.1, we conclud that every R-module N is Gorenstein $(2,1)$-flat. But, if every R-module is Gorenstein $(1,0)$-injective (Gorenstein $(1,0)$-flat), then by Proposition 4.4, it follows that every R-module is $(1,0)$-injective ($(1,0)$-flat). Therefore by [23, Theorem 3.9], R is a regular ring, contradiction.

Lemma 3.5. *Let R be a ring. Then*

(1) *If M is a Gorenstein (n,d)-injective R-module, then $\mathrm{Ext}^1_R(K_{d-1}, M) = 0$ for any special finitely presented R-module K_{d-1} with $\mathrm{pd}_R(K_{d-1}) < \infty$;*

(2) *If M is a Gorenstein (n,d)-flat right R-module, then $\mathrm{Tor}^R_1(M, K_{d-1}) = 0$ for any special finitely presented R-module K_{d-1} with $\mathrm{fd}_R(K_{d-1}) < \infty$.*

Proof. If M is a Gorenstein (n,d)-injective R-module, then there exists the following exact sequence of (n,d)-injective R-module:

$$\mathcal{I}^{\mathcal{N}}_{\mathcal{D}} = \cdots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow C^0 \longrightarrow C^1 \longrightarrow \cdots .$$

Since $M = \mathrm{Ker}(C^0 \to C^1)$ and $\mathrm{Hom}_R(K_{d-1}, \mathcal{I}^{\mathcal{N}}_{\mathcal{D}})$ is exact for any special finitely presented R-module K_{d-1} with $\mathrm{pd}_R(K_{d-1}) < \infty$, it follows that $\mathrm{Ext}^1_R(K_{d-1}, M) = 0$. Similarly, (2) follows. ☐

Lemma 3.6. *Let R be ring. Then*

(1) *Every $(1,0)$-injective R-module is Gorenstein (n,d)-injective;*

(2) *Every $(1,0)$-flat R-module is Gorenstein (n,d)-flat.*

Proof. Let M be a $(1,0)$-injective R-module. Then $\mathrm{Ext}^1_R(N, M) = 0$ for any finitely 1-presented R-module N. Now, if $0 \to K_d \to F_d \to K_{d-1} \to 0$ is a special short exact sequence with respect to finitely n-presented R-module U, then $\mathrm{Ext}^{d+1}_R(U, M) \cong \mathrm{Ext}^1_R(K_{d-1}, M) = 0$, since K_{d-1} is a finitely 1-presented module. So M is (n,d)-injective, and every (n,d)-injective is Gorenstein (n,d)-injective, too. Similarly, (2) is also concluded. ☐

Proposition 3.7. *Let every special finitely presented module with $\mathrm{fd}_R(K_{d-1}) < \infty$ is 2-presented. Then, for any $l \geq 1$:*

(1) *$\mathrm{Ext}^l_R(K_{d-1}, M) = 0$ for any Gorenstein (n,d)-injective R-module M and any special finitely presented R-module K_{d-1} with $\mathrm{pd}_R(K_{d-1}) < \infty$;*

(2) $\mathrm{Tor}_l^R(M, K_{d-1}) = 0$ *for any Gorenstein* (n,d)*-flat right* R*-module* M *and any special finitely presented* R*-module* K_{d-1} *with* $\mathrm{fd}_R(K_{d-1}) < \infty$.

Proof. (1) Assume that K_{d-1} is a special finitely presented module with $\mathrm{pd}_R(K_{d-1}) \leq m$ respect to any n-presented module U. If M is a Gorenstein (n,d)-injective R-module, then, there is a left (n,d)-injective resolution of M. So, we have:

$$0 \longrightarrow N \longrightarrow C_{m-1} \longrightarrow \cdots \longrightarrow C_0 \longrightarrow M \longrightarrow 0,$$

where every C_j is (n,d)-injective for every $0 \leq j \leq m-1$. Since $\mathrm{fd}_R(K_{d-1}) \leq \mathrm{pd}_R(K_{d-1}) < \infty$, K_{d-1} is 2-presented, and so $\mathrm{Ext}_R^{i+1}(K_{d-1}, C_j) = 0$ for any $i \geq 0$. Hence, $\mathrm{Ext}_R^{i+1}(K_{d-1}, M) \cong \mathrm{Ext}_R^{m+i+1}(K_{d-1}, N)$, and since $\mathrm{pd}_R(K_{d-1}) \leq m$, it follows that , $\mathrm{Ext}_R^{i+1}(K_{d-1}, M) = 0$ for any $i \geq 0$.

(2) Assume that K_{d-1} is a special finitely presented module with $\mathrm{fd}_R(K_{d-1}) \leq m$ for any n-presented module U. If M is a Gorenstein (n,d)-flat right R-module, then there is a right (n,d)-flat resolution of M of the form:

$$0 \longrightarrow M \longrightarrow D^0 \longrightarrow \cdots \longrightarrow D^{m-1} \longrightarrow N \longrightarrow 0,$$

where every D^j is (n,d)-flat for every $0 \leq j \leq m-1$. Since U is $(n+1)$-presented, we have $\mathrm{Tor}_{i+1}^R(D^j, K_{d-1}) = 0$ for any $i \geq 0$. If $\mathrm{fd}_R(K_{d-1}) \leq m$, then $\mathrm{Tor}_{i+1}^R(M, K_{d-1}) \cong \mathrm{Tor}_{m+i+1}^R(N, K_{d-1}) = 0$, and so $\mathrm{Tor}_{i+1}^R(M, K_{d-1}) = 0$ for any $i \geq 0$. □

We start with the result which proves that the behaviour of Gorenstein (n,d)-injective (resp. Gorenstein (n,d)-flat) modules in short exact sequences is the same as the one of the classical homological notions.

