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Euler Catalan Riesz Sequence Spaces their Duals and Matrix Transformations

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Abstract. In this article, we introduce and study Euler Catalan Riesz sequence spaces of nonabsolute type. We obtain some topological properties and Schauder basis of the newly formed sequence spaces. Moreover, we compute the $\alpha-, \beta-, \gamma-$duals and characterize some matrix classes on this space. Finally, we prove that this sequence spaces are of Banach-Saks type p and has weak point property.

Keywords: Catalan numbers, Euler mean of unit order, Matrix transformations, Riesz mean, $\alpha-, \beta-, \gamma-$duals.

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1. Introduction

One of the compelling integer sequence in mathematics is

$$1, 1, 2, 5, 14, 42, 132, 429, ...$$

of Catalan numbers. The Catalan sequence was discovered by Leonhard Euler in 1751 when searching for the solution to the question "How many different ways exists to divide a polygon into triangles." The name 'Catalan' is attached to this sequence because the Belgian mathematician Eugene Charles Catalan solved the parenthesization problem (see [10], Example 20.3) which is a situation where these numbers arise. These numbers have unique applications in linear algebra, abstract algebra, analysis, topology, geometry, etc. The Binomial coefficient of ϑ^{th} Catalan number is given by

$$\begin{aligned} C_\vartheta &= \frac{1}{\vartheta+1}\binom{2\vartheta}{\vartheta} \\ &= \frac{(2\vartheta)!}{(\vartheta+1)!\vartheta!}. \end{aligned}$$

The recurrence and the initial condition are given as

$$C_{\vartheta+1} = \sum_{n=0}^{\vartheta} C_n C_{\vartheta-n}, \quad C_0 = 1,$$

which is the most important recurrence verified by the Catalan numbers. Also, the non-linear recurrence relation

$$C_{\vartheta+1} = \frac{2(2\vartheta+1)}{\vartheta+2} C_\vartheta, \quad C_0 = 1,$$

which is again satisfied by the Catalan numbers. For more applications regarding Catalan numbers one can refer to ([10, 22]).

M. İlkhan [12] defined the Catalan matrix $\widehat{\mathcal{C}} = (c_{\vartheta\varrho})$ as

$$c_{\vartheta\varrho} = \begin{cases} \frac{C_\varrho C_{\vartheta-\varrho}}{C_{\vartheta+1}}, & \text{if } 0 \le \varrho \le \vartheta, \\ 0, & \text{if } \varrho > \vartheta. \end{cases}$$

Also, $\mathcal{C}^{-1} = (c^{-1}_{\vartheta\varrho})$ of the matrix $\widehat{\mathcal{C}}$ is computed as

$$c^{-1}_{\vartheta\varrho} = \begin{cases} (-1)^{\vartheta-\varrho} \frac{C_{\varrho+1}}{C_\vartheta C_0^{\vartheta-\varrho+1}} H_{\vartheta-\varrho}, & \text{if } 0 \le \varrho \le \vartheta, \\ 0, & \text{if } \varrho > \vartheta, \end{cases}$$

where $H_0 = 1$ and

$$H_\vartheta = \begin{bmatrix} C_1 & C_0 & 0 & ... & 0 \\ C_2 & C_1 & C_0 & ... & 0 \\ . & . & . & ... & . \\ . & . & . & ... & . \\ C_\vartheta & C_{\vartheta-1} & C_{\vartheta-2} & ... & C_1 \end{bmatrix}$$

$\forall\ \vartheta \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of natural numbers.

Suppose a sequence (q_ϱ) of positive real numbers and $Q_\vartheta = \sum_{\varrho=0}^{\vartheta} p_\varrho$, $\forall\ \vartheta \in \mathbb{N}$.

Then, the Riesz mean $\widehat{\mathcal{R}} = (r_{\vartheta\varrho})$ is defined by

$$r_{\vartheta\varrho} = \begin{cases} \frac{q_\varrho}{Q_\vartheta}, & \text{if } 0 \le \varrho \le \vartheta, \\ 0, & \text{if } \varrho > \vartheta, \end{cases}$$

$\forall\ \vartheta, \varrho \in \mathbb{N}$. For more details about Riesz mean one can refer to (see [2, 3, 14, 24]).

