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# On $\sigma$-Classes of Modules with Applications

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Abstract. In this paper, we introduce some lattices of classes of left R-module relative to a preradical sigma. These lattices are generalizations of the lattices R-TORS, R-tors, R-nat, R-conat, of torsion theories, hereditary torsion theories, natural classes and conatural classes, respectively. We define the lattices $\sigma$-(R-TORS), $\sigma$-(R-tors), $\sigma$-(R-nat), $\sigma$-(R-conat), which reduce to the lattices mentioned above, when one takes sigma as the identity. We characterize the equality between these lattices by means of the ($\sigma$-HH) condition, which we introduce. We also present some results about $\sigma$-retractable rings, $\sigma$-Max rings extending results about Mod-retractable rings and Max rings.

**Keywords:** Preradical, Lattices of module classes, Left Mod-retractable ring, Left semiartinian ring, Left max ring.

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## 1. Introduction

$R$ will denote an associative ring with 1 and $R$-Mod will denote the category of left $R$-modules and $R$-morphisms. Several results of module theory are latticial in nature. The behavior of the lattices associated with a ring determines several properties of the ring. The comparison between several lattices

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associated with a ring often provides interesting information. We will use this method with some lattices of module classes relative to a preradical $\sigma$.

Recall that the preradicals in $R$-Mod are functors $\sigma : R\text{-Mod} \to R\text{-Mod}$ such that for each $R$-module $M$ one has $\sigma(M) \leq M$ and for each $R$-morphism $f : M \to N$ one has that $f(\sigma(M)) \leq \sigma(N)$. Note that a preradical $\sigma$ is a subfuntor of the identity functor in $R$-Mod. Recall that preradicals in $R$-mod conforms the big complete lattice $R$-pr, where order is given by $\sigma \leq \tau \Leftrightarrow \sigma(M) \subseteq \tau(M)$, for each ${}_RM$. A family of preradicals $\{r_i\}_I$ has a an infimum $\wedge r_i$ which evaluated in a module $M$ gives $\bigcap_{i\in I} r_i(M)$, and $\{r_i\}_I$ also has a supremun $\vee r_i$ which evaluated in a module $M$ gives $\sum_{i\in I} r_i(M)$. The identity in $R$-mod functor is the greatest preradical and the preradical $\underline{0}$ which assigns to each module its zero submodule is the least preradical. Remember that in R-pr, there are two binary operations: composition, denoted by $\sigma\tau$ (defined by $\sigma\tau(M) = \sigma(\tau(M))$), and cocomposition, denoted by $\sigma : \tau$ (defined by $(\sigma : \tau)(M)/\sigma(M) = \tau(M/\sigma(M))$).

We say that a preradical $\sigma$ is a left exact preradical if it is left exact as a functor. $\sigma$ is idempotent if $\sigma\sigma = \sigma$. $\sigma$ is stable (costable) if for each injective $R$-module $E$ (for each projective $R$-module $P$) one has that $E = \sigma(E) \oplus E'$ (one has that $P = \sigma(P) \oplus P'$) for some $E' \leq E$ (for some $P' \leq P$).

There are two important classes of modules associated with a preradical $\sigma$, its torsion class $\mathbb{T}_\sigma = \{M \mid \sigma(M) = M\}$ and its torsion free class $\mathbb{F}_\sigma = \{M \mid \sigma(M) = 0\}$. $\mathbb{T}_\sigma$ is a class closed under the taking of coproducts and epimorphic images (i.e. it is a pretorsion class) and $\mathbb{F}_\sigma$ is a class closed under taking submodules and products (i.e is a pretorsion free class).

We say that an ordered pair $(\mathbb{T}, \mathbb{F})$ of module classes is a torsion theory if $\mathbb{T}$ is closed under taking epimorphic images, coproducts and extensions, and $\mathbb{F} = \{N \mid Hom(M, N) = 0,\ \forall M \in \mathbb{T}\}$ or equivalently, if $\mathbb{F}$ is closed under taking submodules, products, extensions and $\mathbb{T} = \{M \mid Hom(M, N) = 0,\ \forall N \in \mathbb{F}\}$. A torsion theory $(\mathbb{T}, \mathbb{F})$ having $\mathbb{T}$ closed under taking submodules, or equivalently, having $\mathbb{F}$ closed under taking injective hulls is called a hereditary torsion theory.

The class of hereditary torsion theories is a complete lattice that has been extensively studied (see, for example [9] or [13]).

For further information about lattices of classes of modules see [7]. Comparisons between lattices of classes of modules may characterize types of rings. By example, in [10], the authors prove that every torsion theory is hereditary, if and only if, all modules are retractable. They called $R$-Mod-retractable ring to a ring with this property. A module $M$ is retractable when $Hom(M, N) \neq 0$ for each nonzero submodule $N$ of $M$. The technique of studying the relationships between various lattices associated with a ring is used, for example, in [14] and [15], where, among many interesting results, the class of co-semisimple modules is characterized. A module M is co-semisimple when the simple submodules subgenerated by M are M-injective. A module M is said to be co-semisimple if the simple submodules subgenerated by M are M-injective. This concept is an extension of the V-ring concept. In these papers, Talaee defines torsion theories, uses module preradicals such as Jacobson's radical and Zhou's preradical $\delta$, and establishes the closure properties of certain classes of modules.

In [6], the lattice $\sigma$-(R-TORS) of $\sigma$-torsion classes, and the lattice $\sigma$-(R-tors) of $\sigma$-hereditary torsion classes are defined for a preradical $\sigma$. We refer to the reader to [13, Chapter VI] and [4] for more information about preradicals on $R$-Mod.

In [5], for an exact left and stable preradical $\sigma$, the lattice $\sigma$-(R-nat) of $\sigma$-natural classes is defined and, for an exact and costable preradical $\sigma$, the lattice $\sigma$-(R-conat) of $\sigma$-conatural classes is defined.

We introduce, for a ring $R$ and a preradical $\sigma$, the condition ($\sigma$-HH). We say that $R$ satisfies the condition ($\sigma$-HH) if for any pair of $R$-modules $M, N$ with $\sigma(M) \neq 0$, and $\sigma(N) \neq 0$, one has that $Hom(\sigma(N), M) \neq 0$ if and only if $Hom(M, \sigma(N)) \neq 0$. This condition is related to the parainjective $R$-modules of $\sigma$-torsion and with the paraprojective $R$-modules of $\sigma$-torsion. This condition is also important with respect to left $\sigma$-semiartinian rings, the left $\sigma$-max rings and the $\sigma$-retractable rings introduced in [6]. Finally, it turns out that the rings satisfying the ($\sigma$-HH)-condition are the rings for which the lattices of $\sigma$-torsion classes, of hereditary $\sigma$-torsion classes, of $\sigma$-natural classes and of $\sigma$-conatural classes are all the same.

## 2. Some kinds of preradicals

Recall the following definitions.

**Definition 2.1.** A preradical $\sigma$ is

(1) Left exact if and only if $\sigma(A) = A \cap \sigma(B)$ whenever $A$ be a submodule of $B$ if and only if $\sigma$ is idempotent and its pretorsion class $\mathbb{T}_\sigma$ is hereditary.
(2) Cohereditary if $\sigma$ preserves epimorphisms, this is equivalent to the pretorsion-free class of $\sigma$, $\mathbb{F}_\sigma$ being closed under quotients and it is also equivalent to $\sigma(M) = \sigma(R)M$, for each $M$.
(3) Splitting in $M$ if $\sigma(M) \leq_\oplus M$.
(4) Splitting if it splits in each module.
(5) Cosplitting if it is hereditary and cohereditary.
(6) Stable if it splits in each injective module.
(7) Costable if it splits in each projective module.
(8) Centrally splitting if it has a complement $\sigma'$ in $R$-pr, or equivalently, if there exists a central idempotent $e \in R$ such that $\sigma(M) = eM$, for each module $M$.

Example 2.2. (1) Let $Z : R\text{-mod} \to R\text{-mod}$ be such that $Z(M) = \{x \in M \mid (0:m) \text{ is essential in } R\}$, where $(0:m) = \{r \in R \mid rm = 0\}$. Then

$Z$ is a left exact preradical. The preradical $Z : Z$, also denoted $Z_2$ or $t_G$, is a left exact radical.

(2) $soc : R\text{-}mod \to R\text{-}mod$ such that

$$soc(M) = \sum\{S \leq M \mid S \text{ is a simple module}\}$$

is a left exact preradical because it is idempotent, and its pretorsion class, the class of semisimple modules, is hereditary.

(3) If $\mathcal{A}$ is a set of simple modules, denote $soc_{\mathcal{A}}$ the left exact preradical such that $\mathrm{soc}_{\mathcal{A}}(M) = \Sigma\{S \leq M \mid$ is isomorphic to a module in $\mathcal{A}\}$. It is easy to see that $\mathrm{soc}_{\mathcal{A}}$ is a left exact preradical.

(4) Let $\mathrm{soc_p}$ denote the preradical that assigns to a module the sum of its projective simple submodules. Then $\mathrm{soc_p}$ is a cohereditary preradical, which we will also call a $t$-radical. This is a consequence of the following commutative diagram:

$$\begin{tikzcd} M \arrow[r, two heads, "f"] & N \\ \mathrm{soc_p}(M) \arrow[u, hook] \arrow[r, two heads, "f|"] & \mathrm{soc_p}(N). \arrow[u, hook] \end{tikzcd}$$

This occurs because, as $\mathrm{soc_p}(N)$ is a projective module, $f|$ splits. Thus, $\mathrm{soc_p}$ is a $t$-radical, and it can be calculated as $\mathrm{soc_p}(R) \cdot M$.

(5) Note that $\mathrm{soc_p}$ is an exact preradical.

(6) A preradical $\sigma$ is centrally splitting when it has a complement in R-pr (see [4] Proposition 1.7.1 (d)). This is equivalent to $\sigma = e \cdot (-)$, where $e$ is a central idempotent in $R$ (see [13], Example 2, page 140).

(7) $\mathrm{soc_p}$ is a centrally splitting preradical, if and only if each semisimple projective module is injective (see [11], Corolario 2.5). In this case, the complement of $\mathrm{soc_p}$ in R-pr is ${\tau_G}^j$, the Jansianization of $\tau_G$. † Under the same conditions, the ring $R$ can be described as $R_1 \times R_2$ , where $R_1$ is a semisimple Artinian ring and $R_2$ is a ring having its left singular ideal, being essential in $R_2$, So that $R_2 = t_G(R_2)$ (see [11], Teorema 2.16).

(8) It can be seen that for a semiperfect ring, $\mathrm{soc_p}$ centrally splits if and only if $\mathrm{soc_p}(Rad(R)) = 0$, where $Rad(R)$ denotes the Jacobson Radical of $R$ (See [12] theorem 16). A consequence of this is that $\mathrm{soc_p}$ is centrally splitting if the ring is perfect and commutative (See [12] Corollary 4).

(9) If $R = \mathbb{Z}$, the ring of integers, denote $div$ the functor assigning to ${}_{\mathbb{Z}}M$ its divisible submodule $div(M)$. $div$ is a splitting preradical,

We include some basic properties of preradicals for the reader's convenience.

† The Jansianization of a preradical $\tau$, denoted as $\tau^j$, is defined as the smallest preradical greater than or equal to $\tau$, with a pretorsion class that is closed under taking products.

**Lemma 2.3.** *The following statements about a preradical $\sigma$ are equivalent.*

(1) *$\sigma$ is cohereditary.*
(2) *$\sigma = \sigma(R) \cdot (-)$, i.e. $\sigma(M) = \sigma(R) \cdot (M)$, for each module $M$.*
(3) *$\sigma$ is a radical and it pretorsion free class $\mathbb{F}_\sigma$ is closed under quotients.*

*Proof.* (1) $\Rightarrow$ (2) Let $M$ be a module and let $f : R \to M$ be a morphism, $f = (-) \cdot x$ for some $x \in M$. Then we have that the epimorphism $R \to Rx$ restricts to an epimorphism $\sigma(R) \to \sigma(Rx)$, thus $\sigma(R)x = \sigma(Rx)$. Now, take the epimorphism $R^{(M)} \twoheadrightarrow M$ induced by the family of morphisms $\{(-) \cdot x : R \to M\}_{x \in M}$ then we have the epimorphism $\sigma(R^{(M)}) \twoheadrightarrow \sigma(M)$. Thus $\sigma(M) = \sum_{x \in M} \sigma(R)x = \sigma(R)M$.

(2) $\Rightarrow$ (1) In general, if $I$ is an ideal of $R$ then $I \cdot (-)$ is a preradical preserving epimorphisms, as it is immediately verified. More over, $I\,(M/IM) = 0$ shows that $I \cdot (-)$ is a radical.

(2) $\Rightarrow$ (3) Assume that $f : M \twoheadrightarrow N$ is an epimorphism with $M \in \mathbb{F}_\sigma$ then we have an epimorphism $0 = \sigma(M) \twoheadrightarrow \sigma(N)$. $\sigma$ is a radical as it was noted in the above argument.

