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On Rough $\phi-$Convergence

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Abstract. In this paper we introduce rough $\phi-$convergence of real numbers as a generalization of rough convergence as well as $\phi-$convergence. We study some of its basic properties and relations of the above convergence concept with already known different types of convergence.

In 2001, H.X.Phu proved that "The diameter of a $r-$limit set is not greater than $2r$". We investigate the above result for rough $\phi-$convergence by introducing $r-\phi$ limit set and surprisingly the result comes out to be not true. So our main aim is to find out the different behaviour of the new convergence concept based on $r-\phi$ limit set.

Keywords: Rough convergence, $\phi-$Convergence, $r-\phi$-Convergence, $r-\phi$ Limit set.

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1. Introduction

The idea of rough convergence was first introduced by H. X. Phu [16] in finite dimensional normed spaces. In [17, 18], he showed that the set $\text{LIM}^r x$ is bounded, closed and convex and studied the basic properties of this interesting concept. It should be mentioned that the idea of rough convergence occurs quite naturally in numerical analysis and has interesting applications there. He also investigated the relations between rough convergence and other convergence types and the dependence of $\text{LIM}^r x$ on the roughness degree r. In [5], Aytar generalizes the concept of rough convergence. For extensive study on rough convergence, one may refer to [1, 2, 3, 6, 7, 8, 9, 10, 11, 12, 14, 15, 21].

An Orlicz function (see [19]) is an even, continuous and non-decreasing on $\mathbb{R}^+$ function $\phi : \mathbb{R} \to \mathbb{R}^+$, such that $\phi(x) = 0 \Leftrightarrow x = 0$ and $\lim_{x\to\infty} \phi(x) = \infty$. Note that $\phi(x) = \phi(|x|), \forall x \in \mathbb{R}$ and $\phi(\mathbb{R}) = \mathbb{R}^+$.

In [19], Rao and Ren describe that Orlicz functions have important roles and applications in many areas such as economics, stochastic problems etc.

An Orlicz function $\phi : \mathbb{R} \to \mathbb{R}$ is said to satisfy the $\triangle_2$-condition, if there exists an $M > 0$ such that $\phi(2x) \leq M\phi(x)$, for every $x \in \mathbb{R}^+$.

Let us follow some basic examples (see also [20]).

(i) The function $\phi : \mathbb{R} \to \mathbb{R}$ with $\phi(x) = |x|$ is an Orlicz function.
(ii) The function $\phi : \mathbb{R} \to \mathbb{R}$ with $\phi(x) = x^3$ is not an Orlicz function.
(iii) The function $\phi : \mathbb{R} \to \mathbb{R}$ with $\phi(x) = x^2$ is an Orlicz function satisfying the $\triangle_2$-condition.
(iv) The function $\phi : \mathbb{R} \to \mathbb{R}$ with $\phi(x) = e^{|x|} - |x| - 1$ is an Orlicz function not satisfying the $\triangle_2$-condition.

In this paper, we introduce a concept of rough $\phi-$convergence on $\mathbb{R}$, as a generalization of rough convergence and $\phi-$convergence, where ϕ an Orlicz function.

2. Definitions and preliminaries

Throughout the paper $\mathbb{R}$, r, ϕ stands for the set of all real numbers, a non-negative real number and Orlicz function respectively.

Now, we recall the concepts of rough convergence, $\phi-$convergence as follows:

Definition 2.1. [16] Let $x = (x_i)$ be a sequence of real numbers and r be a non-negative real number. (x_i) is said to be rough $r-$convergent to $x_* \in \mathbb{R}$, denoted by $x_i \xrightarrow{r} x_*$, if

$$\forall \varepsilon > 0,\ \exists\ i_\varepsilon \in \mathbb{N} : i \geq i_\varepsilon \Rightarrow |x_i - x_*| < r + \varepsilon$$

or, equivalently, if $\limsup_{i\to\infty} |x_i - x_*| \leq r$. This is the rough convergence with r as roughness degree. For $r = 0$, we have the classical convergence.

Definition 2.2. [4] Let $x = (x_i)$ be a sequence of real numbers and r be a non-negative real number. Then, the set

$$\mathrm{LIM}^r x_i = \left\{ x_* \in \mathbb{R} : x_i \xrightarrow{r} x_* \right\}$$

is known as the rough limit set of $x = (x_i)$.