Proposition 3.8. (1) *For every short exact sequence* $0 \to X \to Y \to Z \to 0$ *of* R*-modules,* Y *is Gorenstein* (n,d)*-injective if* X *and* Z *are Gorenstein* (n,d)*-injective;*

(2) *For every short exact sequence* $0 \to X \to Y \to Z \to 0$ *of right* R*-modules,* Y *is Gorenstein* (n,d)*-flat if* X *and* Z *are Gorenstein* (n,d)*-flat.*

Proof. (1) By Definition 3.1, there is an exact sequence $\mathcal{I}_{\mathcal{D}}^{\mathcal{N}} = \cdots \to X_1 \to X_0 \to X^0 \to X^1 \to \cdots$ of (n,d)-injective R-modules, where $X = \mathrm{Ker}(X^0 \to X^1)$. Also, there is an exact sequence $(\mathcal{I}_{\mathcal{D}}^{\mathcal{N}})' = \cdots \to Z_1 \to Z_0 \to Z^0 \to Z^1 \to \cdots$ of (n,d)-injective R-modules, where $Z = \mathrm{Ker}(Z^0 \to Z^1)$. On the other hand, for every finitely n-presented module U we have:

$$\mathrm{Ext}_R^{d+1}(U, X_i \oplus Z_i) = \mathrm{Ext}_R^{d+1}(U, X_i) \oplus \mathrm{Ext}_R^{d+1}(U, Z_i) = 0$$

and

$$\mathrm{Ext}_R^{d+1}(U, X^i \oplus Z^i) = \mathrm{Ext}_R^{d+1}(U, X^i) \oplus \mathrm{Ext}_R^{d+1}(U, Z^i) = 0.$$

Therefore, $X_i \oplus Z_i$ and $X^i \oplus Z^i$ are (n,d)-injective for any $i \geq 0$. Thus, there is an exact sequence

$$\mathcal{I}_{\mathcal{D}}^{\mathcal{N}} \oplus (\mathcal{I}_{\mathcal{D}}^{\mathcal{N}})' = \cdots \longrightarrow X_1 \oplus Z_1 \longrightarrow X_0 \oplus Z_0 \longrightarrow X^0 \oplus Z^0 \longrightarrow X^1 \oplus Z^1 \longrightarrow \cdots$$

of (n,d)-injective R-modules, where $Y = \mathrm{Ker}(X^0 \oplus Z^0 \to X^1 \oplus Z^1)$. Let K_{d-1} be a special finitely presented R-module with $\mathrm{pd}_R(K_{d-1}) < \infty$. Then the following short exact sequence of complexes exists:

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & & \mathrm{Hom}_R(K_{d-1}, X_0) & \longrightarrow & \mathrm{Hom}_R(K_{d-1}, X_0 \oplus Z_0) & \longrightarrow & \mathrm{Hom}_R(K_{d-1}, Z_0) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & & \mathrm{Hom}_R(K_{d-1}, X^0) & \longrightarrow & \mathrm{Hom}_R(K_{d-1}, X^0 \oplus Z^0) & \longrightarrow & \mathrm{Hom}_R(K_{d-1}, Z^0) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & & \mathrm{Hom}_R(K_{d-1}, X^1) & \longrightarrow & \mathrm{Hom}_R(K_{d-1}, X^1 \oplus Z^1) & \longrightarrow & \mathrm{Hom}_R(K_{d-1}, Z^1) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \vdots & & \vdots & & \vdots \\
 & & \| & & \| & & \| \\
0 \longrightarrow & & \mathrm{Hom}_R(K_{d-1}, \mathcal{I}_{\mathcal{D}}^{\mathcal{N}}) & \longrightarrow & \mathrm{Hom}_R(K_{d-1}, \mathcal{I}_{\mathcal{D}}^{\mathcal{N}} \oplus (\mathcal{I}_{\mathcal{D}}^{\mathcal{N}})') & \longrightarrow & \mathrm{Hom}_R(K_{d-1}, (\mathcal{I}_{\mathcal{D}}^{\mathcal{N}})') \longrightarrow 0.
\end{array}$$

By hypothesis, $\mathrm{Hom}_R(K_{d-1}, \mathcal{I}_{\mathcal{D}}^{\mathcal{N}})$ and $\mathrm{Hom}_R(K_{d-1}, (\mathcal{I}_{\mathcal{D}}^{\mathcal{N}})')$ are exact, hence $\mathrm{Hom}_R(K_{d-1}, \mathcal{I}_{\mathcal{D}}^{\mathcal{N}} \oplus (\mathcal{I}_{\mathcal{D}}^{\mathcal{N}})')$ is exact by [18, Theorem 6.10].