(3) $\Rightarrow$ (1) Let $f : M \twoheadrightarrow N$ be an epimorphism. $f$ induces an epimorphism $\overline{f} : M/\sigma(M) \twoheadrightarrow N/f\,(\sigma(M))$. Thus we have the situation of the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & f\,(\sigma(M)) \arrow[r, tail] & N \arrow[r, two heads] & N/f\,(\sigma(M)) \arrow[r] & 0 \\
0 \arrow[r] & \sigma(M) \arrow[r, tail] \arrow[u, two heads, "f|"] & M \arrow[r, two heads] \arrow[u, two heads, "f"] & M/\sigma(M) \arrow[r] \arrow[u, two heads, "\overline{f}"] & 0.
\end{tikzcd}$$

As $\sigma$ is a radical, it follows that $\sigma\,(M/\sigma(M)) = 0$. Now, as $\mathbb{F}_\sigma$ is a class closed under taking quotients, it follows that $\sigma\,(N/f\,(\sigma(M))) = 0$. Then, as $f\,(\sigma(M)) \leq N$and as $\sigma$ is a radical, it follows that $\sigma\,(N/f\,(\sigma(M))) = \sigma(N)/f\,(\sigma(M))$ (see, [13, Lemma 1.1, Chap. VI]). So, $\sigma(N)/f\,(\sigma(M)) = 0$ and hence $\sigma(M) = f\,(\sigma(M))$. Therefore $\sigma$ is cohereditary. $\square$

**Lemma 2.4.** *A preradical $\sigma$ is hereditary and cohereditary if and only if $\sigma = \sigma(R) \cdot (-)$ and the ideal $\sigma(R)$ satisfies that $x \in \sigma(R)x$, for each $x \in \sigma(R)$.*

*Proof.* $\Rightarrow$] Since $\sigma$ is cohereditary, we have $\sigma = \sigma(R) \cdot (-)$. On the other hand, as $\sigma$ is hereditary then it is idempotent thus $\sigma(R)$ is idempotent and $\mathbb{T}_\sigma$ is a hereditary class. So, if $x \in \sigma(R)$ then $Rx \leq \sigma(R) \in \mathbb{T}_\sigma$, applying $\sigma$, $x \in Rx = \sigma(Rx) = \sigma(R)Rx = \sigma(R)x$.

$\Leftarrow$] Conversely, if $\sigma$ is a cohereditary radical and $x \in \sigma(R)x$, for each $x \in \sigma(R)$, then $\sigma(R) \leq \sigma(R)\sigma(R) \leq \sigma(R)$. Thus $\sigma$ is an idempotent radical. We only have to show that $\mathbb{T}_\sigma$ is hereditary class, which is equivalent to $\mathbb{F}_\sigma$ being closed under injective hulls. Suppose that $M \in \mathbb{F}_\sigma$ but $E(M) \notin \mathbb{F}_\sigma$ . Then $\sigma(E(M)) \neq 0$ and then $M \cap \sigma(R)(E(M)) \neq 0$. Then there exists $0 \neq m \in \sigma(R)(E(M))$, then $m = a_1x_1 + \cdots + a_nx_n$, with $a_i \in \sigma(R)$ and $x_i \in E(M)$.

Let us take one such $m$ where $n$ be least possible. Notice that $n \neq 1$, because if $0 \neq m = a_1x_1$ then $a_1 = b_1a_1$ for some $b_1 \in \sigma(R)$, by hypothesis. So, $0 \neq m = b_1a_1x_1 = b_1m = 0$, a contradiction. Now, in $m = a_1x_1 + \cdots + a_nx_n$, we can write $a_i = b_ia_i$ for some $b_i \in \sigma(R)$. Note that $m = b_1a_1x_1 + \cdots + b_na_nx_n$ and that $0 = b_1m = b_1a_1x_1 + \cdots + b_1a_nx_n$ thus making a subtraction we have that $m = (b_1 - b_2)a_2x_2 + \cdots + (b_1 - n_n)a_nx_n$ with less than $n$ summands, contradicting the choice of $m$ and $n$. This proves that $\mathbb{F}_\sigma$ is closed under taking injective hulls, and this completes the proof. □

**Lemma 2.5.** *A preradical $\sigma$ is exact, i.e., $\sigma$ preserves exact sequences of modules if and only if it is hereditary and cohereditary.*

*Proof.* If $\sigma$ is exact, clearly is left exact and right exact, thus it is hereditary and cohereditary. If $\sigma$ is cohereditary and hereditary, then it preserves epimorphisms and it is left exact. Thus $\sigma$ preserves short exact sequences, and it is well known that this implies that $\sigma$ is an exact functor (See [13, Chap. I, §5, Lemma 5.1]). □

**Lemma 2.6.** *For a preradical $\sigma$ the following conditions are equivalent:*

(1) *$\sigma$ is exact and $\sigma(R)$ is a direct summand of $R$.*
(2) *$\sigma$ centrally splits.*

*Proof.* (1) ⇒ (2) As $\sigma$ is exact then $\sigma = \sigma(R) \cdot (-)$ and $\sigma(R)$ is an idempotent ideal having the property that $x \in \sigma(R)x, \forall x \in \sigma(R)$. Let us call $\rho$ the preradical which assigns to a module $M$ the submodule $\{x \in M \mid \sigma(R)x = 0\}$. $\rho$ is a left exact radical whose torsion class $\mathbb{T}_\rho$ is closed under taking products. $\rho(R)$ is a two sided ideal. As $R = \sigma(R) \oplus J$ for some left ideal $J$, by hypothesis, and as $\sigma(R)J \subseteq \sigma(R) \cap J = 0$ then, $J \subseteq \rho(R)$. Thus $R = \sigma(R) \oplus \rho(R)$. Then $\sigma(R) = Re$ with $e$ a central idempotent. Thus $\sigma = \sigma(R) \cdot (-) = e \cdot (-)$. Hence $\sigma$ centrally splits.

(2)⇒ (1) This follows from Lemma 2.4 □

**Lemma 2.7.** *The following statements are equivalent*

(1) *$\sigma$ is exact and stable.*
(2) *$\sigma$ centrally splits.*
(3) *$\sigma$ is exact and costable.*

*Proof.* Is clear that if $\sigma$ centrally splits, then it is exact. Now, as $\sigma(M)$ is a direct summand of $M$, for each module $M$, then $\sigma$ is stable and costable.

Conversely, if $\sigma$ is exact and stable, then $\sigma(E(R)) \hookrightarrow E(R)$ splits, thus $\sigma(R) \hookrightarrow R$ also splits. And if $\sigma$ is exact and costable, then $\sigma(R) \hookrightarrow R$ splits, because $R$ is a projective module. We conclude using Lemma 2.6. □

It is a known fact that the class of left exact preradicals, which we will denote $R$-lep (by left exact preradicals) is in bijective correspondence with the set of linear filters for the ring and it is also in bijective correspondence with the class of Wisbauer classes. A Wisbauer class is a class of modules closed under taking submodules, quotients and direct sums. Thus $R$-lep is a complete lattice. We denote by $R$-pr the big lattice of preradicals in $R$-mod.

## 3. $\sigma$-classes of modules

In [5], the authors introduced lattices of module classes induced by a preradical $\sigma$. Namely they introduced $\sigma$-hereditary classes, $\sigma$-cohereditary classes, $\sigma$-natural classes and $\sigma$-conatural classes and showed that they conform (big) lattices, which were denoted $\sigma$-($R$-her), $\sigma$-($R$-coher), $\sigma$-($R$-nat) and $\sigma$-($R$-conat), respectively. We recall the definitions and some relevant results for the sequel.

**Definition 3.1.** Let $\sigma \in R$-pr. A class $\mathcal{C}$ of $R$-modules is a **$\boldsymbol{\sigma}$-hereditary class** if it satisfies the following conditions:

(1) $\mathbb{F}_\sigma \subseteq \mathcal{C}$.
(2) For each $M \in \mathcal{C}$ and each $N \leq M$ it follows that $\sigma(N) \in \mathcal{C}$.

We denote by **$\boldsymbol{\sigma}$-(R-her)** to the collection of all $\sigma$-hereditary classes.

Following Golan, we call skeleton of a lattice $\mathcal{L}$ with a smallest element, to the set of pseudocomplements of $\mathcal{L}$ and denote it Skel($\mathcal{L}$).

**Lemma 3.2.** [5, Lemma 1] *Let $\sigma \in R$-pr. The collection $\sigma$-(R-her) is a pseudocomplemented big lattice. Moreover, if $\sigma$ is idempotent, then $\sigma$-(R-her) is strongly pseudocomplemented and for each $\mathcal{C} \in \sigma$-(R-her) its pseudocomplement, denoted by $\mathcal{C}^{\perp_{\leq\sigma}}$, is given by*

$$\mathcal{C}^{\perp_{\leq\sigma}} = \{M \in R\text{-}Mod \mid \forall\ N \leq M,\ \sigma(N) \in \mathcal{C} \Rightarrow N \in \mathbb{F}_\sigma\}.$$

*In addition, Skel$(\sigma$-(R-her)) is a boolean lattice.*

Given the above lemma, we can make the following definition.

**Definition 3.3.** Let $R$ be a ring and let $\sigma \in R$-pr be left exact and stable. We define the lattice of $\sigma$-natural classes, denoted by **$\boldsymbol{\sigma}$-(R-nat)**, as

$$\sigma\text{-}(R\text{-nat}) = Skel(\sigma\text{-(R-her)}).$$

Example 3.4. Let $R$ be a Noetherian $V$-ring. Then each semisimple projective module is injective. Consider the left exact preradical

$$soc_{\mathcal{A}} = \sum\{S \leq M \mid S \text{ is isomorphic to an element of } \mathcal{A}\}.$$

A class $\mathcal{C}$ of $R$-modules is $soc_{\mathcal{A}}$-hereditary if each ${}_RM \in \mathcal{C}$ has all its simple submodules belonging to $\mathcal{C}$. Let us denote $\mathcal{C}^{\perp_{soc_{\mathcal{A}}}}$ the pseudocomplement of $\mathcal{C}$ in the lattice of $soc_{\mathcal{A}}$-hereditary classes. If $\mathcal{A}$ consists of projective simple modules, then $soc_{\mathcal{A}}$ is a left exact stable preradical. Then $\mathcal{C}^{\perp_{soc_{\mathcal{A}}}}$ is a $soc_{\mathcal{A}}$- natural class. We can describe it as $\{N \mid N$ has no simple submodules belonging to $\mathcal{A}\}$. From this, it follows that there is an order-inverting bijective correspondence between $(\wp(R-simp),\subseteq)$ and the lattice of $soc_{\mathcal{A}}$-natural classes. This illustrates that $soc_{\mathcal{A}}\,(R\text{-}nat)$ is a Boolean lattice.

**Definition 3.5.** Let $\sigma \in R$-pr. A class $\mathcal{C}$ of $R$-modules is a **$\sigma$-cohereditary class** if it satisfies the following conditions:

(1) $\mathbb{F}_\sigma \subseteq \mathcal{C}$.

(2) For each $M \in \mathcal{C}$ and each epimorphism $M \twoheadrightarrow L$ we have that $\sigma(L) \in \mathcal{C}$.

We denote the collection of all $\sigma$-cohereditary classes by **$\sigma$-(R-coher)**.

The following proposition is a generalization of [2, Proposition 3.6].

**Proposition 3.6.** [5, Proposition 11] *Let $\sigma \in R$-pr. If $\sigma$ is idempotent and cohereditary, then $\sigma$-(R-coher) is a strongly pseudocomplemented big lattice and for each $\mathcal{C} \in \sigma$-(R-coher) the pseudocomplement, of $\mathcal{C}$, denoted by $\mathcal{C}^{\perp/\sigma}$, is given by*

$$\mathcal{C}^{\perp/\sigma} = \{M \in R\text{-}Mod \mid \forall\ M \twoheadrightarrow L,\ \sigma(L) \neq 0 \Rightarrow \sigma(L) \notin \mathcal{A}\} \cup \mathbb{F}_\sigma.$$

*In addition, $Skel\,(\sigma$-(R-coher)) is a boolean lattice.*

**Definition 3.7.** Let $R$ be a ring and let $\sigma \in R$-pr be exact and costable. We define the lattice of $\sigma$-conatural classes, denoted by **$\sigma$-(R-conat)** as

$$\sigma\text{-(R-conat)} = Skel(\sigma\text{-(R-coher)}).$$

Later, in [6], the authors introduced the lattices $\sigma$-(R-TORS) and $\sigma$-(R-tors) as follows:

**Definition 3.8.** Let $\sigma \in R$-pr. A class $\mathcal{C}$ of $R$-modules **$\sigma$-torsion class** if $\mathcal{C}$ is a $\sigma$-cohereditary class, closed under taking arbitrary direct sums and under taking extensions, if it is also a $\sigma$-hereditary class, then we will say that $\mathcal{C}$ is an **hereditary $\sigma$-torsion class**.