Definition 2.3. [13] Let $\phi : \mathbb{R} \to \mathbb{R}$ be an Orlicz function. A sequence $x = (x_n)$ is said to be $\phi-$convergent to x_0 if $\underset{n}{lim}\, \phi(x_n - x_0) = 0$. In this case, x_0 is called the $\phi-$limit of (x_n) and denoted by $\phi - lim\, x = x_0$.
If a sequence (x_n) is $\phi-$convergent to x_0, we denote it by $x_n \xrightarrow{\phi} x_0$.

Remark 2.4. If we take $\phi(x) = |x|$, then, $\phi-$convergent concepts coincide with usual convergence. Also, it is easy to check, if $x = (x_n)$ is $\phi-$convergent to x_0, then any of its subsequence is $\phi-$convergent to x_0 as well.

3. Main results

Definition 3.1. Let $\phi : \mathbb{R} \to \mathbb{R}$ be an Orlicz function and r be a non-negative real number. A sequence $x = (x_i)$ is said to be rough $r - \phi-$convergent to $x_* \in \mathbb{R}$, denoted by $x_i \overset{r-\phi}{\to} x_*$, if

$$\forall \varepsilon > 0, \exists\, i_\varepsilon \in \mathbb{N} : i \geq i_\varepsilon \Rightarrow \phi(x_i - x_*) < r + \varepsilon \tag{3.1}$$

or, equivalently, if $\limsup_{i \to \infty} \phi(x_i - x_*) \leq r$.

This is the rough $\phi-$convergence with r as roughness degree. For $r = 0$ we have the $\phi-$convergence. But our proper interest is the case $r > 0$.

If (3.1) holds, x_* is an $r - \phi$ limit point of (x_i), which is usually no more unique (for $r > 0$). So, we have to consider the so-called $r - \phi$ limit set (or, shortly: $r - \phi$ limit) of (x_i) defined by

$$\mathrm{LIM}^{r-\phi} x_i = \left\{ x_* \in \mathbb{R} : x_i \overset{r-\phi}{\to} x_* \right\}.$$

A sequence $x = (x_i)$ is said to be rough $\phi-$convergent if $\mathrm{LIM}^{r-\phi} x_i \neq \emptyset$. In this case, r is called a convergence degree of (x_i).

This is illustrated by following examples:

Example 3.2. Let ϕ: $\mathbb{R} \to \mathbb{R}$ be an Orlicz function, defined by $\phi(x) = x^2$. Consider a sequence (x_i), defined by

$$x_1 = -1, x_2 = 2, x_{2j-1} = 0 \text{ and } x_{2j} = 1 \text{ for } j = 2, 3, 4, \ldots$$

Obviously, the sequence (x_i) is not $\phi-$convergent. But by definition we have $x_i \overset{r-\phi}{\to} \frac{1}{2}$ for $r = \frac{1}{4}$, and altogether

$$\mathrm{LIM}^{r-\phi} x_i = \begin{cases} \emptyset, & \text{for } r < \frac{1}{4} \\ [1\text{-}\sqrt{r}, \sqrt{r}] & \text{for } r \geq \frac{1}{4}. \end{cases}$$

Example 3.3. Let ϕ: $\mathbb{R} \to \mathbb{R}$ be an Orlicz function, defined by $\phi\,(x) = x^2$. Take a sequence (x_i), where $x_i = (-1)^{i-1}$, $i = 1, 2, 3, 4, \ldots$

Then, (x_i) is not convergent. But by definition we have $x_i \overset{r-\phi}{\to} 1$ for $r \geq 4$, and altogether

$$\mathrm{LIM}^{r-\phi} x_i = \begin{cases} \emptyset & \text{for } r < 4 \\ [1\text{-}\sqrt{r},\ \text{-}1\text{+}\sqrt{r}] & \text{for } r \geq 4. \end{cases}$$

Sometimes we are interested in the set of $r-\phi$ limit points lying in a given subset $S \subset \mathbb{R}$, which is called $r-\phi$ limit in S and denoted by

$$\mathrm{LIM}^{S,r-\phi} x_i = \left\{ x_* \in S : x_i \overset{r-\phi}{\to} x_* \right\}.$$

It is obvious that,

$$\mathrm{LIM}^{\mathbb{R},r-\phi} x_i = \mathrm{LIM}^{r-\phi} x_i \text{ and } \mathrm{LIM}^{S,r-\phi} x_i = S \cap \mathrm{LIM}^{r-\phi} x_i.$$

It is clear that when $\phi\,(x) = |x|$, rough $\phi-$convergence coincides with rough convergence, i.e., rough convergence is a special type of rough $\phi-$convergence, or, more generally, rough $\phi-$convergence is a generalization of rough convergence.