Now, we define Euler mean $\widehat{\mathcal{E}} = (e_{\vartheta\varrho})$ of unit order as

$$e_{\vartheta\varrho} = \begin{cases} \frac{\binom{\vartheta}{\varrho}}{2^\vartheta}, & \text{if } 0 \le \varrho \le \vartheta, \\ 0, & \text{if } \varrho > \vartheta, \end{cases}$$

$\forall\ \vartheta, \varrho \in \mathbb{N}$. For more details about Euler mean one can refer to (see [4, 5, 6]).

By Ω, we expressed the space of all real or complex valued sequences. Also by c, c_0 and ℓ_∞ we denote the spaces of all convergent, null and bounded sequences, respectively. We denote the spaces of all absolutely, p-absolutely, bounded and convergent series by ℓ_1, ℓ_p, bs and cs, respectively. Consider $\mathcal{U}$ and $\mathcal{V}$ be two sequence spaces and $\mathcal{L} = (l_{\vartheta\varrho})$ be an infinite matrix of real numbers, where $\vartheta, \varrho \in \mathbb{N}$. Define a matrix transformation $\mathcal{L} : \mathcal{U} \to \mathcal{V}$ as the sequence $\mathcal{L}z = \{\mathcal{L}_\vartheta z\}$, if for every sequence $z = (z_\varrho) \in \mathcal{U}$. The $\mathcal{L}$-transform of z is in $\mathcal{V}$, where

$$\mathcal{L}_\vartheta z = \sum_{\varrho=0}^{\infty} l_{\vartheta\varrho} z_\varrho \quad (\vartheta \in \mathbb{N}). \tag{1.1}$$

By $(\mathcal{U}, \mathcal{V})$, we denote the class of all matrices $\mathcal{L}$ such that $\mathcal{L} : \mathcal{U} \to \mathcal{V}$. Thus, $\mathcal{L} \in (\mathcal{U}, \mathcal{V})$ if and only if the series on the right-hand side of (1.1) converges for each $\vartheta \in \mathbb{N}$. Let $\mathcal{U}$ be a sequence space. The matrix domain $\mathcal{U}_\mathcal{L}$ of an infinite matrix $\mathcal{L}$ is defined by

$$\mathcal{U}_\mathcal{L} = \{z = (z_\varrho) \in \Omega : \mathcal{L}z \in \mathcal{U}\} \tag{1.2}$$

which is a sequence space. To know more about infinite matrices one can refer to (see [1, 13, 16, 19, 20, 23]) and many others.
Consider a sequence space $\mathcal{U}$ with linear topology. Then $\mathcal{U}$ is said to be a $\mathcal{K}-$space provided that each of the maps $p_i : \mathcal{U} \to \mathbb{R}$ given by $p_i(z) = z_i$ is continuous, $\forall\, i \in \mathbb{N}$. A $\mathcal{K}-$space $\mathcal{U}$, where $\mathcal{U}$ is a complete linear space is called an $F\mathcal{K}$ space. An $F\mathcal{K}-$space whose topology is normal is called a $BK-$space. For example, the space l_p $(1 \leq p < \infty)$ and the spaces c_0, c, l_∞ are BK-space with

$$\|z\|_p = \Big(\sum_{\varrho=0}^{\infty} |z_\varrho|^p\Big)^{\frac{1}{p}} \quad \text{and} \quad \|z\|_\infty = \sup_\varrho |x_\varrho|,$$

respectively. Consider a normed space $\mathcal{U}$ contains a sequence (b_ϱ) such that for every $z \in \mathcal{U}$, there is a unique sequence (a_n) of scalars as

$$\lim_v \left\| z - \sum_{\varrho=0}^{v} a_\varrho b_\varrho \right\| = 0,$$

then (b_ϱ) is said to be a Schauder basis for $\mathcal{U}$. For a detailed study about recent results in summability theory one can refer to ([11, 17, 25, 27, 28]).