**Definition 3.9.** Let $\sigma \in R$-pr. We denote the collection of all $\sigma$-torsion classes by **$\sigma$-(R-TORS)** and by **$\sigma$-(R-tors)** the collection of all hereditary $\sigma$-torsion classes.

In [6], the authors define the following assignations relative to a preradical $\sigma$:

(1) $\sigma^* : \mathcal{P}(R\text{-Mod}) \to \mathcal{P}(R\text{-Mod})$ given by $\sigma^*(\mathcal{C}) = \{\sigma(M) \mid M \in \mathcal{C}\}$ and

(2) $\overleftarrow{\sigma} : \mathcal{P}(R\text{-Mod}) \to \mathcal{P}(R\text{-Mod})$ given $\overleftarrow{\sigma}(\mathcal{C}) = \{M \mid \sigma(M) \in \mathcal{C}\}$.

In addition, it holds that $\sigma^*(\overleftarrow{\sigma}(\sigma^*(\mathcal{C}))) = \sigma^*(\mathcal{C})$ and $\overleftarrow{\sigma}(\sigma^*(\overleftarrow{\sigma}(\mathcal{C}))) = \overleftarrow{\sigma}(\mathcal{C})$, for each $\mathcal{C} \in \mathcal{P}(R\text{-Mod})$.

The authors showed the following results.

**Proposition 3.10.** [6, Corollary 3.10] *If $\sigma$ is a radical, then*

$$\sigma\text{-(R-}tors) = \{\mathcal{C} \in R\text{-}tors \mid \mathbb{F}_\sigma \subseteq \mathcal{C}\}.$$

**Theorem 3.11.** [6, Theorem 3.13] *If $\sigma$ is an exact preradical, then*

$$\sigma\text{-}(\text{R-}TORS) = \{\overleftarrow{\sigma}(\mathcal{C}) \mid \mathcal{C} \in R\text{-}TORS\ \}.$$

$\sigma$-retractable modules and $\sigma$- (R-Mod)-retractable rings also introduced in [6].

**Definition 3.12.** Let $R$ be a ring and $\sigma \in R\text{-}pr$. We say that an $R$-module $M$ is **$\boldsymbol{\sigma}$-retractable** if for every submodule $N \leq M$ with $\sigma(N) \neq 0$ we have that $Hom(M, \sigma(N)) \neq 0$ and we will say that $R$ is a **$\boldsymbol{\sigma}$-($\boldsymbol{R\text{-}Mod}$)-retractable ring** if every $R$-module is $\sigma$-retractable.

In [8], the authors introduced the condition $(HH)$ and characterized the rings satisfying it, via the retractability and properties of lattices of module classes, namely, the lattice of natural classes, the lattice of conatural classes, and the lattice of hereditary torsion theories. In the following, we extend some of these concepts and some of these results.

**Definition 3.13.** Let $R$ be a ring and $\sigma \in R\text{-}pr$. We will say that $R$ satisfies the **condition ($\boldsymbol{\sigma}$-HH)** if for any modules $M, N$ with $\sigma(M) \neq 0,\ \sigma(N) \neq 0$, it holds that $Hom(\sigma(N), M) \neq 0 \Leftrightarrow Hom(M, \sigma(N)) \neq 0$.

*Remark* 3.14. Note that if $\sigma = 1_{R\text{-}Mod}$, then we have condition $(HH)$ (see [8]), i.e. for any $R$-modules $M$ and $N$ we have that

$$Hom(N, M) \neq 0 \Leftrightarrow Hom(M, N) \neq 0.$$

Recall that a module $M$ is a **parainjective module** if for each module $N$, whenever there is some monomorphism $f \in \text{Hom}(M, N)$ there exists an epimorphism $g \in \text{Hom}(N, M)$. A module $M$ is a **paraprojective module** if for each module $N$, whenever there is some epimorphism $f \in \text{Hom}(N, M)$ there exists a monomorphism $g \in \text{Hom}(M, N)$.

*R-Simp* will denote a family of representatives of isomorphism classes of left simple modules.

*Remark* 3.15. Let $R$ be a ring and $\sigma \in R$-pr. If $R$ satisfies the ($\sigma$-HH), condition, then:

(1) $R$ is a $\sigma$- (R-Mod)-retractable ring.
(2) For any $S \in R\text{-}Simp \cap \mathbb{T}_\sigma$, it follows that $S$ is parainjective.
(3) For any $S \in R\text{-}Simp \cap \mathbb{T}_\sigma$, it follows that $S$ is paraprojective.

*Proof.*

(1) It is clear.
(2) Let $S \in R\text{-}Simp \cap \mathbb{T}_\sigma$ and let $M$ be an $R$-module such that $S$ embeds in $M$. Since $S \in \mathbb{T}_\sigma$, it follows that $\sigma(S) = S$. Since $S \in \mathbb{T}_\sigma$, then $\sigma(S) = S$, so $(Hom(\sigma(S), M) \neq 0$. Then, $Hom(M, \sigma(S)) \neq 0$, since $R$ satisfies the $\sigma\text{-}HH$ condition. Thus, $Hom(M, S) \neq 0$, i.e., $S$ is a quotient of $M$, whence S is parainjective.

(3) Let $S \in R\text{-}Simp \cap \mathbb{T}_\sigma$ and let $M$ be an $R$-module such that $S$ is a quotient of $M$. It follows that $Hom(M, \sigma(S)) \neq 0$, since $S = \sigma(S)$ because $S \in \mathbb{T}_\sigma$. Then, from the ($\sigma$-HH) condition, it follows that $Hom(\sigma(S), M) \neq 0$ , i.e. $Hom(S, M) \neq 0$, whence $S$ embeds in $M$. Therefore, $S$ is paraprojective.

□

**Definition 3.16.** Let $R$ be a ring and $\sigma \in R\text{-}lep$. We will say that an $R$-module $M \neq 0$ is $\sigma$-atomic if any nonzero submodule $N$ of $M$ with $\sigma(N) \neq 0$ contains a submodule $S \in R\text{-}Simp \cap \mathbb{T}_\sigma$, and we will say that $R$ is a left $\sigma$-semiartinian ring if any $R$-module is $\sigma$-atomic.

Recall the following definition which were introduced in [6].

**Definition 3.17.** Let $R$ be a ring and take an idempotent preradical $\sigma$. We say that a non zero $R$-module $M$ is $\sigma$-coatomic if for each non zero quotient $L$ of $M$ with $\sigma(L) \neq 0$, $L$ has a quotient $S$, $S \in R\text{-}Simp \cap \mathbb{T}_\sigma$. We will say that $R$ is a left $\sigma$-max ring, if each $R$-module is $\sigma$-coatomic.

**Theorem 3.18.** *Let $R$ be a ring and let $\sigma$ be a left exact preradical. If each $S$, such that $S \in R\text{-}Simp \cap \mathbb{T}_\sigma$ is parainjective, then $R$ is a left $\sigma$-max ring.*

*Proof.* Let $M \neq 0$ be an $R$-module and $L$ be a non zero quotient of $M$, with $\sigma(L) \neq 0$. Let us consider a cyclic submodule $Rx$ of $\sigma(L)$. As $\sigma$ is left exact, then $\sigma$ is idempotent and $\mathbb{T}_\sigma$ is a hereditary class. So $\sigma(L) \in \mathbb{T}_\sigma$, and thus $Rx \in \mathbb{T}_\sigma$. Now, if we consider a simple quotient of $Rx$, $S$, say, we have that $S \in \mathbb{T}_\sigma$, thus $S \in R\text{-}Simp \cap \mathbb{T}_\sigma$. Taking the inclusion of $S$ in its injective hull $E(S)$, we obtain a non zero morphism from $Rx$ to $E(S)$. This morphism has a non zero extension $f : L \to E(S)$. Note that $S \leq f(L)$ because $S$ is essential in $E(S)$. As $S$ is parainjective, then it is a quotient of $f(L)$ and thus $S$ is a quotient of $L$. See the below diagram.

$$\begin{tikzcd}
M \arrow[r, two heads] & L \arrow[rrdd, dashed, "f"] & & \\
& \sigma(L) \arrow[u, hook] & & \\
& Rx \arrow[u, hook] \arrow[r, two heads] & S \arrow[r, tail] & E(S).
\end{tikzcd}$$

□

EXAMPLE 3.19. Consider the ring $R = \mathbb{Z}_n \times \mathbb{Z}$, which is the product of the ring of integers module $n$ and the ring of integers. Let us define $\sigma = (\overline{1}, 0) \cdot (-)$.

It can be easily observed that $R$ is a left $\mathrm{soc_p}$-max ring, but not a left max ring. This is because divisible Abelian groups do not have proper maximal subgroups. However, it is a left $\mathrm{soc_p}$-paraprojective ring, which means that each projective simple module is parainjective. In fact, a left projective simple $R$-module ${}_RS$ is also a simple projective $\mathbb{Z}_n$-module. Furthermore, since every Abelian group of order dividing $n$ is a direct sum of cyclic modules, it follows that ${}_RS$ is a quotient of each ${}_RM$ that contains one copy of ${}_RS$.

**Theorem 3.20.** *Let $R$ be a ring and let $\sigma$ be a left exact preradical. If each simple module $S$ in $\mathbb{T}_\sigma$ is paraprojective, then $R$ is a left $\sigma$-semiartinian ring.*

*Proof.* Let $M$ be an $R$-module and let $N$ be a non zero submodule of $M$ with $\sigma(N) \neq 0$. Let us take a non zero cyclic submodule $Rx$ of $\sigma(N)$. As $\sigma$ is left exact, then $\sigma(N) \in \mathbb{T}_\sigma$, thus $Rx \in \mathbb{T}_\sigma$.

$$\begin{tikzcd}
\sigma(N) \arrow[r, tail] & N \arrow[r, tail] & M \\
Rx \arrow[u, tail] \arrow[r, two heads] & S &
\end{tikzcd}$$

Now, if $S$ is a simple quotient of $Rx$, then we have that $S \in \mathbb{T}_\sigma$. By hypothesis, $S$ is paraprojective, thus $S$ embeds in $Rx$ and from this, $S$ embeds in $M$. □

**Corollary 3.21.** *Let $R$ be a ring and let $\sigma \in R$ be a left exact preradical. If $R$ is a ring satisfying the* ($\sigma$-HH)*-condition, then $R$ is a left $\sigma$-max and a left $\sigma$-semiartinian ring.*

*Proof.* It follows from (2) and (3) of Remark 3.15 and from Theorems 3.18 and 3.20. □

**Theorem 3.22.** *Let $R$ be a ring and let $\sigma$ be a left exact preradical. The following statements are equivalent:*

(1) *$R$ has the* ($\sigma$-HH)*-condition.*
(2) *$R$ is a $\sigma$-* (R-*Mod*)*-retractable ring and each $S \in R$-Simp $\cap\, \mathbb{T}_\sigma$ is paraprojective.*
(3) *Any $S \in R$-Simp $\cap\, \mathbb{T}_\sigma$ is paraprojective and parainjective.*

*Proof.* (1) ⇒ (2) It follows from (1) and (3) of Remark 3.15.
(2) ⇒ (3) This is clear.
(3) ⇒ (1) Let $M$ and $N$ be two $R$-modules such that $\sigma(M), \sigma(N) \neq 0$ and let us suppose that $f \in Hom\,(\sigma(N), M) \neq 0$. We have the following diagram

$$\begin{tikzcd}
\sigma(N) \arrow[r, two heads, "f\restriction"] & f\left(\sigma(N)\right) \arrow[r, tail] & M \arrow[dl, dashed, two heads] \\
f^{-1}(S) \arrow[u, tail] \arrow[r, two heads, "f\restriction"] & S \arrow[u, tail] &
\end{tikzcd}$$

As $\sigma$ is left exact, we have that $\sigma(N) \in \mathbb{T}_\sigma$, and thus $0 \neq f\left(\sigma(N)\right) \in \mathbb{T}_\sigma$. Now, from Theorem 3.20, there exists $S \in R$-*Simp* $\cap\, \mathbb{T}_\sigma$, a submodule of $f\left(\sigma(N)\right)$, which is parainjective. Thus $S$ is a quotient of $M$. Besides, $S$ is a quotient of $f^{-1}(S)$, hence $S$ embeds in $f^{-1}(S)$, because $S$ is paraprojective. Thus we have that $Hom\,(M, \sigma(N)) \neq 0$.