Example 3.4. Let ϕ: $\mathbb{R} \to \mathbb{R}$ be an Orlicz function, defined by

$$\phi\,(x) = \sqrt{|x|}.$$

Consider a sequence (x_i), defined in the Example 3.2. By definition of $r-\phi$ limit set,

$$\mathrm{LIM}^{r-\phi} x_i = \begin{cases} \emptyset & \text{for } r < \frac{1}{\sqrt{2}} \\ [1\text{-}r^2,\ r^2] & \text{for } r \geq \frac{1}{\sqrt{2}}. \end{cases}$$

But in case of rough convergence, the $r-$limit set is given by

$$\mathrm{LIM}^{r} x_i = \begin{cases} \emptyset & \text{for } r < \frac{1}{2} \\ [1\text{-}r,\ r] & \text{for } r \geq \frac{1}{2} \end{cases}$$

i.e., for $\frac{1}{2} \leq r < \frac{1}{\sqrt{2}}$, the sequence is rough convergent but not rough $\phi-$convergent. If $r \geq \frac{1}{\sqrt{2}}$, the sequence seems to be both rough convergent and rough $\phi-$convergent.

Definition 3.5. A sequence $x = (x_i)$ is said to be $\phi-$bounded if there exists $B > 0$ such that

$$|\phi\,(x_i)| < B,\ \forall i \in \mathbb{N}.$$

Proposition 3.6. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a convex Orlicz function which satisfies $\triangle_2$-condition. Then, a sequence $x = (x_i)$ is $\phi-$bounded iff there exists an $r \geq 0$ s.t.* $\mathrm{LIM}^{r-\phi} x \neq \emptyset$.

Proof. Let $x = (x_i)$ be $\phi-$bounded. Then, there exists a $B > 0$ such that $|\phi\,(x_i)| < B$ $\forall i \in \mathbb{N}$. Define $r = \sup\{|\phi\,(x_i)| : i \in \mathbb{N}\}$. Then, we have $|\phi\,(x_i)| \leq r$ $\forall i \in \mathbb{N}$, i.e., $|\phi\,(x_i - 0)| \leq r$, $\forall i \in \mathbb{N}$. Then, the set $\mathrm{LIM}^{r-\phi} x$

contains origin and hence $\text{LIM}^{r-\phi}x \neq \emptyset$.
Conversely, if $\text{LIM}^{r-\phi}x_i \neq \emptyset$ for some $r \geq 0$, then, $\exists\ x_*$ such that $x_* \in \text{LIM}^{r-\phi}x$. Since ϕ satisfies $\triangle_2$-condition so there exists $M \in \mathbb{R}^+$ such that $\phi(2x) \leq M\phi(x)$. Further $x_* \in \text{LIM}^{r-\phi}x$, so for every $\varepsilon > 0$, there exists $i_\varepsilon \in \mathbb{N}$ such that

$$i \geq i_\varepsilon \Rightarrow \phi(x_i - x_*) < r + \frac{2\varepsilon}{M}. \tag{3.2}$$

Now for any $i \geq i_\varepsilon$,

$$\begin{aligned}
\phi(x_i) &= \phi(x_i - x_* + x_*) \\
&= \phi\left(\frac{1}{2}(2(x_i - x_*)) + \frac{1}{2}(2x_*)\right) \\
&\leq \frac{1}{2}(\phi(2(x_i - x_*))) + \frac{1}{2}(\phi(2x_*)) \\
&\leq \frac{M}{2}(\phi(x_i - x_*)) + \frac{M}{2}(\phi(x_*)) \\
&< \frac{M}{2}(r + \phi(x_*)) + \varepsilon, \text{ using (3.2).}
\end{aligned}$$