(2) Similar to the proof of (1). □

Theorem 3.9. *Let R be a ring. Then, the following statements are equivalent:*

(1) *Every special finitely presented module with $\mathrm{pd}_R(K_{d-1}) < \infty$ is 2-presented;*
(2) *For every short exact sequence $0 \to X \to Y \to Z \to 0$ of R-modules, Z is Gorenstein (n,d)-injective if X and Y are Gorenstein (n,d)-injective.*

Proof. (1) $\Longrightarrow$ (2) If Y is a Gorenstein (n,d)-injective R-module, then by Definition 3.1 and Remark 3.2, there is an exact sequence $0 \to K \to Y_0 \to$

$Y \to 0$ of R-modules, where Y_0 is (n,d)-injective and K is Gorenstein (n,d)-injective. Consider the following commutative diagram with exact rows exists:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & K \arrow[r, equal] \arrow[d] & K \arrow[d] & & \\
0 \arrow[r] & M \arrow[r] \arrow[d] & Y_0 \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

By Proposition 3.8(1), M is Gorenstein (n,d)-injective, and so we have a commutative diagram of R-modules:

$$\begin{tikzcd}
\mathcal{I}_{\mathcal{D}}^{\mathcal{N}} = \cdots \arrow[r] & C_1 \arrow[rr] \arrow[dr] & & C_0 \arrow[rr] \arrow[dr] & & Y_0 \arrow[rr] \arrow[dr] & & I^0 \arrow[rr] \arrow[dr] & & I^1 \arrow[r] & \cdots, \\
 & & K_0 \arrow[ur] \arrow[dr] & & M \arrow[ur] \arrow[dr] & & Z \arrow[ur] \arrow[dr] & & K^1 \arrow[ur] \arrow[dr] & & \\
 & 0 \arrow[ur] & & 0 \arrow[ur] & & 0 \arrow[ur] & & 0 \arrow[ur] & & 0 &
\end{tikzcd}$$

where C_i and Y_0 are (n,d)-injective, I^i is injective, $Z = \mathrm{Ker}(I^0 \to I^1)$, $M = \mathrm{Ker}(Y_0 \to Z)$, $K_i = \mathrm{Ker}(C_i \to C_{i-1})$ and $K^i = \mathrm{Ker}(I^i \to I^{i+1})$. By Remark 3.2, Y_0, I^i and K_i are Gorenstein (n,d)-injective and hence by Lemma 3.7(1), $\mathrm{Ext}^l_R(K_{d-1}, K_i) = \mathrm{Ext}^l_R(K_{d-1}, M) = \mathrm{Ext}^l_R(K_{d-1}, Y_0) = \mathrm{Ext}^l_R(K_{d-1}, I^i) = 0$ for any $l \geq 1$ and every special finitely presented K_{d-1} module with $\mathrm{pd}_R(K_{d-1}) < \infty$. Therefore, it follows that $\mathrm{Ext}^l_R(K_{d-1}, Z) = 0 = \mathrm{Ext}^l_R(K_{d-1}, K^i)$, and hence we have the following exact commutative diagram :

$$\begin{tikzcd}
\mathrm{Hom}(K_{d-1}, Y_0) \arrow[rr] \arrow[dr] & & \mathrm{Hom}(K_{d-1}, I^0) \arrow[rr] \arrow[dr] & & \cdots \\
 & \mathrm{Hom}(K_{d-1}, Z) \arrow[ur] \arrow[dr] & & \mathrm{Hom}(K_{d-1}, K^1) \arrow[ur] \arrow[dr] & \\
0 \arrow[ur] & & 0 \arrow[ur] & & 0
\end{tikzcd}$$

Consequently, Z is Gorenstein (n,d)-injective.

(2) $\Longrightarrow$ (1) Let K_{d-1} be a special finitely presented R-module with respect to any finitely n-presented module U with $\mathrm{pd}_R(K_{d-1}) < \infty$, and let $0 \to K_d \to F_d \to K_{d-1} \to 0$ be a special short exact sequence, where K_d is a finitely generated module. We show that K_d is special finitely presented. Let M be

a Gorenstein (n,d)-injective module and $0 \to M \to E \to L \to 0$ an exact sequence of R-modules, where E is injective. Then we have:

$$0 = \mathrm{Ext}^1_R(F_d, M) \longrightarrow \mathrm{Ext}^1_R(K_d, M) \longrightarrow \mathrm{Ext}^2_R(K_{d-1}, M) \longrightarrow 0.$$

So, $\mathrm{Ext}^1_R(K_d, M) \cong \mathrm{Ext}^2_R(K_{d-1}, M)$. On the other hand,

$$0 = \mathrm{Ext}^1_R(K_{d-1}, E) \longrightarrow \mathrm{Ext}^1_R(K_{d-1}, L) \longrightarrow \mathrm{Ext}^2_R(K_{d-1}, M) \longrightarrow 0.$$

Hence, $\mathrm{Ext}^1_R(K_{d-1}, L) \cong \mathrm{Ext}^2_R(K_{d-1}, M)$. By (2), L is Gorenstein (n,d)-injective and consequently by Lemma 3.5, $\mathrm{Ext}^1_R(K_{d-1}, L) = 0$. This implies that $\mathrm{Ext}^1_R(K_d, M) = 0$ for any Gorenstein (n,d)-injective R-module M. Since by Lemma 3.6, every $(1,0)$-injective (or FP-injective) module is Gorenstein (n,d)-injective, $\mathrm{Ext}^1_R(K_d, N) = 0$ for any FP-injective module N and so K_d is 1-presented. Hence, K_{d-1} is 2-presented. □