Let us now see that if $Hom\,(M, \sigma(N)) \neq 0$, then $Hom\,(\sigma(N), M) \neq 0$. To do so, let us consider $0 \neq f \in Hom\,(M, \sigma(N))$. Given that $\sigma \in R$-lep and $0 \neq f(M) \leq \sigma(N)$, then $f(M) \in \mathbb{T}_\sigma$, i.e., $\sigma\,(f(M)) = f(M) \neq 0$. Thus, by Theorem 3.20, there exists $S \in R\text{-Simp} \cap \mathbb{T}_\sigma$ such that $S$ embeds in $\sigma\,(f(M))$.

$$\begin{tikzcd}
M \arrow[r, two heads, "f\restriction"] & f\,(M) \arrow[r, tail] & \sigma(N) \arrow[dl, dashed, two heads] \\
f^{-1}(S) \arrow[u, tail] \arrow[r, two heads, "f\restriction"] & S \arrow[u, tail] &
\end{tikzcd}$$

Then, taking into account that every $\sigma$-torsion simple module is parainjective, we have that is a quotient of $\sigma(N)$. Finally, using the fact that every $\sigma$-torsion simple module is paraprojective, we have that $S$ embeds in $f^{-1}(S)$,Thus we obtain a nonzero morphism from $\sigma(N)$ to $M$. □

A ring $R$ is called a ***BKN*-ring** if between any two nonzero modules there exists a nonzero morphism. Such rings are introduced and characterized in [4]. In the following, we generalize these rings.

**Definition 3.23.** Let $R$ be a ring and let $\sigma \in R$-pr. We say that $R$ is a **$\sigma$- (BKN)-ring** if for any $R$-modules $M$ and $N$ with $\sigma(M), \sigma(N) \neq 0$, it holds that
$$Hom\,(\sigma(N), \sigma(M)) \neq 0.$$

Recall that a ring $R$ is called **left local ring** if there is only one isomorphism type of simple $R$-modules. In the following definition we generalize this concept.

**Definition 3.24.** Let $R$ be a ring and $\sigma \in R$-pr. We say that a ring $R$ is **left $\sigma$-local ring** if
$$|R\text{-Simp} \cap \mathbb{T}_\sigma| = 1.$$

**Theorem 3.25.** *Let $R$ be a ring and let $\sigma$ be an exact and costable preradical. The following statements are equivalent:*

(1) *$R$ is a $\sigma$-* (BKN)*-ring.*
(2) *$R$ satisfies the* ($\sigma$-HH)*-condition and it is a left $\sigma$-local ring.*

*Proof.* (1) $\Rightarrow$ (2) Let $M$ and $N$ be two $R$-modules such that $\sigma(M), \sigma(N) \neq 0$ and with $Hom\,(\sigma(N), M) \neq 0$. As $R$ is a $\sigma$- (BKN)-ring, we have that $Hom(\sigma(M), \sigma(N)) \neq 0$. As $\sigma$ is exact and costable, then $\sigma(M)$ is a quotient of $M$. Hence $Hom(M, \sigma(N)) \neq 0$.

Now, if $S_1, S_2 \in R\text{-Simp} \cap \mathbb{T}_\sigma$, we have that $Hom\,(\sigma(S_1), \sigma(S_2)) \neq 0$, thus $Hom(S_1, S_2) \neq 0$. Then $S_1 \cong S_2$. Hence $R$ is a left $\sigma$-local ring .

(2) $\Rightarrow$ (1) Let $M$ and $N$ be two $R$-modules such that $\sigma(M), \sigma(N) \neq 0$. From Corollary 3.21, we have that $R$ is $\sigma$-semiartinian, so there exist $S_1, S_2 \in R\text{-}Simp \cap \mathbb{T}_\sigma$ such that $S_1$ embeds in $\sigma(M)$ and $S_2$ embeds in $\sigma(N)$. As $R$ satisfies the ($\sigma$-HH) condition, it follows that $Hom\,(\sigma(M), S_1) \neq 0$. Finally, as $R$ is a left $\sigma$-local ring, we see that $S_1 \cong S_2$ and from this, we get $Hom\,(\sigma(M), \sigma(N)) \neq 0$. $\square$

The following definition appears in [3, Definition 22].

**Definition 3.26.** A module class $\mathcal{C}$ satisfies the condition *(CN)* if whenever each nonzero quotient of an arbitrary module $M$ shares a nonzero quotient with some element of $\mathcal{C}$, it happens that $M$ belongs to $\mathcal{C}$.

In the following definition we generalize the ($CN$) condition.

**Definition 3.27.** Assume $\mathcal{C} \subseteq R$-Mod. $\mathcal{C}$ satisfies the **($\boldsymbol{\sigma}$-$\boldsymbol{CN}$)-condition** if

$$\left(\begin{array}{c}\text{for each epimorphism } \; M \twoheadrightarrow L \text{ with } \sigma(L) \neq 0,\\ \text{there exist } C \in \mathcal{C}, N \in R\text{-Mod}, \text{ with } \sigma(N) \neq 0,\\ \text{and epimorphims } \; L \twoheadrightarrow N \twoheadleftarrow C\end{array}\right) \Longrightarrow M \in \mathcal{C}.$$

**Lemma 3.28.** *Let $\mathcal{C}$ be the class of $R$-modules satisfying the ($\sigma$-$CN$)-condition and let $C \in \mathcal{C}$. If $L$ is a quotient of $C$ with $\sigma(L) \neq 0$, then $L \in \mathcal{C}$.*

*Proof.* Let $H$ be a quotient of $L$ with $\sigma(H) \neq 0$. Note that $H$ is also a quotient of $C$. As $\mathcal{C}$ satisfies ($\sigma$-$CN$), it follows that $L \in \mathcal{C}$. $\square$

The following theorem is a generalization of [3, Theorem 22].

**Theorem 3.29.** *Let $\sigma$-pr be an exact and costable preradical and $\mathcal{C} \subseteq R$-Mod. The following statements are equivalent for $\mathcal{C}$:*

(1) $\mathcal{C} \in \sigma$-(R-*conat*).
(2) $\mathcal{C}$ *satisfies* ($\sigma$-$CN$).
(3) $\mathcal{C} \in \mathcal{L}_{/_\sigma}$ *and* $\mathcal{C} = \left(\mathcal{C}^{\perp_{/\sigma}}\right)^{\perp_{/\sigma}}$.

*Proof.* (1) $\Rightarrow$ (2) Let $\mathcal{C} \in \sigma$-($R$-conat) and $M$ be an $R$-module. As $\mathcal{C} \in \sigma$-($R$-conat), then $\mathcal{C} = \mathcal{A}^{\perp_{/\sigma}}$, for some $\mathcal{A} \in \mathcal{L}_{/_\sigma}$, thus by [5, Proposition 11], we have

$$\mathcal{A}^{\perp_{/\sigma}} = \{M \in R\text{-Mod} \mid \forall\; M \twoheadrightarrow L,\; \sigma(L) \neq 0 \Rightarrow \sigma(L) \notin \mathcal{A}\} \cup \mathbb{F}_\sigma.$$

Note that if for each non zero quotient $L$ of $M$, with $\sigma(L) \neq 0$ it holds that $\sigma(L) \notin \mathcal{A}$, then $M \in \mathcal{A}^{\perp_{/\sigma}}$, $M \in \mathcal{C}$. Let us assume that $L$ is a non zero quotient of $M$, with $\sigma(L) \neq 0$, and such that $\sigma(L) \in \mathcal{A}$ and that there exist $C \in \mathcal{C}$ and $N \in R$-Mod, with $\sigma(N) \neq 0$, with epimorphisms $L \twoheadrightarrow N \twoheadleftarrow C$. As $C \in \mathcal{C}$ and $C \twoheadrightarrow N$, is an epimorphism with $\sigma(N) \neq 0$, we get that $\sigma(N) \notin \mathcal{A}$.

Besides for the epimorphism $f : L \twoheadrightarrow N$, as $\sigma$ is exact, then $\sigma$ is idempotent and cohereditary, thus $\sigma(N) = \sigma\,(\sigma(N))$ and $f_{\downharpoonleft} : \sigma(L) \twoheadrightarrow \sigma(N)$ is also an epimorphism. As $\mathcal{A} \in \mathcal{L}_{/\sigma}$ it follows that $\sigma(N) \in \mathcal{A}$, a contradiction. Thus $\sigma(L) \notin \mathcal{A}$ and hence $M \in \mathcal{C}$.

(2) $\Rightarrow$ (3) Let $\mathcal{C}$ be a class of $R$-modules satisfying ($\sigma$-$CN$). Firstly, we show that $\mathcal{C} \in \mathcal{L}_{/\sigma}$. For this, let us consider $C \in \mathcal{C}$ and let $L \in R$-Mod be a quotient of $M$ with $\sigma(L) \neq 0$. As $\sigma$ is an exact costable preradical, we have that

$$C = \sigma(R)C \oplus C' = \sigma(C) \oplus C', \tag{3.1}$$

where $C' = \{c \in C \mid \sigma(R)c = 0\}$. It follows that $\sigma(C)$ is a quotient of $C$. Then, by Lemma 3.28, $\sigma(L) \in \mathcal{C}$.

Now, to show that $\mathcal{C} = \left(\mathcal{C}^{\perp/\sigma}\right)^{\perp/\sigma}$, let us recall the following description of the double pseudocomplement of $\mathcal{C}$,

$$\begin{aligned}\left(\mathcal{C}^{\perp/\sigma}\right)^{\perp/\sigma} = \{&M \in R\text{-Mod} \mid \forall M \twoheadrightarrow K,\ \sigma(K) \neq 0 \\ &\Rightarrow \exists \sigma(K) \twoheadrightarrow L,\ \sigma(L) \neq 0 \text{ and } \sigma(L) \in \mathcal{C}\} \cup \mathbb{F}_\sigma,\end{aligned} \tag{3.2}$$

see [5, Definition 4]. So, if $C \in \mathcal{C}$ and $K$ is a quotient of $C$ with $\sigma(K) \neq 0$, then as $\sigma$ is idempotent and cohereditary, we get an epimorphism

$$\sigma(C) \twoheadrightarrow \sigma(K).$$

In the same way that in (3.1), $\sigma(C)$ is a quotient of $C$, thus from Lemma 3.28 $\sigma(C) \in \mathcal{C}$. Applying Lemma 3.28 one more time, we obtain that $\sigma(K) \in \mathcal{C}$. From (3.2) we get that $C \in \left(\mathcal{C}^{\perp/\sigma}\right)^{\perp/\sigma}$. This shows that $\mathcal{C} \subseteq \left(\mathcal{C}^{\perp/\sigma}\right)^{\perp/\sigma}$.

For the other inclusion, let us consider $M \in \left(\mathcal{C}^{\perp/\sigma}\right)^{\perp/\sigma}$ and a quotient $L$ of $M$ with $\sigma(L) \neq 0$. As $\sigma(L)$ is a quotient of $L$, then $\sigma(L)$ is a quotient of $M$. Also $\sigma(\sigma(L)) = \sigma(L) \neq 0$. So as $M \in \left(\mathcal{C}^{\perp/\sigma}\right)^{\perp/\sigma}$, there exists a quotient $T$ of $\sigma(L)$ with $\sigma(T) \neq 0$ and $\sigma(T) \in \mathcal{C}$. Finally, as $\mathcal{C}$ satisfies ($\sigma$-$CN$), we get that $M \in \mathcal{C}$. So, $\left(\mathcal{C}^{\perp/\sigma}\right)^{\perp/\sigma} \subseteq \mathcal{C}$, hence $\mathcal{C} = \left(\mathcal{C}^{\perp/\sigma}\right)^{\perp/\sigma}$.

(3) $\Rightarrow$ (1) This is clear. □

*Remark* 3.30. Let $\sigma$ be an exact and costable preradical and let $\mathcal{C} \subseteq R$-Mod. If $\xi_{\sigma\text{-conat}}(\mathcal{C})$ denotes the least $\sigma$-conatural class containing $\mathcal{C}$, then

$$\begin{aligned}\xi_{\sigma\text{-conat}}(\mathcal{C}) = \{&M \in R\text{-Mod} \mid \forall M \twoheadrightarrow L,\ \sigma(L) \neq 0 \text{ there exist } C \in \mathcal{C}, \\ &N \in R\text{-Mod}, \text{with } \sigma(N) \neq 0, \text{ such that } L \twoheadrightarrow N \twoheadleftarrow C\} \cup \mathbb{F}_\sigma\end{aligned}$$

In [3, Theorem 42] left max rings are characterized via conatural classes. The following theorem generalizes this result.