Let $B = \max\left\{\phi(x_1), \phi(x_2), ..., \phi(x_{i_\varepsilon - 1}), \frac{M}{2}(r + \phi(x_*)) + 1\right\}$. Then, since ϕ is even and non decreasing on $\mathbb{R}^+$, $\phi(x_i) < B\ \forall i \in \mathbb{N}$. Hence, $x = (x_i)$ is $\phi-$bounded. □

Proposition 3.7. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a convex Orlicz function. If $y_0 \in \text{LIM}^{r_0-\phi}x_i$ and $y_1 \in \text{LIM}^{r_1-\phi}x_i$, then,*

$$y_\lambda = (1-\lambda)y_0 + \lambda y_1 \in \text{LIM}^{(1-\lambda)r_0+\lambda r_1-\phi}x_i \text{ for } \lambda \in [0,1].$$

Proof. Let $y_0 \in \text{LIM}^{r_0-\phi}x_i$. Then, for each $\varepsilon > 0$, $\exists\ i_\varepsilon \in \mathbb{N}$ s.t. $\phi(x_i - y_0) < r_0 + \varepsilon, \forall\ i \geq i_\varepsilon$.

Again, $y_1 \in \text{LIM}^{r_1-\phi}x_i$. Then, for each $\varepsilon > 0$, $\exists\ i_\varepsilon^{'} \in \mathbb{N}$ s.t. $\phi(x_i - y_1) < r_1 + \varepsilon, \forall\ i \geq i_\varepsilon^{'}$.

Now,

$$\begin{aligned}
\phi(x_i - y_\lambda) &= \phi(x_i - (1-\lambda)y_0 - \lambda y_1) \\
&= \phi((1-\lambda)x_i + \lambda x_i - (1-\lambda)y_0 - \lambda y_1) \\
&= \phi((1-\lambda)(x_i - y_0) + \lambda(x_i - y_1)) \\
&\leq (1-\lambda)\phi(x_i - y_0) + \lambda\phi(x_i - y_1) \\
&< (1-\lambda)(r_0 + \varepsilon) + \lambda(r_1 + \varepsilon) \\
&= r + \varepsilon, r = (1-\lambda)r_0 + \lambda r_1, \forall i \geq max\left(i_\varepsilon, i_\varepsilon^{'}\right).
\end{aligned}$$

Hence, $y_\lambda \in \text{LIM}^{(1-\lambda)r_0+\lambda r_1-\phi}x_i$, $\lambda \in [0,1]$. □

Since $\phi(\mathbb{R}) = \phi(\mathbb{R}^+) = \mathbb{R}^+$, we can define the function $\phi^{-1} : \mathbb{R}^+ \to \mathbb{R}^+$ by $\phi^{-1}(y) = \max\{x \in \mathbb{R}^+ : \phi(x) = y\}$, $\forall y \geq 0$. The following result holds:

Theorem 3.8. *Let* (x_i) *be a bounded sequence and* $r \geq 0$ *such that*

$$\phi^{-1}(r) \geq \tfrac{1}{2}\left(\limsup_{i\to\infty} x_i - \liminf_{i\to\infty} x_i\right).$$

Then, $\mathrm{LIM}^{r-\phi}x_i = \left[\limsup_{i\to\infty} x_i - \phi^{-1}(r), \liminf_{i\to\infty} x_i + \phi^{-1}(r)\right]$.

Corollary 3.9. *The diameter of a* $r-\phi-$*limit set is not greater than* $2\phi^{-1}(r)$.

Example 3.10. Let $\phi : \mathbb{R} \to \mathbb{R}$ be an Orlicz function and $\phi(x) = |x|^n$, $n \in \mathbb{N}$ and r be a non-negative real number. Then, the $r - \phi$ *limit set* of a sequence $x = (x_i)$ is given by:

$$\mathrm{LIM}^{r-\phi}x_i = \left[\limsup_{i\to\infty} x_i - \sqrt[n]{r}, \liminf_{i\to\infty} x_i + \sqrt[n]{r}\right].$$

Justification. Let $x = (x_i)$ be $r - \phi$ converges to $x_* \in \mathbb{R}$. Then, for each $\varepsilon > 0$, $\exists\ i_\varepsilon \in \mathbb{N}$ such that

$$\begin{aligned}
&\phi(x_i - x_*) < r + \varepsilon, \forall i \geq i_\varepsilon \\
&\Rightarrow |x_i - x_*|^n < r + \varepsilon \\
&\Rightarrow |x_i - x_*| < \sqrt[n]{r+\varepsilon} \\
&\Rightarrow x_i - \sqrt[n]{r+\varepsilon} < x_* < x_i + \sqrt[n]{r+\varepsilon}.
\end{aligned}$$

Therefore, $x_i - \sqrt[n]{r} \leq x_* \leq x_i + \sqrt[n]{r}$, $\forall\ i \geq i_\varepsilon$ (since ε is arbitrary).