The multiplier space $\mathcal{R}(\mathcal{U}, \mathcal{V})$ of $\mathcal{U}$ and $\mathcal{V}$ is defined as

$$\mathcal{R}(\mathcal{U}, \mathcal{V}) = \{\lambda = (\lambda_\varrho) \in \Omega : \mu\lambda = (\mu_\varrho \lambda_\varrho) \in \mathcal{V} \text{ for all } \mu = (\mu_\varrho) \in \mathcal{U}\}.$$

From this notation, we can define $\alpha-, \beta-$ and $\gamma-$ duals of the sequence space $\mathcal{U}$ as

$$\mathcal{U}^\alpha = \mathcal{R}(\mathcal{U}, \ell_1),\;\; \mathcal{U}^\beta = \mathcal{R}(\mathcal{U}, cs) \text{ and } \;\mathcal{U}^\gamma = \mathcal{R}(\mathcal{U}, bs).$$

Inspired essentially by the above mentioned studies, we introduce the concept of Euler Catalan Riesz sequence spaces by means of Catalan matrix, Euler mean and Riesz mean. First, we discuss some algebraic and topological structure. Furthermore, we obtain basis for the sequence space and compute $\alpha-, \beta-, \gamma-$ duals. In addition, we characterize some matrix mappings on the space. Also, we examine some geometric properties of the newly defined sequence space in the last section.

2. Euler catalan-riesz sequence space and its toeplitz duals

In this section, we introduce the composition matrix $\widetilde{C} = (\tilde{c}_{\vartheta\varrho})$ obtained by combining the Catalan matrix, Euler mean of unit order and Riesz mean defined by

$$\tilde{c}_{\vartheta\varrho} = \begin{cases} \frac{\binom{\vartheta}{\varrho} C_\varrho C_{\vartheta-\varrho} q_\varrho}{2^\vartheta C_{\vartheta+1} Q_\vartheta}, & \text{if } 0 \leq \varrho \leq \vartheta, \\ 0, & \text{if } \varrho > \vartheta. \end{cases}$$

Next, we define Euler Catalan-Riesz sequence space as

$$\ell_p(\widehat{\mathcal{E}},\widehat{\mathcal{C}},\widehat{\mathcal{R}}) = \left\{x=(x_\varrho)\in\Omega : \sum_{\vartheta=0}^{\infty}\left|\sum_{\varrho=0}^{\vartheta}\frac{\binom{\vartheta}{\varrho}C_\varrho C_{\vartheta-\varrho}q_\varrho}{2^\vartheta C_{\vartheta+1}Q_\vartheta}x_\varrho\right|^p<\infty\right\},$$

for $1\leq p<\infty$. By using (1.2), the sequence space $\ell_p(\widehat{\mathcal{E}},\widehat{\mathcal{C}},\widehat{\mathcal{R}})$ may be redefined as

$$\ell_p(\widehat{\mathcal{E}},\widehat{\mathcal{C}},\widehat{\mathcal{R}}) = (\ell_p)_{\widetilde{C}}.$$

Let $y=(y_\varrho)$ be the $\widetilde{C}-$transform of a sequence $x=(x_\varrho)$, $\forall\ \varrho\in\mathbb{N}$ which is given as

$$y_\varrho=\sum_{i=0}^{\varrho}\frac{\binom{\varrho}{i}C_iC_{\varrho-i}q_i}{2^\varrho C_{\varrho+1}Q_\varrho}x_i.$$

Theorem 2.1. *Euler Catalan-Riesz sequence space $\ell_p(\widehat{\mathcal{E}},\widehat{\mathcal{C}},\widehat{\mathcal{R}})$ is a linear space which is also a $BK-$space with the norm $||x||_{\ell_p(\widehat{\mathcal{E}},\widehat{\mathcal{C}},\widehat{\mathcal{R}})}=||\widetilde{C}x||_p$.*

Proof. The proof is straight forward, hence we omit the proof. □

Therefore, the absolute type property does not hold on the sequence space $\ell_p(\widehat{\mathcal{E}},\widehat{\mathcal{C}},\widehat{\mathcal{R}})$ because $||x||_{\ell_p(\widehat{\mathcal{E}},\widehat{\mathcal{C}},\widehat{\mathcal{R}})}\neq |||x|||_{\ell_p(\widehat{\mathcal{E}},\widehat{\mathcal{C}},\widehat{\mathcal{R}})}$, for atleast one sequence in the space $\ell_p(\widehat{\mathcal{E}},\widehat{\mathcal{C}},\widehat{\mathcal{R}})$, where $|x|=(|x_\varrho|)$. Thus, we say that the sequence space $\ell_p(\widehat{\mathcal{E}},\widehat{\mathcal{C}},\widehat{\mathcal{R}})$ is of non-absolute type.