**Proposition 3.31.** *Let $\sigma$ be an exact and costable preradical. The following statements are equivalent:*

(1) *$R$ is a left $\sigma$-max ring.*
(2) *Each $\sigma$-conatural class is generated by a family of simple $\sigma$-torsion modules.*

*Proof.* (1) $\Rightarrow$ (2) Let $\mathcal{C}$ be a non trivial $\sigma$-conatural class and let $\mathcal{S} \subseteq \mathcal{C}$ be the class of all simple $\sigma$-torsion modules in $\mathcal{C}$. Notice that $\mathcal{S}$ is non empty because $R$ is a $\sigma$-max ring. Now, if $0 \neq M \in \mathcal{C}$ and $L$ is a non zero quotient of $M$ with $\sigma(L) \neq 0$, as $R$ is $\sigma$-max, then $L$ has a simple quotient $S \in R\text{-Simp} \cap \mathbb{T}_\sigma$. By Lemma 3.28, $S \in \mathcal{C}$, by Remark 3.30, we have that $M \in \xi_{\sigma\text{-conat}}(\mathcal{S})$. It follows that $\mathcal{C} = \xi_{\sigma\text{-conat}}(\mathcal{S})$.

(2) $\Rightarrow$ (1) Let $M \in R$-Mod and $L$ be a non zero quotient of $M$ with $\sigma(L) \neq 0$. By hypothesis, there exists $\mathcal{S} \subseteq R\text{-Simp}\cap\mathbb{T}_\sigma$ such that $\xi_{\sigma\text{-conat}}(M) = \xi_{\sigma\text{-conat}}(\mathcal{S})$. As $L \in \xi_{\sigma\text{-conat}}(M)$, $L$ has a quotient $S \in \mathcal{S}$, Thus $R$ is a left $\sigma$-max ring. $\square$

The following theorem generalizes [1, Proposition 2.11].

**Theorem 3.32.** *Let $\sigma$ be an exact and costable preradical. The following statements are equivalent:*

(1) *Each $S \in R\text{-}Simp \cap \mathbb{T}_\sigma$ is parainjective.*
(2) $\sigma$-(R-*conat*) $\subseteq \sigma$-(R-*tors*).
(3) $\sigma$-(R-*conat*) $\subseteq \mathcal{L}_{\leq_\sigma}$.

*Proof.* (1) $\Rightarrow$ (2) In [6, Proposition 6.5] it is shown that for an exact and costable preradical $\sigma$, and for a left $\sigma$-max ring, each $\sigma$-conatural class is closed under taking direct sums. Now, by hypothesis, each $S \in R\text{-Simp} \cap \mathbb{T}_\sigma$ is parainjective, from Theorem 3.18, we obtain that $R$ is a left $\sigma$-max ring. Thus it suffices to show that any $\sigma$-conatural class is a $\sigma$-hereditary class.

For this, let us consider $\mathcal{C} \in \sigma$-(R-conat), $M \in \mathcal{C}$ and $N \leq M$ with $\sigma(N) \neq 0$. Let $S \in \text{Simp}\cap\mathbb{T}_\sigma$ be a quotient of $\sigma(N)$, which exists because $R$ is a left $\sigma$-max ring. If $T$ be the push-out of $\sigma(N) \twoheadrightarrow S$ and $\sigma(N) \hookrightarrow M$, as in the following diagram

$$\begin{tikzcd} \sigma(N) \arrow[r, tail] \arrow[d, two heads] & N \arrow[r, tail] & M \arrow[d, two heads, dashed] \\ S \arrow[rr, tail, dashed] & & T, \end{tikzcd}$$

then $S$ embeds in $T$. Thus, as $S$ is parainjective by hypothesis, then $S$ is a simple $\sigma$-torsion quotient of $T$. Then $S$ is also a simple $\sigma$-torsion quotient of $M$. Then any simple $\sigma$-torsion quotient of $\sigma(N)$ belongs to $\mathcal{C}$. Thus, by Proposition 3.31, $\sigma(N) \in \mathcal{C}$. Hence $\mathcal{C}$ is a $\sigma$-hereditary class.

(2) $\Rightarrow$ (3) This is clear.

(3) $\Rightarrow$ (1) Let $S \in \text{Simp}\cap\mathbb{T}_\sigma$ and let $M$ be an $R$-module such that $S$ embeds in $M$. By hypothesis, we have that $S \in \xi_{\sigma\text{-conat}}(M)$. From Remark 3.30 it follows that $S$ is a quotient of $M$, then $S$ is parainjective. $\square$

*Remark* 3.33. If $\sigma$ is an exact and costable preradical, then for each $R$-module $M$, we have a decomposition $M = \sigma(M) \oplus M'$, where $M' = \{m \in M \mid \sigma(R)m =$

$0\}$. Notice than $\sigma$ has a complement $\sigma'$, in $R$-pr satisfying

$$\sigma'(M) = M',$$

for each $M \in R$-Mod.

**Theorem 3.34.** *Let $\sigma$ be an exact and costable preradical. If $R$ satisfies* ($\sigma$-HH) *and* ($\sigma'$-$HH$)*, then $R$ satisfies* ($HH$).

*Proof.* Let $M$ and $N$ be two $R$-modules and suppose that $Hom(N, M) \neq 0$. Let us take a non zero $f \in Hom(N, M)$. If $\nu : M = \sigma(M) \oplus \sigma'(M) \to \sigma(M)$ denotes the natural projection on $\sigma(M)$ and $\nu f \neq 0$, then $Hom(N, \sigma(M)) \neq 0$. So, by ($\sigma$-HH),there exists a non zero morphism $g : \sigma(M) \to N$, thus $g\nu : M \to N$ is a non zero morphism, $Hom(M, N) \neq 0$.

Now, if $\nu f = 0$, then taking the co-restriction of $f$ to its image, we have that $f \restriction: N \to \sigma'(M)$ is a non zero morphism. From condition ($\sigma'$-$HH$), there exists a non zero morphism $h : \sigma'(M) \to N$. Finally, $h\nu' : M \to N$ is non zero morphism, where $\nu' : M \to \sigma'(M)$ is the natural projection over $\sigma'(M)$. Thus, $Hom(M, N) \neq 0$. □

In [8, Theorem 1.13] it is shown that if $R$ satisfies condition ($HH$), then each torsion class is a conatural class. The following corollary is a consequence.

**Corollary 3.35.** *Let $\sigma \in R$-pr be exact and costable. If $R$ satisfies condition* ($\sigma$-HH) *and condition* ($\sigma'$-$HH$)*, then each torsion class is a conatural class.*

**Theorem 3.36.** *Let $\sigma \in R$-pr be exact and costable. If $R$ satisfies condition* ($\sigma$-HH) *and condition* ($\sigma'$-$HH$)*, then*

$$\sigma\text{-}(\text{R-}TORS) \subseteq \sigma\text{-}(\text{R-}conat)\,.$$

*Proof.* Let $\mathcal{C} \in \sigma$-(R-TORS). As $\sigma$ is exact, there exists $\mathcal{D} \in R$-TORS such that $\mathcal{C} = \overleftarrow{\sigma}(\mathcal{D})$, see [6, Theorem 3.13]. Now, from Corollary 3.35, we have that $\mathcal{D} \in R$-conat. Then, as $\sigma$ is an exact and costable preradical, we have that $\overleftarrow{\sigma}(\mathcal{D}) \in R$-($\sigma$-conat), see [5, Proposition 15], i.e., $\mathcal{C} \in \sigma$-(R-conat). □

**Theorem 3.37.** *Let $\sigma \in R$-pr be exact and costable. If $R$ satisfies condition* ($\sigma$-HH) *and* ($\sigma'$-$HH$)*, then*

$$\sigma\text{-}(\text{R-}TORS) = \sigma\text{-}(\text{R-}conat) = \sigma\text{-}(\text{R-}tors)\,.$$

*Proof.* It follows from Theorem 3.36, from Remark 3.15, from Theorem 3.32 and from the fact that $\sigma$-(R-tors) $\subseteq \sigma$-(R-TORS). □

**Lemma 3.38.** *Let $\sigma$ be an exact and costable preradical, let $R$ be a ring with ($\sigma$-$HH$), $\mathcal{C} \in \sigma$-(R-$TORS$) and $M \in R$-Mod. If $M$ has a quotient $L$ with $L \in \mathcal{C}$ and $\sigma(L) \neq 0$, then $M$ has a non zero submodule $N$, such that $N \in \mathcal{C} \cap \mathbb{T}_\sigma$. Moreover, there exist a largest submodule $K$ of $M$ with the property that $K \in \mathcal{C} \cap \mathbb{T}_\sigma$.*

*Proof.* As $\sigma$ is an exact and costable preradical, then $\sigma(L)$ is a quotient of $L$. Now, as $\mathcal{C}$ is a $\sigma$-torsion class and $\sigma(L) \neq 0$, it follows that $\sigma(L) \in \mathcal{C}$.

As $\sigma(L)$ is a nonzero quotient of $M$, then $Hom(M, \sigma(L)) \neq 0$. Thus, from the ($\sigma$-$HH$)-condition, there exists an morphism $0 \neq h : \sigma(L) \to M$. Let $N = h(\sigma(L))$, and note that $N$ is a nonzero submodule of $M$. Moreover, as $N$ is a quotient of $\sigma(L)$, then $\sigma(N)$ is a quotient of $\sigma(\sigma(L))$, because $\sigma$ is cohereditary, as $\sigma(\sigma(L)) = \sigma(L)$, then $\sigma(N) = N$ with $\sigma(N) \in \mathcal{C}$. Hence, $N \in \mathcal{C} \cap \mathbb{T}_\sigma$.

Now, as $\mathcal{C}$ is a class closed under taking direct sums, then $\mathcal{C}_M = \{N \leq M \mid 0 \neq N \in \mathcal{C} \cap \mathbb{T}_\sigma\} \neq \emptyset$ and as $\sigma$ is left exact, we get that

$$\bigoplus_{C \in \mathcal{C}_M} C \in \mathcal{C} \cap \mathbb{T}_\sigma.$$

As $\sum_{C \in \mathcal{C}_M} C$ is a quotient of $\bigoplus_{C \in \mathcal{C}_M} C$ and as $\sigma$ is a cohereditary preradical, it follows that $\sum_{C \in \mathcal{C}_M} C \in \mathcal{C} \cap \mathbb{T}_\sigma$. Hence, $K = \sum_{C \in \mathcal{C}_M} C$ is the largest $\sigma$-torsion submodule of $M$ belonging to $\mathcal{C}$. □

**Lemma 3.39.** *Let $\sigma$ be an exact and costable preradical, let $R$ be a ring with the* ($\sigma$-HH)*-condition and let $\mathcal{C} \subseteq R$-Mod. If $\mathcal{C} \in \sigma$-*(R-*TORS*)*, then $\sigma^*(\mathcal{C}) \in R$-conat.*

*Proof.* Let us show that $\sigma^*(C)$ has $(CN)$. For this, let us take an $R$-module $M$ such that for any non zero quotient $N$ there exist $C \in \mathcal{C}$ and $L \in R$-Mod as in the following diagram:

$$M \twoheadrightarrow N \twoheadrightarrow L \twoheadleftarrow \sigma(C)$$

and we will see that $M \in \sigma^*(\mathcal{C})$.

In this situation, $\sigma(C) \in \mathcal{C}$, because $\sigma(C)$ is a quotient of $C$, as $\sigma$ is exact and costable, and $\mathcal{C}$ is a cohereditary class. Now, note that, $L = \sigma(L)$ because $\sigma$ is cohereditary and $\sigma(C) = \sigma(\sigma(C))$, thus $L \in \mathcal{C}$. So, by Lemma 3.38, we consider the largest submodule, $K$ of $M$ such that $K \in \mathcal{C} \cap \mathbb{T}_\sigma$. Note that $K \neq 0$ and that if $K = M$, then we are done, because $M = \sigma(K)$ with $K \in \mathcal{C}$, that is, $M \in \sigma^*(\mathcal{C})$. Assume to the contrary that $K \lneq M$, then $M/K \neq_R 0$. By hypothesis, $M/K$ also has a non zero submodule $U \in \mathcal{C} \cap \mathbb{T}_\sigma$, as in the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, tail] \arrow[d, equal] & M \arrow[r, two heads, "\pi"] & M/K \arrow[r] & 0 \\
0 \arrow[r] & K \arrow[r, tail] & \pi^{-1}(U) \arrow[r, two heads] \arrow[u] & U \arrow[r] \arrow[u] & 0.
\end{tikzcd}$$

But as $\mathcal{C} \cap \mathbb{T}_\sigma$ is a class closed under extensions, then $\pi^{-1}(U)$ is a submodule of $M$ in $\mathcal{C} \cap \mathbb{T}_\sigma$, properly containing $K$, a contradiction. Then $K = M$. □

Let us recall that for any class $\mathcal{D} \subseteq R$-Mod, we have that $\overleftarrow{\sigma}(\mathcal{D}) = \overleftarrow{\sigma}\sigma^*\overleftarrow{\sigma}(\mathcal{D})$; and that if $\sigma$ is an exact preradical, then a class $\mathcal{C}$ is $\sigma$-torsion if and only if $\mathcal{C} = \overleftarrow{\sigma}(\mathcal{D})$, for a torsion class $\mathcal{D}$. Besides, if $\sigma$ is an exact and costable preradical and $\mathcal{D}$ is a conatural class, then $\overleftarrow{\sigma}(\mathcal{D})$ is a $\sigma$-conatural class, see [5] and [6].