Let $t = \limsup_{i\to\infty} x_i$ and $A_i = \{x_k : k \geq i\}$ and $t_i = sup\ \{x_k : k \geq i\} = sup\ A_i$. $t-\varepsilon$ is less than the greatest lower bound of t_i's and hence is certainly a lower bound of t_i's. Hence, for any $k \in \mathbb{N}$, $t-\varepsilon$ is less than t_k, the *lub* of $\{t_i : i \geq k\}$. Therefore, $t - \varepsilon$ is not an upper bound of A_i $, i \in \mathbb{N}$. Thus, $\exists\ i_k$ such that $x_{i_k} > t - \varepsilon$. For $k = 1$, let i_1 be such that $x_{i_1} > t - \varepsilon$. Since $t - \varepsilon$ is not an upper bound of A_{i_1+1}, $\exists\ i_2 \geq i_1 + 1 > i_1$ such that $t - \varepsilon < x_{i_2}$. Proceeding in this way, we get a subsequence (x_{i_k}) s.t. $t - \varepsilon < x_{i_k}$, $\forall\ k \in \mathbb{N}$, i.e., $t - \varepsilon < x_i$ for infinitely many i.

Similarly, if $s = \liminf_{i\to\infty} x_i$, then it can be shown that $s+\varepsilon > x_i$, for infinitely many i.

Hence, $t - \varepsilon - \sqrt[n]{r} < x_* < s + \varepsilon + \sqrt[n]{r}$ i.e. $t - \sqrt[n]{r} \leq x_* \leq s + \sqrt[n]{r}$ or, $\limsup_{i\to\infty} x_i - \sqrt[n]{r} \leq x_* \leq \liminf_{i\to\infty} x_i + \sqrt[n]{r}$.

In [16], Phu had proved that the diameter of a r-limit set cann't be strictly larger than $2r$, but in case of $r - \phi$ convergence this may not be true. We present an example given below to illustrate the validity of our claim.

Example 3.11. Let ϕ: $\mathbb{R} \to \mathbb{R}$ be an Orlicz function, defined by $\phi(x) = \sqrt{|x|}$. Consider the sequence (x_i), defined in the Example 3.2. It is easy to show that

$$\mathrm{LIM}^{r-\phi}x_i = \begin{cases} \emptyset & \text{for } r < \frac{1}{\sqrt{2}} \\ [1\text{-}r^2,\ r^2] & \text{for } r \geq \frac{1}{\sqrt{2}}. \end{cases}$$

Therefore,

$$\begin{aligned} diam\left(\mathrm{LIM}^{r-\phi}x_i\right) &= \sup\left\{|p-q| : p,q \in \mathrm{LIM}^{r-\phi}x_i\right\} \\ &= 2r^2-1 > 2r, \text{ for any } r > \frac{1+\sqrt{3}}{2}. \end{aligned}$$

Definition 3.12. Let $\phi : \mathbb{R} \to \mathbb{R}$ be an Orlicz function. A real number γ is said to be $\phi-$cluster point of a sequence $x = (x_i)$, if for every $\varepsilon > 0$, there exists infinitely many i's in $\mathbb{N}$ such that $\phi(x_i - \gamma) < \varepsilon$.

The set of all $\phi-$cluster points of a sequence $x = (x_k)$ is denoted by $\phi(C_x)$.