Corollary 3.10. *Let every special finitely presented module with* $\mathrm{pd}_R(K_{d-1}) < \infty$ *be* 2*-presented. Then, a module* M *is Gorenstein* (n,d)*-injective if and only if every pure submodule and any pure epimorphic image of* M *are Gorenstein* (n,d)*-injective.*

Proof. $(\Longrightarrow)$ Let M be a Gorenstein (n,d)-injective module. If the exact sequence $0 \to K \to M \to \frac{M}{K} \to 0$ is pure, then, $\mathrm{Ext}^1_R(K_{d-1}, K) = 0$ for every special finitely presented module K_{n-1}. So, we have $\mathrm{Ext}^1_R(K_{d-1}, K) \cong \mathrm{Ext}^{d+1}_R(U, K) = 0$ for any finitely n-presented module U. Thus, K is (n,d)-injective, and hence K is Gorenstein (n,d)-injective by Remark 3.2. Therefore, by Theorem 3.9, $\frac{M}{K}$ is Gorenstein (n,d)-injective.

$(\Longleftarrow)$ Assume that the exact sequence $0 \to K \to M \to Z \to 0$ is pure, where Z and K are Gorenstein (n,d)-injective. Then, by Proposition 3.8(1), M is Gorenstein (n,d)-injective. □

Proposition 3.11. *Let every special finitely presented module with* $\mathrm{fd}_R(K_{d-1}) < \infty$ *be* 2*-presented. Then, for every short exact sequence* $0 \to X \to Y \to Z \to 0$ *of right* R*-modules,* X *is Gorenstein* (n,d)*-flat if* Y *and* Z *are Gorenstein* (n,d)*-flat.*

Proof. If Y is a Gorenstein (n,d)-flat module, then by Definition 3.1 and Remark 3.2, there is an exact sequence $0 \to Y \to Y^0 \to L \to 0$ of right R-modules, where Y^0 is (n,d)-flat and L is Gorenstein (n,d)-flat. We have the following pushout diagram with exact rows:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & X \arrow[r] \arrow[d, equal] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X \arrow[r] & Y^0 \arrow[r] \arrow[d] & T \arrow[r] \arrow[d] & 0 \\
 & & L \arrow[r, equal] \arrow[d] & L \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

By Proposition 3.8(2), T is Gorenstein (n, d)-flat, and so we have the following commutative diagram:

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[r] & F_1 \arrow[rr] \arrow[dr] & & F_0 \arrow[rr] \arrow[dr] & & Y^0 \arrow[rr] \arrow[dr] & & T^0 \arrow[rr] \arrow[dr] & & T^1 \arrow[r] & \cdots, \\
 & & K_0 \arrow[ur] \arrow[dr] & & X \arrow[ur] \arrow[dr] & & T \arrow[ur] \arrow[dr] & & K^1 \arrow[ur] \arrow[dr] & & \\
 & 0 \arrow[ur] & & 0 \arrow[ur] & & 0 \arrow[ur] & & 0 \arrow[ur] & & 0 &
\end{tikzcd}$$

where T^i and Y^0 are (n, d)-flat modules, F_i is flat (or, $(1, 0)$-flat) for any $i \geq 0$, and also, $X = \mathrm{Ker}(Y^0 \to T^0)$. Similar to the proof of the Lemma 3.6, every F_i is (n, d)-flat, too. By Remark 3.2 K_i and K^i are Gorenstein (n, d)-flat. Hence by Lemma 3.7(2), $\mathrm{Tor}_l^R(K_i, K_{d-1}) = \mathrm{Tor}_l^R(K^i, K_{d-1}) = \mathrm{Tor}_l^R(T, K_{d-1}) = 0$ for any special finitely presented module K_{d-1} with $\mathrm{fd}_R(K_{d-1}) < \infty$ and any $l \geq 0$. So, we have the following exact commutative diagram :

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[r] & F_0 \otimes_R K_{d-1} \arrow[rr] \arrow[dr] & & Y^0 \otimes_R K_{d-1} \arrow[rr] \arrow[dr] & & \cdots \\
 & & X \otimes_R K_{d-1} \arrow[ur] \arrow[dr] & & T \otimes_R K_{d-1} \arrow[ur] \arrow[dr] & \\
 & 0 \arrow[ur] & & 0 \arrow[ur] & & 0
\end{tikzcd}$$

Consequently, X is Gorenstein (n, d)-flat. $\square$

Corollary 3.12. *Let every special finitely presented module with* $\mathrm{fd}_R(K_{d-1}) < \infty$ *be* 2*-presented. Then, a right* R*-module* M *is Gorenstein* (n, d)*-flat if and only if every pure submodule and any pure epimorphic image of* M *are Gorenstein* (n, d)*-flat.*

Proof. ($\Longrightarrow$) Let M be a Gorenstein (n,d)-flat right R-module and K a pure submodule in M. Then, the exact sequence $0 \to K \to M \to \frac{M}{K} \to 0$ is pure. So, if K_{d-1} is special finitely presented module with respect to every finitely n-presented R-module U, then $0 = \mathrm{Tor}_1^R(\frac{M}{K}, K_{d-1}) \cong \mathrm{Tor}_{d+1}^R(\frac{M}{K}, U)$ and consequently $\frac{M}{K}$ is (n,d)-flat. Hence by Remark 3.2, $\frac{M}{K}$ is Gorenstein (n,d)-flat. Now, if K_{d-1} is special finitely presented module with $\mathrm{fd}_R(K_{d-1}) < \infty$, then by Proposition 3.11, the exact sequence $0 \to K \to M \to \frac{M}{K} \to 0$ implies that K is Gorenstein (n,d)-flat.