**Theorem 3.40.** *Let $\sigma$ be exact and costable. If $R$ satisfies the* ($\sigma$-HH) *condition, then*

$$\sigma\text{-}(\text{R-}\mathit{TORS}) \subseteq \sigma\text{-}(\text{R-}\mathit{conat})\,.$$

*Proof.* Let $\mathcal{C} \in \sigma$-(R-TORS). There exists a torsion class $\mathcal{D}$ such that $\mathcal{C} = \overleftarrow{\sigma}(\mathcal{D})$. Thus, we have,

$$\mathcal{C} = \overleftarrow{\sigma}(\mathcal{D}) = \overleftarrow{\sigma}\sigma^*\overleftarrow{\sigma}(\mathcal{D}).$$

It suffices to show that $\sigma^*\overleftarrow{\sigma}(\mathcal{D})$ is a conatural class. To see this, by Lemma 3.39 it suffices to show that $\overleftarrow{\sigma}(\mathcal{D})$ is a $\sigma$-torsion class. But this holds by hypothesis. □

**Theorem 3.41.** *Let $\sigma$ be an exact and costable preradical. If $R$ satisfies the* ($\sigma$-HH) *condition, then*

$$\sigma\text{-}(\text{R-}\mathit{TORS}) = \sigma\text{-}(\text{R-}\mathit{conat}) = \sigma\text{-}(\text{R-}\mathit{tors})\,.$$

*Proof.* It follows from Theorem 3.32, from Theorem 3.40 and the fact that $\sigma$-(R-tors) $\subseteq \sigma$-(R-TORS). □

The following definition is dual to Definition 3.27.

**Definition 3.42.** Let $\mathcal{C} \subseteq R$-Mod. We will say that $\mathcal{C}$ satisfies **condition ($\boldsymbol{\sigma}$-$\boldsymbol{N}$)** if

$$\left(\begin{array}{c} \text{for each monomorphism } L \rightarrowtail M \text{ with } \sigma(L) \neq 0, \\ \text{there exist } C \in \mathcal{C}, N \in R\text{-Mod}, \text{ with } \sigma(N) \neq 0 \\ \text{and monomorphisms } L \leftarrowtail N \rightarrowtail C \end{array}\right) \Longrightarrow M \in \mathcal{C}.$$

**Lemma 3.43.** *Let $\mathcal{C}$ be a class of $R$-modules satisfying the* ($\sigma$-$N$) *condition and let $C \in \mathcal{C}$. If $L$ embeds in $C$, with $\sigma(L) \neq 0$, then $L \in \mathcal{C}$.*

*Proof.* Let $K$ be an $R$-module embedded in $L$, with $\sigma(K) \neq 0$. Then $K$ embeds in $C$ and as $C \in \mathcal{C}$, we conclude that $K \in \mathcal{C}$. □

*Remark* 3.44. In [5, Corollary 2] it is shown that

$$\sigma\text{-}\left(R\text{-nat}\right) = \mathcal{L}_{\{\leq_\sigma, \oplus, \sigma(E()), ext\}}.$$

**Theorem 3.45.** *Let $\mathcal{C} \subseteq R$-Mod and let $\sigma \in R$-pr be a left exact and stable preradical. Then $\mathcal{C} \in \sigma$-(R-nat) if and only if $\mathcal{C}$ satisfies ($\sigma$-$N$).*

*Proof.* Suppose that $\mathcal{C} \subseteq R$-Mod is a $\sigma$-natural class. By definition of $\sigma$- ($R$-*nat*), one has that $\mathcal{C} = \mathcal{A}^{\perp_{\leq_\sigma}}$, for some class $\mathcal{A} \in \mathcal{L}_{\leq_\sigma}$. Then, by [5, Lemma 1], we have that

$$\mathcal{A}^{\perp/\sigma} = \{M \in R\text{-Mod} \mid \forall\ L \rightarrowtail M,\ \sigma(L) \in \mathcal{A} \Rightarrow L \in \mathbb{F}_\sigma\}.$$

Let $M$ be an $R$-module. Note that, if for any $L \rightarrowtail M$ it holds that $\sigma(L) \in \mathcal{A} \Rightarrow L \in \mathbb{F}_\sigma$, then $M \in \mathcal{A}^{\perp_{\leq_\sigma}} = \mathcal{C}$. Suppose then that there exists $L \rightarrowtail M$ with $\sigma(L) \in \mathcal{A}$, but that $L \notin \mathbb{F}_\sigma$. Additionally, suppose that there exists $N \in R$-Mod, with $\sigma(N) \neq 0$ and $C \in \mathcal{C}$ such that $C \leftarrowtail N \rightarrowtail L$.

Note that $\sigma(N) \in \mathcal{C}$, since $\mathcal{C} \in \mathcal{L}_{\leq_\sigma}$ and $C \in \mathcal{C}$. It follows that $\sigma(N) \notin \mathcal{A}$, since otherwise $N \in \mathbb{F}_\sigma$.

On the other hand, since $\sigma$ is left exact, and hence it is idempotent, it follows that. $\sigma(N) = \sigma(\sigma(N)) \rightarrowtail \sigma(\sigma(L)) = \sigma(L)$. Finally, since $\mathcal{A} \in \mathcal{L}_{\leq_\sigma}$ and $\sigma(L) \in \mathcal{A}$ it follows that $\sigma(N) \in \mathcal{A}$, which is a contradiction. Thus, there does not exist $L \rightarrowtail M$ with $\sigma(L) \in \mathcal{A}$ and such that $L \notin \mathbb{F}_\sigma$. Therefore $M \in \mathcal{C}$.

Suppose now that $\mathcal{C} \subseteq R$-Mod is a class satisfying condition ($\sigma$-$N$) and let us see that, indeed, $\mathcal{C}$ is a $\sigma$-natural class.

Let us begin by showing that $\mathcal{C} \in \mathcal{L}_{\leq_\sigma}$. Let $M \in \mathcal{C}$ and $N \leq M$. Note that if $\sigma(N) = 0$, then $N \in \mathcal{C}$, since $\mathbb{F}_\sigma \subseteq \mathcal{C}$. Suppose then that $\sigma(N) \neq 0$ and let $L \rightarrowtail \sigma(N)$, with $\sigma(L) \neq 0$. Since $L \rightarrowtail M$, there exist $H \in R$-Mod, with $\sigma(H) \neq 0$, and $C \in \mathcal{C}$ such that $L \leftarrowtail H \rightarrowtail C$.

Now let us show that $\mathcal{C} \in \mathcal{L}_\oplus$. Let $\{M_i\}_{i\in I} \subseteq \mathcal{C}$ and $L \rightarrowtail \oplus_{i\in I} M_i$, with $\sigma(L) \neq 0$. Since $\sigma$ is left exact, one has that $\sigma(L) \rightarrowtail \sigma(\oplus_{i\in I} M_i) = \oplus_{i\in I}\sigma(M_i)$. Now, by the projection argument, we have that there exist $l \in \sigma(L)$ and $0 \neq m_i \in \sigma(M_i)$, for some $i \in I$, such that $Rl \cong Rm_i$. Note also that $\sigma$ is idempotent and $\mathbb{T}_\sigma$ is a hereditary class, since $\sigma$ is left exact. It follows that $\sigma(Rl) = Rl \neq 0$, since $Rl \cong Rm_i \leq \sigma(M_i)$ and $\sigma(M_i) \in \mathbb{T}_\sigma$. Now, since $M_i \in \mathcal{C}$ and $Rm_i \rightarrowtail M_i$, with $\sigma(Rm_i) \neq 0$, then there exist $C \in \mathcal{C}$ and $N \in R$-Mod, with $\sigma(N) \neq 0$, such that

$$L \longleftarrowtail Rl \cong Rm_i \longleftarrowtail N \rightarrowtail C.$$

That is, $\oplus_{i\in I} M_i \in \mathcal{C}$.

Now, let us see that $\mathcal{C} \in \mathcal{L}_{\sigma(E())}$. Let $M \in \mathcal{C}$ and $L \rightarrowtail \sigma(E(M))$. Since $\sigma$ is left exact and stable, one has that $\sigma(E(M)) = E(\sigma(M))$. Now, given that $\sigma(M) \leq_e E(\sigma(M))$ and $\sigma(L) \neq 0$, it follows that $\sigma(L) \cap \sigma(M) \neq 0$. But, note that $\sigma(\sigma(L) \cap \sigma(M)) = \sigma(E(M)) \cap \sigma(L) \cap \sigma(M) = \sigma(L) \cap \sigma(M) \neq 0$, since $\sigma$ is left exact. Therefore, exist $C \in \mathcal{C}$ and $N \in R$-Mod, with $\sigma(N) \neq 0$, such that

$$L \longleftarrowtail \sigma(L) \cap \sigma(M) \longleftarrowtail N \rightarrowtail C.$$

Whereupon, $\sigma(E(M)) \in \mathcal{C}$.

Finally, let us see that $\mathcal{C} \in \mathcal{L}_{ext}$. Let

$$0 \longrightarrow M' \overset{f}{\rightarrowtail} M \overset{g}{\twoheadrightarrow} M'' \longrightarrow 0$$

be an exact sequence, with $M', M'' \in \mathcal{C}$, and $L \rightarrowtail M$ with $\sigma(L) \neq 0$. Without loss of generality, suppose that $f$ is the inclusion and that $L \leq M$. Note that, if $\sigma(L) \cap M' \neq 0$, then $\sigma(\sigma(L) \cap M') = \sigma(L) \cap \sigma(L) \cap M' = \sigma(L) \cap M' \neq 0$, since $\sigma$ is left exact. In this case, we have that there exists $C \in \mathcal{C}$ and $N \in R$-Mod, such that $L \leftarrowtail \sigma(L) \cap M' \leftarrowtail N \rightarrowtail C$, since $M' \in \mathcal{C}$.

Now, if $\sigma(L) \cap M' = 0$, then $g\rfloor : \sigma(L) \rightarrowtail M''$ is a monomorphism, whence , there exist $C \in \mathcal{C}$ and $N \in R$-Mod, such that $L \leftarrowtail \sigma(L) \cong g(\sigma(L)) \leftarrowtail N \rightarrowtail C$, since $M'' \in \mathcal{C}$. □

*Remark* 3.46. Let $\sigma \in R$-pr be left exact and stable, and $\mathcal{C} \subseteq R$-Mod. If $\xi_{\sigma\text{-nat}}(\mathcal{C})$ denotes the smallest class $\sigma$-natural containing $\mathcal{C}$, then

$$\begin{aligned}\xi_{\sigma\text{-nat}}(\mathcal{C}) = \{M \in R\text{-Mod} \mid \forall L \rightarrowtail M,\ \sigma(L) \neq 0 \text{ there exist } C \in \mathcal{C}, N \in R\text{-Mod}, \\ \text{con } \sigma(N) \neq 0,\ \text{such that } L \leftarrowtail N \rightarrowtail C\} \cup \mathbb{F}_\sigma.\end{aligned}$$

**Proposition 3.47.** *Let $\sigma \in R$-pr be left exact and stable. The following statements are equivalent:*

(1) *$R$ is a left $\sigma$-semiartinian ring.*
(2) *Each $\sigma$-natural class is generated by a family $\sigma$-torsion simple modules.*

*Proof.* (1) ⇒ (2) Let $\mathcal{C}$ be a non trivial $\sigma$-natural class and $\mathcal{S} \subseteq \mathcal{C}$ be the class of all simple modules of $\sigma$-torsion of $\mathcal{C}$. The class $\mathcal{S}$ is non empty since $R$ is a left $\sigma$-semiartinian ring. Consider $0 \neq M \in \mathcal{C}$, and ${}_RL \neq 0$ embedding in $M$, with $\sigma(L) \neq 0$. Since $R$ is a left $\sigma$-semiartinian ring, there exists $S \in R\text{-Simp} \cap \mathbb{T}_\sigma$ a submodule of $L$. By Lemma 3.43, the submodules of modules in $\mathcal{C}$ belong to $\mathcal{C}$, so $S \in \mathcal{C}$. Then, by Remark 3.46, one has that $M \in \xi_{\sigma\text{-nat}}(\mathcal{S})$. It follows that $\mathcal{C} = \xi_{\sigma\text{-nat}}(\mathcal{S})$.

(2) ⇒ (1) Let $M \in R$-Mod and $L$ be a nonzero submodule of $M$ with $\sigma(L) \neq 0$. By hypothesis one has that there exists $\mathcal{S} \subseteq R\text{-Simp} \cap \mathbb{T}_\sigma$ such that $\xi_{\sigma\text{-nat}}(M) = \xi_{\sigma\text{-nat}}(\mathcal{S})$. Now, since $L \in \xi_{\sigma\text{-nat}}(M)$, one has that there exists $S \in \mathcal{S}$ such that $S$ embeds in $L$. That is, $R$ is a left $\sigma-$semiartinian ring. □

The following theorem generalizes [1, Proposition 2.12].