Theorem 3.13. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a convex Orlicz function which satisfies $\triangle_2$-condition. Then, there exists $K > 0$ such that for an arbitrary bounded sequence $x = (x_i)$, a $\phi-$cluster point c of x and a $r-\phi$ limit point x_* of x, we have*

$$\phi\left(x_* - c\right) \leq Kr.$$

Proof. Since $x_* \in \mathrm{LIM}^{r-\phi}x$, so for every $\varepsilon > 0$ there exists $i_\varepsilon \in \mathbb{N}$ such that

$$\phi\left(x_i - x_*\right) < r + \varepsilon \;\; \forall i \geq i_\varepsilon. \tag{3.3}$$

Since ϕ satisfies $\triangle_2$-condition, so there exists $M \in \mathbb{R}^+$ such that $\phi\left(2x\right) \leq M\phi\left(x\right)$ for every $x \in \mathbb{R}$. Also, $c \in \phi\left(C_x\right)$ implies the existence of infinitely many i's in $\mathbb{N}$ satisfying $\phi\left(x_i - c\right) < \varepsilon$ for every $\varepsilon > 0$. Choose $i = i_0 \geq i_\varepsilon$ in such a way that

$$\phi\left(x_{i_0} - c\right) < \varepsilon. \tag{3.4}$$

We prove that $\phi\left(x_* - c\right) \leq Kr$ where $K = \frac{M}{2}$.

Suppose that

$$\phi\left(x_* - c\right) > \frac{Mr}{2}. \tag{3.5}$$

Choose $\varepsilon = \frac{2\left(\phi(x_*-c) - \frac{Mr}{2}\right)}{3M}$. Then, we have from Equation (3.4),

$$\phi\left(x_{i_0} - c\right) < \frac{2\left(\phi\left(x_* - c\right) - \frac{Mr}{2}\right)}{3M}. \tag{3.6}$$

As ϕ is even, convex, and satisfies $\triangle_2$-condition, we have

$$\begin{aligned} &\phi\left(x_* - c\right) \leq \frac{M}{2}\phi\left(x_* - x_{i_0}\right) + \frac{M}{2}\phi\left(x_{i_0} - c\right) \\ &\Rightarrow \phi\left(x_* - c\right) < \frac{M}{2}\phi\left(x_{i_0} - x_*\right) + \frac{\phi\left(x_* - c\right)}{3} - \frac{Mr}{6}, \text{ by Equation (3.6)} \\ &\Rightarrow 2\phi\left(x_* - c\right) < \frac{3M}{2}\phi\left(x_{i_0} - x_*\right) - \frac{Mr}{2} \\ &\Rightarrow \frac{Mr}{2} + \phi\left(x_* - c\right) < \frac{3M}{2}\phi\left(x_{i_0} - x_*\right) - \frac{Mr}{2}, \text{ by Equation (3.5)} \\ &\Rightarrow \frac{3M}{2}\phi\left(x_{i_0} - x_*\right) > Mr + \frac{3M\varepsilon}{2} + \frac{Mr}{2} \\ &\Rightarrow \phi\left(x_{i_0} - x_*\right) > r + \varepsilon, \end{aligned}$$

which is a contradiction to Equation (3.3). This completes the proof. □

Theorem 3.14. *Let $r > 0$ and $\phi : \mathbb{R} \to \mathbb{R}$ be a convex Orlicz function. Then, a sequence $x = (x_i)$ is rough $\phi-$convergent with roughness degree r if there exists a sequence $y = (y_i)$ such that $\phi - \lim y = x_*$ and $\phi(x_i - y_i) \leq r$ for each $i \in \mathbb{N}$.*

Proof. Assume that $x_i \overset{r-\phi}{\to} x_*$. Then, for every $\varepsilon > 0$, there exists an $i_\varepsilon \in \mathbb{N}$ such that

$$\phi\left(x_i - x_*\right) < r + \varepsilon \quad \forall i \geq i_\varepsilon. \tag{3.7}$$

Consider the sequence $y = (y_i)$ defined by

$$y_i = \begin{cases} x_*, & \text{if } \phi(x_i - x_*) \leq r \\ x_i - r\frac{x_i - x_*}{\phi(x_i - x_*)}, & \text{otherwise.} \end{cases}$$

Then, using the fact that ϕ is convex, it is easy to prove that

$$\phi\left(y_i - x_*\right) = \begin{cases} 0, & \text{if } \phi\left(x_i - x_*\right) \leq r \\ \text{a number} \leq \phi\left(x_i - x_*\right) - r, & \text{otherwise.} \end{cases}$$

Now using Equation (3.7), we can say that $\phi\left(y_i - x_*\right) \to 0$. In other words $\phi - \lim y = x_*$.

Again using the fact that ϕ is convex it is easy to verify using the definition of $y = (y_i)$ that $\phi\left(x_i - y_i\right) \leq r$ for each $i \in \mathbb{N}$. □

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