($\Longleftarrow$) Let K be a pure submodule in M. Then, the exact sequence $0 \to K \to M \to \frac{M}{K} \to 0$ is pure. Then by hypothesis and Proposition 3.8(2), M is Gorenstein (n,d)-flat. $\square$

As for the classical injective (resp. flat) notion, the class $\mathcal{GI}_d^n(R)$ (resp. $\mathcal{GF}_d^n(R^{op})$) is closed under direct products (resp. direct sums).

Proposition 3.13. (1) *The class $\mathcal{GI}_d^n(R)$ is closed under direct products;*
(2) *The class $\mathcal{GF}_d^n(R^{op})$ is closed under direct sums.*

Let $\mathcal{E}$ be the class of finite injective modules and the symbol $\mathcal{P}$ denotes the class of finite projective right modules. If $\mathcal{I}$ is a class of modules, then $\mathcal{I}$ is called injectively resolving if $\mathcal{E} \subseteq \mathcal{I}$, and for every short exact sequence $0 \to X \to Y \to Z \to 0$ with $X \in \mathcal{I}$ the conditions $Y \in \mathcal{I}$ and $Z \in \mathcal{I}$ are equivalent. Also, $\mathcal{I}$ is called projectively resolving if $\mathcal{P} \subseteq \mathcal{I}$, and for every short exact sequence $0 \to X \to Y \to Z \to 0$ with $Z \in \mathcal{I}$ the conditions $X \in \mathcal{I}$ and $Y \in \mathcal{I}$ are equivalent, (see [14, 1.1. Resolving classes]).

By Propositions 3.8, 3.13, Theorem 3.9 and [14, Proposition 1.4], we have the following easy observations.

Corollary 3.14. *Let every special finitely presented module with* $\mathrm{pd}_R(K_{d-1}) < \infty$ *be 2-presented. Then,*

(1) *The class class $\mathcal{GI}_d^n(R)$ is closed under direct sums;*
(2) *The class $\mathcal{GF}_d^n(R^{op})$ is closed under direct products;*
(3) *The class $\mathcal{GI}_d^n(R)$ is injectively resolving;*
(4) *The class $\mathcal{GI}_d^n(R)$ is closed under direct summands;*
(5) *The class $\mathcal{GF}_d^n(R^{op})$ is projectively resolving;*
(6) *The class $\mathcal{GF}_d^n(R^{op})$ is closed under direct summands.*

4. Applications on special coherent rings

Let R be a ring. Then, a duality pair over R is a pair $(\mathcal{M}, \mathcal{C})$ [15, Definition 2.1], where $\mathcal{M}$ is a class of left (resp. right) R-modules and $\mathcal{C}$ is a class of right (resp. left) R-modules, subject to the following conditions: (1) For any module M, one has $M \in \mathcal{M}$ if and only if $M^* \in \mathcal{C}$; (2) $\mathcal{C}$ is closed under direct summands and finite direct sums.

First in this section, according to previous section results, we investigate the relationships between Gorenstein (n,d)-injective and Gorenstein (n,d)-flat modules over special coherent rings. Then by using of it, we obtain that the pairs $(\mathcal{GI}_d^n(R), \mathcal{GF}_d^n(R^{op}))$ and $(\mathcal{GF}_d^n(R^{op}), \mathcal{GI}_d^n(R))$ are duality pair.

Theorem 4.1. *Let R be a special coherent ring. Then,*

(1) *An R-module M is Gorenstein (n,d)-injective if and only if M^* is a Gorenstein (n,d)-flat right R-module;*

(2) *A right R-module M is Gorenstein (n,d)-flat if and only if M^* is a Gorenstein (n,d)-injective R-module.*

Proof. (1) ($\Longrightarrow$) By Definition 3.1, there exists the exact sequence $\cdots \to C_1 \to C_0 \to M \to 0$, where every C_i is (n,d)-injective. By [24, Proposition 3.1], every C_i^* is an (n,d)-flat right R-module. So by [18, Lemma 3.53], there is an exact sequence $0 \to M^* \to C_0^* \to C_1^* \to \cdots$ of (n,d)-flat right R-modules. Hence, we have:

$$\mathcal{F}_{\mathcal{D}}^{\mathcal{N}} = \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow C_0^* \longrightarrow C_1^* \longrightarrow \cdots,$$

where every F_i is projective and (n,d)-flat, and also $M^* = \mathrm{Ker}(C_0^* \to C_1^*)$. By definition, it suffices to show that $\mathcal{F}_{\mathcal{D}}^{\mathcal{N}} \otimes_R K_{d-1}$ is an exact sequence whenever K_{d-1} is a special finitely presented R-module with $\mathrm{fd}_R(K_{d-1}) < \infty$. Let $\mathrm{fd}_R(K_{d-1}) = m$. We prove by induction on m. The case $m = 0$ is clear. Assume that $m \geq 1$. There exists a special exact sequence $0 \to K_d \to F_d \to K_{d-1} \to 0$ with respect to every finitely n-presented R-module U, where F_d is projective. Now, we deduce that K_d is special finitely presented, since R is a special coherent ring. Also, $\mathrm{fd}_R(K_d) \leq m-1$. So, the following short exact sequence of complexes exists:

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots & \\
 & & \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & & F_1 \otimes_R K_d & \longrightarrow & F_1 \otimes_R F_d & \longrightarrow & F_1 \otimes_R K_{d-1} & \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & & F_0 \otimes_R K_d & \longrightarrow & F_0 \otimes_R F_d & \longrightarrow & F_0 \otimes_R K_{d-1} & \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & & C_0^* \otimes_R K_d & \longrightarrow & C_0^* \otimes_R F_d & \longrightarrow & C_0^* \otimes_R K_{d-1} & \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & & C_1^* \otimes_R K_d & \longrightarrow & C_1^* \otimes_R F_d & \longrightarrow & C_1^* \otimes_R K_{d-1} & \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow & \\
 & & \vdots & & \vdots & & \vdots & \\
 & & \| & & \| & & \| & \\
0 \longrightarrow & & \mathcal{F}_{\mathcal{D}}^{\mathcal{N}} \otimes_R K_d & \longrightarrow & \mathcal{F}_{\mathcal{D}}^{\mathcal{N}} \otimes_R F_d & \longrightarrow & \mathcal{F}_{\mathcal{D}}^{\mathcal{N}} \otimes_R K_{d-1} & \longrightarrow 0.
\end{array}$$

By induction, $\mathcal{F}_{\mathcal{D}}^{\mathcal{N}} \otimes_R F_d$ and $\mathcal{F}_{\mathcal{D}}^{\mathcal{N}} \otimes_R K_d$ are exact, hence $\mathcal{F}_{\mathcal{D}}^{\mathcal{N}} \otimes_R K_{d-1}$ is exact by [18, Theorem 6.10].

($\Longleftarrow$) Let M^* be a Gorenstein (n,d)-flat right R-module. Then, by (2)($\Longrightarrow$), M^{**} is Gorenstein (n,d)-injective. By [21, Proposition 2.3.5], M is pure in M^{**}, and so by Corollary 3.10, M is Gorenstein (n,d)-injective.
(2) ($\Longrightarrow$) Let M be a Gorenstein (n,d)-flat right R-module. Then by Definition 3.1, there is an (n,d)-flat right resolution of M as form:

$$0 \longrightarrow M \longrightarrow D^0 \longrightarrow D^1 \longrightarrow \cdots .$$

By [24, Proposition 2.3], $(D^i)^*$ is (n,d)-injective for any $i \geq 0$. So by [18, Lemma 3.53], there is an exact sequence $\cdots \to (D^1)^* \to (D^0)^* \to M^*$ of (n,d)-injective R-modules. Also, for module M^*, there is an exact sequence $0 \to M^* \to E_0 \to E_1 \to \cdots 0$, where each E_i is an injective R-module. So there exists the exact sequence

$$\mathcal{I}_{\mathcal{D}}^{\mathcal{N}} = \cdots \longrightarrow (D^1)^* \longrightarrow (D^0)^* \longrightarrow E_0 \longrightarrow E_1 \longrightarrow \cdots$$

of (n,d)-injective R-modules with $M^* = \mathrm{Ker}(E_0 \to E_1)$. Now, let K_{d-1} be a special finitely presented module with $\mathrm{pd}_R(K_{d-1}) < \infty$ and $0 \to K_d \to F_d \to K_{d-1} \to 0$ a special exact sequence, then K_d is special finitely presented module by hypothesis, and so similar to the proof of [11, Theorem 2.3], we obtain that M^* is Gorenstein (n,d)-injective.

($\Longleftarrow$) Let M^* be a Gorenstein (n,d)-injective R-module. Then, by (1)($\Longrightarrow$), M^{**} is a Gorenstein (n,d)-flat right R-module. By [21, Proposition 2.3.5], M is pure in M^{**}, and so by Corollary 3.12,M is Gorenstein (n,d)-flat. □

Proposition 4.2. *Let R be a special coherent ring. Then*

(1) *The pair $(\mathcal{GI}_d^n(R), \mathcal{GF}_d^n(R^{op}))$ is a duality pair;*
(2) *The pair $(\mathcal{GF}_d^n(R^{op}), \mathcal{GI}_d^n(R))$ is a duality pair.*

Proof. (1) Let M be an R-module. Then by Theorem 4.1(1), $M \in \mathcal{GI}_d^n(R)$ if and only if $M^* \in \mathcal{GF}_d^n(R^{op})$. By Proposition 3.13(2), any finite direct sum of Gorenstein (n,d)-flat right modules is Gorenstein (n,d)-flat. Also, by Proposition 3.14(6), $\mathcal{GF}_d^n(R^{op})$ is closed under direct summands, and hence the pair $(\mathcal{GI}_d^n(R), \mathcal{GF}_d^n(R^{op}))$ is a duality pair.