**Theorem 3.48.** *Let $\sigma \in R$-pr be exact and costable. The following statements are equivalent::*

(1) *Each $S \in R$-Simp $\cap\, \mathbb{T}_\sigma$ is paraprojective.*
(2) *$\sigma$-(R-nat) $\subseteq$ $\sigma$-(R-TORS).*
(3) *$\sigma$-(R-nat) $\subseteq \mathcal{L}_{/_\sigma}$.*

*Proof.* (1) ⇒ (2) Let $\mathcal{C} \in R$-($\sigma$-nat). Since $\sigma$-$(R\text{-nat}) = \mathcal{L}_{\{\leq_\sigma, \oplus, \sigma(E()), ext\}}$ and $\sigma$-$(R\text{-TORS}) = \mathcal{L}_{\{/_\sigma, \oplus, ext\}}$ it only remains to show that $\mathcal{C} \in \mathcal{L}_{/_\sigma}$. To do so, let us consider $M \in \mathcal{C}$ and $N \leq M$ with $\sigma(N) \neq 0$. Let $S \in \mathrm{Simp} \cap \mathbb{T}_\sigma$ be a submodule of $\sigma(N)$, which exists since $R$ is a left $\sigma$-semiartinian ring. If in the following diagram $P$ is the pull-back of $M \twoheadrightarrow \sigma(N)$ and $S \rightarrowtail \sigma(N)$,

$$\begin{tikzcd} P \arrow[rr, dashed, two heads] \arrow[d, dashed, tail] & & S \arrow[d, tail] \\ M \arrow[r, two heads] & N \arrow[r, two heads] & \sigma(N). \end{tikzcd}$$

Then $S$ is a quotient of $P$, but by hypothesis $S$ is paraprojective, so $S$ is a simple module of $\sigma$-torsion, which embeds in $P$. Then $S$ is a $\sigma$-torsion simple module which embeds in $M$. The above shows that any $\sigma$-torsion simple submodule of $\sigma(N)$ belongs to $\mathcal{C}$. As well, by Proposition 3.47, $\sigma(N) \in \mathcal{C}$. Therefore $\mathcal{C}$ is a $\sigma$-hereditary class.

(2) $\Rightarrow$ (3) It is clear.

(3) $\Rightarrow$ (1) Let $M$ be an $R$-module and $S \in \mathrm{Simp} \cap \mathbb{T}_\sigma$ be a quotient of $M$. By hypothesis, one has that $S \in \xi_{\sigma\text{-nat}}(M)$. It follows from Remark 3.46 that $S$ embeds in $M$. That is, $S$ is paraprojective. $\square$

**Corollary 3.49.** *Let $\sigma \in R$-pr be exact and costable. If $R$ satisfies condition* ($\sigma$-HH), *then*

$$\sigma\text{-(R-}nat) \subseteq \sigma\text{-(R-}TORS)\,.$$

*Proof.* It follows from statement 3 of Remark 3.15, and from Theorem 3.48. $\square$

**Definition 3.50.** Let $\sigma \in R$-pr and let ${}_RM$ be an $R$-module. We will say that a submodule $N$ of $M$ is $\sigma$-essential in $M$ if $N \cap L \neq 0$ for any $L$ of $M$ with $\sigma(L) \neq 0$.

*Remark* 3.51. Let $\sigma \in R$-pr and let ${}_RM$ be an $R$-module. Let us denote $Soc_\sigma(M)$ the sum of the simple $\sigma$-torsion submodules of $M$. If $R$ is a left $\sigma$-semiartinian ring then $Soc_\sigma(M)$ is essential in $\sigma(M)$, thus it is $\sigma$-essential in ${}_RM$. Thus we have for a left $\sigma$-semiartinian ring that

$$\sigma(M) \neq 0 \Leftrightarrow Soc_\sigma(M) \neq 0.$$

**Theorem 3.52.** *Let $\sigma \in R$-pr be exact and costable. If $R$ satisfies condition* ($\sigma$-HH), *then $\sigma$-(R-tors) $\subseteq$ $\sigma$-(R-nat).*

*Proof.* Let $\mathcal{C} \in \sigma$-(R-tors). To see that $\mathcal{C} \in \sigma$-(R-nat) recall that $\sigma$-(R-nat) $= \mathcal{L}_{\leq_\sigma, \oplus, \sigma E}$ and that $\sigma$-(R-tors) $= \mathcal{L}_{\leq_\sigma, \oplus, /_\sigma, ext}$. Thus, it suffices to show that if $M \in \mathcal{C}$, then $\sigma E(M) \in \mathcal{C}$. Let us first note that for any module ${}_RN$ one has that

$$Soc_\sigma(N) = Soc_\sigma(E(N)),$$

since $N$ is an essential submodule of $E(N)$. On the other hand, as $\sigma$ is an exact and stable preradical, and in particular it is left exact and stable, we have that

$$\sigma(E(M)) = \sigma(E(M)).$$

From Lemma 1.42 and the equations one has that

$$\begin{aligned} M \in \mathcal{C} &\Longleftrightarrow Soc_\sigma(M) \in \mathcal{C} \\ &\Longleftrightarrow Soc_\sigma(E(M)) \in \mathcal{C} \\ &\Longleftrightarrow Soc_\sigma(\sigma(E(M))) \in C \\ &\Longleftrightarrow \sigma(E(M)) \in \mathcal{C}. \end{aligned}$$

Thus, if $M \in \mathcal{C}$, then $E(M) \in \mathcal{C}$. □

**Definition 3.53.** Let $\sigma \in R - pr$ exact an stable and assume that $R$ has $(\sigma\text{-HH})$. We are going to define for each ordinal $\alpha$, a preradical $Soc_\sigma^\alpha$. Let us define $Soc_\sigma^0$ as the zero preradical $\underline{0}$. Now, let us define $Soc_\sigma^{\alpha+1}(M)/Soc_\sigma^\alpha(M) = Soc_\sigma(M/Soc_\sigma^\alpha(M))$, as in the diagram

$$\begin{tikzcd} 0 \arrow[r] & Soc_\sigma^\alpha(M)) \arrow[r] \arrow[d, equal] & M \arrow[r] & M/Soc_\sigma^\alpha(M) \arrow[r] & 0 \\ 0 \arrow[r] & Soc_\sigma^\alpha(M)) \arrow[r] & Soc_\sigma^{\alpha+1}(M) \arrow[r] \arrow[u, hook] & Soc_\sigma(M/Soc_\sigma^\alpha(M)) \arrow[r] \arrow[u, hook] & 0. \end{tikzcd} \tag{3.3}$$

Let us define $Soc_\sigma^\alpha(M) = \sum_{\beta<\alpha} Soc_\sigma^\beta(M)$ if $\alpha$ is a limit ordinal.

**Lemma 3.54.** *Let $\sigma \in R$-$pr$ be exact and stable and let $\mathcal{C}$ be a $\sigma$-hereditary torsion class. If $R$ has the* $(\sigma\text{-HH})$ *condition, then $M \in \mathcal{C}$ if and only if each of its simple $\sigma$-torsion subquotients belong to $\mathcal{C}$.*

*Proof.* Let us first assume that $M \in \mathcal{C}$. If $\sigma(M) \neq 0$ and $S \in R - simp \cap \mathbb{T}_\sigma$ is a subquotient of $\sigma(M)$, then by Theorem 1,10, $S$ embeds in $\sigma(M)$. As $\mathcal{C}$ is a $\sigma$-hereditary class, one has that $S \in \mathcal{C}$.

On the other hand, if $\sigma(M) = 0$, then $M \in \mathcal{C}$, because $\mathbb{F}_\sigma \subseteq \mathcal{C}$, for any $\sigma$-hereditary torsion class $\mathcal{C}$.

Now suppose that any simple $\sigma$-torsion subquotient of $\sigma(M)$ belongs to $\mathcal{C}$ and let us show that $M$ in $\mathcal{C}$.

We can assume that $\sigma(M) \neq 0$. Since $\sigma$ is exact and stable, then it centrally splits, so that $M = \sigma(M) \oplus M'$, with $M' \in \mathbb{F}_\sigma$. Thus, it suffices to show that $\sigma(M) \in \mathcal{C}$. As $R$ has $(\sigma\text{-HH})$, then by Corollary 3.21, we have that $R$ is left $\sigma$-semiartinian. Then by Remark 3.51, $0 \neq Soc_\sigma \leqslant_{ess} \sigma(M)$. Note also that $Soc_\sigma(M) \in \mathcal{C}$, since $\mathcal{C}$ is a $\sigma$-hereditary torsion class. Therefore, if it happens that $\sigma(M) = Soc_\sigma(M)$, then $\sigma(M)$ belongs to $\mathcal{C}$. In this case we are done. Suppose then that $\sigma(M)/Soc_\sigma(M) \neq 0$. Note that we can consider two cases: $\sigma(M)/Soc_\sigma(M) \in \mathbb{F}_\sigma$ and $\sigma(M)/Soc_\sigma(M) \notin \mathbb{F}_\sigma$.

In the first case, $\sigma(M)/Soc_\sigma(M) \in \mathcal{C}$ because $\mathbb{F}_\sigma \subseteq \mathcal{C}$. Since $\mathcal{C}$ is closed under extensions, the exact sequence $0 \to Soc_\sigma(M) \to \sigma(M) \to \sigma(M)/Soc_\sigma(M) \to 0$ implies that $\sigma(M) \in \mathcal{C}$.

In the second case, we are going to show by induction that for any ordinal $\alpha$, the module $Soc_\sigma^\alpha(\sigma(M))$ belongs to $\mathcal{C}$.

It is clear that

$$\underline{0}\left(\sigma\left(M\right)\right)=Soc_{\sigma}^{0}\left(\sigma\left(M\right)\right)=_{R}0$$

belongs to $\mathcal{C}$. Now suppose that $Soc_{\sigma}^{\alpha}\left(\sigma\left(M\right)\right)$ belongs to $\mathcal{C}$, then the exact sequence

$$0\to Soc_{\sigma}^{\alpha}\left(\sigma\left(M\right)\right)\to Soc_{\sigma}^{\alpha+1}\left(\sigma\left(M\right)\right)\to Soc_{\sigma}\left(\sigma\left(M\right)/Soc_{\sigma}^{\alpha}\left(\sigma\left(M\right)\right)\right)\to 0$$

has the ends in $C$, because any simple submodule of $Soc_{\sigma}\left(\sigma\left(M\right)/Soc_{\sigma}^{\alpha}\left(\sigma\left(M\right)\right)\right)$ is a $\sigma$-torsion simple subquotient of $\sigma\left(M\right)$, which is in $\mathcal{C}$, by hypothesis.

So, $Soc_{\sigma}\left(\sigma\left(M\right)/Soc_{\sigma}^{\alpha}\left(\sigma\left(M\right)\right)\right)$ belongs to $\mathcal{C}$, because $\mathcal{C}$ is closed under direct sums and quotients of $\sigma$-torsion modules. If $\alpha$ is a limit ordinal, suppose that $Soc_{\sigma}^{\beta}\left(\sigma\left(M\right)\right)\in\mathcal{C}$ for each $\beta<\alpha$, then $Soc_{\sigma}^{\alpha}\left(\sigma\left(M\right)\right)=\sum_{\beta<\alpha}Soc_{\sigma}^{\beta}\left(\sigma\left(M\right)\right)$ is also in $\mathcal{C}$, because this sum is a quotient of $\oplus_{\beta<\alpha}Soc_{\sigma}^{\beta}\left(\sigma\left(M\right)\right)$.