(2) Let M be a right R-module. Then by Theorem 4.1(2), $M \in \mathcal{GF}_d^n(R^{op})$ if and only if $M^* \in \mathcal{GI}_d^n(R)$. By Proposition 3.13(1), any finite direct sum of Gorenstein (n,d)-injective modules is Gorenstein (n,d)-injective and by Proposition 3.14(4), $\mathcal{GI}_d^n(R)$ is closed under direct summands, and so the pair $(\mathcal{GF}_d^n(R^{op}), \mathcal{GI}_d^n(R))$ is a duality pair. □

Let $\mathcal{F}$ be a class of R-modules and M an R-module. Following [8, 16, 20], we say that a morphism $f : F \to M$ is an $\mathcal{F}$-precover of M if $F \in \mathcal{F}$ and $\mathrm{Hom}_R(F', F) \to \mathrm{Hom}_R(F', M) \to 0$ is exact for all $F' \in \mathcal{F}$. Moreover, if whenever a morphism $g : F \to F$ such that $fg = f$ is an automorphism of F, then $f : F \to M$ is called an $\mathcal{F}$-cover of M. The class $\mathcal{F}$ is called (pre)covering, if each object in R has an $\mathcal{F}$-(pre)cover. Dually, the notions of $\mathcal{F}$-preenvelopes, $\mathcal{F}$-envelopes and (pre)enveloping are defined.

In the following theorem on special coherent rings, by using of duality pairs, we show that the classes $\mathcal{GI}_d^n(R)$ and $\mathcal{GF}_d^n(R^{op})$ are covering and preenveloping.

Theorem 4.3. *Let R be a special coherent ring. Then*

(1) *The class $\mathcal{GI}_d^n(R)$ is covering and preenveloping;*
(2) *The class $\mathcal{GF}_d^n(R^{op})$ is covering and preenveloping.*

Proof. (1) Every direct sum of Gorenstein (n,d)-injective modules and every direct product of Gorenstein (n,d)-injective modules are Gorenstein (n,d)-injective by Propositions 3.14(1) and 3.13(1), respectively. Also, by Corollary 3.10, the class of Gorenstein (n,d)-injective modules is closed under pure submodules, pure quotients and pure extensions. So, by Proposition 4.2(1) and [15, Theorem 3.1], we deduce that every R-module has a Gorenstein (n,d)-injective cover and a Gorenstein (n,d)-injective preenvelope.

(2) Every direct sum of Gorenstein (n,d)-flat right modules and every direct product of Gorenstein (n,d)-flat right modules are Gorenstein (n,d)-flat by Propositions 3.13(2) and 3.14(2), respectively. Also, by Corollary 3.12, the class of Gorenstein (n,d)-flat right modules is closed under pure submodules, pure quotients and pure extensions. So, by Proposition 4.2(2) and [15, Theorem 3.1], we deduce that every R-module has a Gorenstein (n,d)-flat cover and a Gorenstein (n,d)-flat preenvelope. □

In the following proposition, we show that Gorenstein (n,d)-injective and Gorenstein (n,d)-flat R-modules are (n,d)-injective and (n,d)-flat, respectively if $l.(n,d)\text{-}dim(R) < \infty$.

Proposition 4.4. *If $l.(n,d)\text{-}dim(R) < \infty$, then the following hold:*

(1) *Every Gorenstein (n,d)-injective R-module is (n,d)-injective;*
(2) *Every Gorenstein (n,d)-flat right R-module is (n,d)-flat.*

Proof. Let $l.(n,d)\text{-}dim(R) < \infty$. Then, for all special finitely presented modules K_{d-1} with respect to every finitely n-presented module U, it follows that $\mathrm{pd}_R(K_{d-1}) < \infty$. So by Lemma 3.5(1), $0 = \mathrm{Ext}_R^1(K_{d-1}, M) \cong \mathrm{Ext}_R^{n+1}(U, M)$ for every Gorenstein (n,d)-injective R-module M, and consequently M is (n,d)-injective. Also, we have $\mathrm{fd}_R(K_{d-1}) < \infty$. So by Lemma 3.5(2), $0 = \mathrm{Tor}_1^R(N, K_{d-1}) \cong \mathrm{Tor}_{n+1}^R(N, U)$ for every Gorenstein (n,d)-flat right R-module N, and consequently N is (n,d)-flat. □

Now we give some equivalent characterizations for ${}_RR$ being Gorenstein (n,d)-injective in terms of the properties of Gorenstein (n,d)-injective and Gorenstein (n,d)-flat modules.

Proposition 4.5. *Let R be a special coherent ring. Then the following statements are equivalent:*

(1) ${}_RR$ *is Gorenstein* (n, d)*-injective;*
(2) *Every* R*-module has an epic Gorenstein* (n, d)*-injective cover;*
(3) *Every flat* R*-module is Gorenstein* (n, d)*-injective;*
(4) *Every* (n, d)*-injective right* R*-module is Gorenstein* (n, d)*-flat;*
(5) *Every injective right* R*-module is Gorenstein* (n, d)*-flat;*
(6) *Every right* R*-module has a monic Gorenstein* (n, d)*-flat preenvelope.*

Proof. (3) $\Longrightarrow$ (1) Trivial.

(2) $\Longrightarrow$ (1) By hypothesis, R has an epic Gorenstein (n, d)-injective cover $f : C \to R$, then we have a split exact sequence $0 \to \mathrm{Ker} f \to C \to R \to 0$ with C is a Gorenstein (n, d)-injective R-module. So, by Proposition 3.14(4), ${}_RR$ is Gorenstein (n, d)-injective.

(1) $\Rightarrow$ (2) By Theorem 4.3(1), there is a Gorenstein (n, d)-injective cover $\phi : A \to M$ for any R-module M. Also, there is an exact sequence $0 \to K \to F \stackrel{h}{\to} M \to 0$ of R-modules, where F is free module. Since ${}_RR$ is Gorenstein (n, d)-injective, then by Proposition 3.14(1), it follows that $F \cong \bigoplus_{i\in I} R$ is Gorenstein (n, d)-injective. So there exists a map $g : F \to A$ such that $\phi g = h$. Since h is epic, we deduce that $\phi : A \to M$ is also epic.

(6) $\Longrightarrow$ (5) Let I be an injective right R-module. then I has a monic Gorenstein (n, d)-flat preenvelope $f : I \to F$ by assumption. So, the split exact sequence $0 \to I \to F \to \frac{F}{I} \to 0$ exists, and so I is direct summand of F. Hence, by Proposition 3.14(6), I is Gorenstein (n, d)-flat.

(5) $\Longrightarrow$ (1) By hypothesis, R^* is a Gorenstein (n, d)-flat right R-module, since R^* is injective. So, ${}_RR$ is Gorenstein (n, d)-injective by Theorem 4.1(1).

(1) $\Longrightarrow$ (6) By Theorem 4.3(2), every right R-module M has a Gorenstein (n, d)-flat preenvelope $f : M \to D$. By Theorem 4.1(1), R^* is Gorenstein (n, d)-flat, and so $\prod_{i\in I} R^*$ is Gorenstein (n, d)-flat by Proposition 3.14(2). On the other hand, $({}_RR)^*$ is a cogenerator. Therefore, exact sequence of the form $0 \to M \stackrel{g}{\to} \prod_{i\in I} R^*$ exists, and hence homomorphism $0 \to D \stackrel{h}{\to} \prod_{i\in I} R^*$ such that $hf = g$ shows that f is monic.

(5) $\Longrightarrow$ (4) Let M be an (n, d)-injective right R-module. Then the exact sequence $0 \to M \to E(M) \to \frac{E(M)}{M} \to 0$ is special pure, since $\mathrm{Ext}^1_R(K_{d-1}, M) = 0$ for every special finitely presented K_{d-1} with respect to any finitely n-presented U. Since by (5), $E(M)$ is Gorenstein (n, d)-flat, from Corollary 3.12, we deduce that M is Gorenstein (n, d)-flat.

(4) $\Longrightarrow$ (3) Let N be a flat R-module. Then, N^* is an injective right R-module, so N^* is Gorenstein (n, d)-flat by hypothesis, and hence N is Gorenstein (n, d)-injective by Theorem 4.1(1). $\square$

Corollary 4.6. *Let* R *be a special coherent ring with* $l.(n, d)$*-dim*$(R) < \infty$*. Then the following statements are equivalent:*

(1) ${}_RR$ *is Gorenstein* (n, d)*-injective;*

(2) *Every Gorenstein flat R-module is Gorenstein (n, d)-injective;*
(3) *Every R-module is Gorenstein (n, d)-injective;*
(4) *Every Gorenstein injective right R-module is Gorenstein (n, d)-flat.*

Proof. (3) $\Longrightarrow$ (2) and (2) $\Longrightarrow$ (1) are clear.

(1) $\Longrightarrow$ (3) Let M be an R-module. Then, there is an exact sequence $\cdots \to P_1 \to P_0 \to M \to 0$, where each P_i is flat. If ${}_RR$ is Gorenstein (n, d)-injective, then by Proposition 4.4(1), R is (n, d)-injective. Hence, by Proposition 3.14(1), we deduce that every P_i is (n, d)-injective. Also, for module M, there is an exact sequence $0 \to M \to E_0 \to E_1 \to \cdots 0$, where every E_i is injective. So, we have:

$$\mathcal{I}_{\mathcal{D}}^{\mathcal{N}} = \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow E_0 \longrightarrow E_1 \longrightarrow \cdots,$$

where P_i and E_i are (n, d)-injective and $M = \ker(E_0 \to E_1)$. Similar to the proof of the Theorem 4.1((2) ($\Longrightarrow$)), we get that $\mathrm{Hom}_R(K_{d-1}, \mathcal{I}_{\mathcal{D}}^{\mathcal{N}})$ is an exact sequence for every special finitely presented R-module K_{d-1} with $\mathrm{pd}_R(K_{d-1}) < \infty$, and consequently, M is Gorenstein (n, d)-injective.

(3) $\Longrightarrow$ (4) If M is a Gorenstein injective right R-module, then by hypothesis, M^* is Gorenstein (n, d)-injective, and hence by Theorem 4.1(2), it follows that M is Gorenstein (n, d)-flat.

(4) $\Longrightarrow$ (1) If every Gorenstein injective right R-module is Gorenstein (n, d)-flat, it follows that every injective right R-module is Gorenstein (n, d)-flat, too. So by Proposition 4.5, we obtain that ${}_RR$ is Gorenstein (n, d)-injective. □

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