As $\{Soc_{\sigma}^{\alpha}\left(\sigma\left(M\right)\right)\}$ is an ascending chain of submodules of $\sigma\left(M\right)$ which can not be all distinct, thus there exists a first ordinal $\gamma$ such that $Soc_{\sigma}^{\gamma}\left(\sigma\left(M\right)\right)=Soc_{\sigma}^{\gamma+1}\left(\sigma\left(M\right)\right)\}\}$. This means that

$$0=Soc_{\sigma}^{\gamma+1}\left(\sigma\left(M\right)\right)/Soc_{\sigma}^{\gamma}\left(\sigma\left(M\right)\right)=Soc_{\sigma}(M/Soc_{\sigma}^{\gamma}\left(\sigma\left(M\right)\right)).$$

This can only occur if $M/Soc_{\sigma}^{\gamma}\left(\sigma\left(M\right)\right)=0$. Therefore $M=Soc_{\sigma}^{\gamma}\left(\sigma\left(M\right)\right)$, which belongs to $\mathcal{C}$. It is clear that $\underline{0}\left(\sigma\left(M\right)\right)=Soc_{\sigma}^{0}\left(\sigma\left(M\right)\right)=_{R}0$ belongs to $\mathcal{C}$. Now suppose that $Soc_{\sigma}^{\alpha}\left(\sigma\left(M\right)\right)$, belongs to $\mathcal{C}$, then the exact sequence $0\to Soc_{\sigma}^{\alpha}\left(\sigma\left(M\right)\right)\to Soc_{\sigma}^{\alpha+1}\left(\sigma\left(M\right)\right)\to Soc_{\sigma}\left(\sigma\left(M\right)/Soc_{\sigma}^{\alpha}\left(\sigma\left(M\right)\right)\right)\to 0$ has the ends in $\mathcal{C}$, because any simple submodule of $Soc_{\sigma}\left(\sigma\left(M\right)/Soc_{\sigma}^{\alpha}\left(\sigma\left(M\right)\right)\right)$ is a $\sigma$-torsion simple subquotient of $\left(M\right)$, which is in $\mathcal{C}$, by hypothesis. So $Soc_{\sigma}\left(\sigma\left(M\right)/Soc_{\sigma}^{\alpha}\left(\sigma\left(M\right)\right)\right)$ belongs to $\mathcal{C}$, because $\mathcal{C}$ is closed under direct sums and quotients of $\sigma$-torsion modules. If $\alpha$ is a limit ordinal, suppose that $Soc_{\sigma}^{\beta}\left(\sigma\left(M\right)\right)\in\mathcal{C}$ for each $\beta<\alpha$, then $Soc_{\sigma}^{\alpha}\left(\sigma\left(M\right)\right)=\sum_{\beta<\alpha}Soc_{\sigma}^{\beta}\left(\sigma\left(M\right)\right)$ is also in $\mathcal{C}$, because the sum is a quotient of $\oplus_{\beta<\alpha}Soc_{\sigma}^{\beta}\left(\sigma\left(M\right)\right)$.

As $\{Soc_{\sigma}^{\alpha}\left(\sigma\left(M\right)\right)\}$ is an ascending chain of submodules of $\sigma\left(M\right)$ which can not all be distinct there exists a first ordinal $\gamma$ such that $Soc_{\sigma}^{\gamma}\left(\sigma\left(M\right)\right)=Soc_{\sigma}^{\gamma+1}\left(\sigma\left(M\right)\right)$. This means that

$$0=Soc_{\sigma}^{\gamma+1}\left(\sigma\left(M\right)\right)/Soc_{\sigma}^{\gamma}\left(\sigma\left(M\right)\right)=Soc_{\sigma}(M/Soc_{\sigma}^{\gamma}\left(\sigma\left(M\right)\right)).$$

This can only occur if $M/Soc_{\sigma}^{\gamma}\left(\sigma\left(M\right)\right)=0$. Therefore $M=Soc_{\sigma}^{\gamma}\left(\sigma\left(M\right)\right)$, which belongs to $\mathcal{C}$. □

**Theorem 3.55.** *Let $\sigma\in R$-pr be an exact costable preradical. If $R$ satisfies condition* ($\sigma$-HH), *then*

$$\sigma\text{-}\left(R\text{-}nat\right)=\sigma\text{-}\left(R\text{-}conat\right).$$

*Proof.* $\subseteq$] Let us take a $\sigma$-natural class $\mathcal{C}$, i.e. a module class satisfying $\left(\sigma\text{-}N\right)$. We will show that $\mathcal{C}$ satisfies $\left(\sigma\text{-}CN\right)$, that is, we are going to show that if for an $R$-module $_{R}M$ and for each non zero epimorphism $M\twoheadrightarrow L$, with $\sigma(L)\neq 0$, there exist $C \in \mathcal{C}$ and $N \in R$-Mod, with $\sigma(N) \neq 0$, and two epimorphisms $L \twoheadrightarrow N \twoheadleftarrow C$, then $M \in \mathcal{C}$.

Let $f : K \rightarrowtail M$ be a monomorphism with $\sigma(K) \neq 0$. Since $R$ satisfies condition ($\sigma$-HH), then by Corollary 3.21, one has that $R$ is a left $\sigma$-semiartinian ring, so that there exists $S \in R\text{-}Simp \cap \mathbb{T}_\sigma$ such that $S$ embeds in $K$ and hence embeds in $M$. By Remark 3.15, one has that $S$ is parainjective, so $S$ is a quotient of $M$. Now, by hypothesis, we have that there exists $C \in \mathcal{C}$ such that $S$ is a quotient of $C$. By Remark 3.15, we have that $S$ is paraprojective, so $S$ embeds in $C$. Note that we find ourselves in the following situation

$$\begin{tikzcd} K \arrow[r, tail] & M \\ & S \arrow[u, tail] \arrow[r, tail] & C. \end{tikzcd} \tag{3.4}$$

Since $\mathcal{C}$ satisfies the condition $(\sigma\text{-}N)$, from (3.4), it follows that $M \in \mathcal{C}$. That is to say, $\mathcal{C} \in \sigma\text{-}(R\text{-conat})$.

Let $\mathcal{C}$ be a $\sigma$-conatural class. We will show that $\mathcal{C}$ satisfies the ($\sigma$-$N$) condition, i.e, we will show that if $M$ is a $R$-module such that for any monomorphism $L \rightarrowtail M$, with $\sigma(L) \neq 0$ there exist $C \in \mathcal{C}$ and $N \in R$-Mod, with $\sigma(N) \neq 0$, which satisfy that $L \leftarrowtail N \rightarrowtail C$, then $M \in \mathcal{C}$.

Consider then an epimorphism $f : M \twoheadrightarrow K$, with $\sigma(K) \neq 0$. Since $R$ satisfies condition ($\sigma$-HH), by Corollary 3.21, $R$ is a left $\sigma$-max ring, so there exists $S \in R\text{-}Simp \cap \mathbb{T}_\sigma$ such that $S$ is a quotient of $K$ and hence a quotient of $M$. Now, by Remark 3.15, one has that $S$ is paraprojective, so $S$ embeds in $M$. Then, by hypothesis, we have that there exists $C \in \mathcal{C}$ such that $S$ embeds in $C$. Note that, from Remark 3.15, one has that $S$ is parainjective, so $S$ is a quotient of $C$. Thus, we have the following situation

$$\begin{tikzcd} M \arrow[r, two heads] & K \arrow[d, two heads] \\ & S & C. \arrow[l, two heads] \end{tikzcd} \tag{3.5}$$

Now, since $\mathcal{C} \in \sigma\text{-}(R\text{-conat})$, $\mathcal{C}$ satisfies condition $(\sigma\text{-}CN)$, so, from (3.5), one has that $M \in \mathcal{C}$. Thus, $\mathcal{C} \in \sigma\text{-}(R\text{-nat})$. □

**Theorem 3.56.** *Let $\sigma \in R$-pr be exact and costable. If $\sigma\text{-}(R\text{-}nat) = \sigma\text{-}(R\text{-}conat)$, then any $S \in R\text{-}Simp \cap \mathbb{T}_\sigma$ is parainjective and paraprojective.*

*Proof.* Let $M$ be a $R$-module and $S \in R\text{-}Simp \cap \mathbb{T}_\sigma$. If $S$ embeds in $M$, since $S = \sigma(S)$, then $S \in \xi_{\sigma\text{-}nat}(M)$. Now, since $\sigma\text{-}(R\text{-nat}) = \sigma\text{-}(R\text{-conat})$, it follows that $\xi_{\sigma\text{-}nat}(M) = \xi_{\sigma\text{-}conat}(M)$. Thus, $S \in \xi_{\sigma\text{-}conat}(M)$, then, by 3.30, $S$ is a quotient of $M$, i.e., $S$ is parainjective.

On the other hand, if $S$ is a quotient of $M$, then $S \in \xi_{\sigma\text{-}conat}(M)$, since $S = \sigma(S)$. It follows from the fact that $\sigma\text{-}(R\text{-nat}) = \sigma\text{-}(R\text{-conat})$, that $S \in \xi_{\sigma\text{-}nat}(M)$. By 3.46, one has that $S$ embeds in $M$, i.e., $S$ is paraprojective. $\square$

*Remark* 3.57. Let $\sigma \in R$-pr. If $\sigma$ centrally splits , then there exists a central idempotent $e \in R$, such that $\sigma(\_) = e \cdot \_$. Now, as $R = Re \oplus R(1-e)$, it follows that $\sigma(R) \cong R/(1-e)R$, so a $\sigma(R)$-module is an $R$-module annulled by $1-e$. As well, if $M$ i s a $\sigma(R)$-module, then $M = \sigma(M)$.

**Lemma 3.58.** *Let $\sigma \in R$-pr. If $\sigma$ centrally splits, then the following statements are equivalent:*

(1) *$R$ satisfies the condition* ($\sigma$-HH).
(2) *$\sigma(R)$ satisfies the condition* $(HH)$.

*Proof.* (1) $\Rightarrow$ (2) Let $M, N$ be two $\sigma(R)$-modules and let $0 \neq f \in Hom_{\sigma(R)}(M, N)$. Since $M$ and $N$ are also $R$-modules, it follows that $0 \neq f \in Hom_R(M, N)$. Now, by Remark 3.57, it follows that $M = \sigma(M)$, so that $0 \neq f \in Hom_R(\sigma(M), N)$. Since $R$ satisfies the condition ($\sigma$-HH), it follows that there exists $0 \neq g \in Hom_R(N, \sigma(M))$, whence $0 \neq g \in Hom_{\sigma(R)}(N, M)$.That is , $\sigma(R)$ satisfies the condition ($\sigma$-HH).

(2) $\Rightarrow$ (1) Let $M, N$ be two $R$-modules such that $\sigma(M), \sigma(N) \neq 0$ and $0 \neq f \in Hom_R(\sigma(M), N)$. Note that $\sigma(M)$ and $\sigma(N)$ are two $\sigma(R)$-modules. Now, since $\sigma$ is a preradical, then $f$ can be corestricted to $\sigma(N)$, i.e., $f\lceil\colon \sigma(M) \to \sigma(N)$ is a $\sigma(R)$ nonzero morphism. It follows that there exists $0 \neq g \in Hom_{\sigma(R)}(\sigma(N), \sigma(M))$, since $\sigma(R)$ satisfies the condition $(HH)$. Thus, we find ourselves in the following situation.

$$\begin{tikzcd}
\sigma(M) \arrow[r, "f\lceil"] \arrow[d, equal] & \sigma(N) \arrow[dl, "h\neq 0"] \arrow[d, hook] \\
\sigma(M) & N = \sigma(N) \oplus (1-e)N. \arrow[l, dashed, "h\oplus\bar{0}"]
\end{tikzcd}$$

Then, if $\bar{0}$ denotes the zero morphism, then we have that $h \oplus \bar{0}$ is a nonzero morphism from $N$ to $\sigma(M)$. Therefore, $R$ satisfies the condition ($\sigma$-HH). $\square$

Example 3.59. For each prime integer $p$ and for each natural number $n > 1$, we have that $\mathbb{Z}_{p^n}$ is a commutative perfect local ring. Thus it satisfies the conditions $(HH)$, then the ring $(\mathbb{Z}_{p_1^{n_1}} \times \mathbb{Z}_{p_2^{n_2}} \times ...\mathbb{Z}_{p_k^{n_k}}) \times R$ satisfies the condition ($\sigma$-HH), for the preradical $\sigma = e \cdot (-)$, where $e = (\bar{1}, ..., \bar{1})$ (there are $k$ $\bar{1}s$).

The next theorem generalizes [8, Theorem 2.18].

**Theorem 3.60.** *Let $R$ be a ring and $\sigma \in R$-pr be exact and costable. The following statements are equivalent:*

(1) *$R$ satisfies condition* ($\sigma$-HH).
(2) *$R$ is a $\sigma$-* (R-*Mod*)*-retractable ring and each $S \in R$-Simp $\cap \mathbb{T}_\sigma$ is paraprojective.*
(3) *Each $S \in R$-Simp $\cap \mathbb{T}_\sigma$ is paraprojective and parainjective.*
(4) $\sigma$-(R-*TORS*) $= \sigma$-(R-*conat*) $= \sigma$-(R-*tors*) $= \sigma$-(R-*nat*).
(5) $\sigma$-(R-*nat*) $= \sigma$-(R-*conat*).
(6) *$R = \sigma(R) \times R_2$, where $\sigma(R)$ is a ring that satisfies the $(HH)$ condition.*
(7) *$R = \sigma(R) \times R_2$, where $\sigma(R)$ is isomorphic to a finite product of full matrix rings over local left and right perfect rings.*

*Proof.* (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) is Theorem 3.22. (1) $\Rightarrow$ (5) is Theorem 3.55. (5) $\Rightarrow$ (1) It follows from Theorem 3.22. (1) $\Rightarrow$ (4) is Theorem 3.41. (4) $\Rightarrow$ (5) This is clear. (1) $\Leftrightarrow$ (6) This follows from Lemma 3.58 and because $\sigma$ centrally splits. (6) $\Leftrightarrow$ (7) It follows from [8, Theorem 2.18]. $\square$

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