Theorem 2.2. *The sequence space $\ell_p(\widehat{\mathcal{E}},\widehat{\mathcal{C}},\widehat{\mathcal{R}})$ is linearly isomorphic to l_p.*

Proof. To prove this theorem, we should show the existence of a linear bijection between the spaces $\ell_p(\widehat{\mathcal{E}},\widehat{\mathcal{C}},\widehat{\mathcal{R}})$ and l_p. Define a mapping $\widehat{T}:\ell_p(\widehat{\mathcal{E}},\widehat{\mathcal{C}},\widehat{\mathcal{R}})\to l_p$ given by $y=\widehat{T}x$. Clearly, $\widehat{T}$ is linear and it is true that for every $\widehat{T}x=0$ implies that $x=0$ which shows that $\widehat{T}$ is injective.
Consider $y\in l_p$ and define the sequence $x=(x_\vartheta)$ as

$$x_\vartheta=\sum_{i=0}^{\vartheta}\frac{\binom{\vartheta}{i}(-1)^{\vartheta-i}2^iC_{i+1}H_{\vartheta-i}Q_i}{C_\vartheta q_\vartheta}y_i. \tag{2.1}$$

Then, we have

$$||x||_{\ell_p(\widehat{\mathcal{E}},\widehat{\mathcal{C}},\widehat{\mathcal{R}})}=\left(\sum_{\vartheta=0}^{\infty}\left|\sum_{\varrho=0}^{\vartheta}\frac{\binom{\vartheta}{\varrho}C_\varrho C_{\vartheta-\varrho}q_\varrho}{2^\vartheta C_{\vartheta+1}Q_\vartheta}x_\varrho\right|^p\right)^{1/p}.$$

Taking (2.1) into account, we have

$$||x||_{\ell_p(\widehat{\mathcal{E}},\widehat{\mathcal{C}},\widehat{\mathcal{R}})}=\left(\sum_{\varrho}|y_\varrho|^p\right)^{1/p}=||y||_{l_p}=||\widehat{T}x||_{l_p}<\infty.$$

Thus, $\widehat{T}$ is norm preserving and $x\in\ell_p(\widehat{\mathcal{E}},\widehat{\mathcal{C}},\widehat{\mathcal{R}})$. Hence $\widehat{T}$ is surjective and $\ell_p(\widehat{\mathcal{E}},\widehat{\mathcal{C}},\widehat{\mathcal{R}})\cong l_p$. □

Let us define a sequence $b^{(\vartheta)} = \{b_{\varrho}^{(\vartheta)}\}_{\varrho\in\mathbb{N}}$ of the elements of $\ell_p(\widehat{\mathcal{E}},\widehat{\mathcal{C}},\widehat{\mathcal{R}})$ for every $\vartheta \in \mathbb{N}$,

$$b_{\varrho}^{(\vartheta)} = \begin{cases} \frac{\binom{\varrho}{\vartheta}(-1)^{\varrho-\vartheta}2^{\vartheta}C_{\vartheta+1}H_{\varrho-\vartheta}Q_{\vartheta}}{C_{\varrho}q_{\varrho}}, \text{ if } 0 \leq \vartheta < \varrho, \\ 0, \quad \text{otherwise.} \end{cases}$$

Theorem 2.3. *The sequence* $(b^{(\vartheta)}(t))_{\varrho\in\mathbb{N}}$ *is a Schauder basis for* $\ell_p(\widehat{\mathcal{E}},\widehat{\mathcal{C}},\widehat{\mathcal{R}})$ *and every* $x \in \ell_p(\widehat{\mathcal{E}},\widehat{\mathcal{C}},\widehat{\mathcal{R}})$ *has a unique representation of the form*

$$x = \sum_{\vartheta} \lambda_{\vartheta} b^{(\vartheta)},$$

where $\lambda_{\vartheta} = (\widetilde{C}x)_{\vartheta}$, $\forall\, \vartheta \in \mathbb{N}$.

Remark 2.4. It is well known that every Banach space X with a Schauder basis is separa- ble.

From Theorem 2.3 and Remark 2.4, we can give following corollary:

Corollary 2.5. *The sequence space* $\ell_p(\widehat{\mathcal{E}},\widehat{\mathcal{C}},\widehat{\mathcal{R}}), 1 \leq p < \infty$ *is separable.*

To define the duals of this newly formed sequence space, we first study the following lemma defined by Stielglitz and Tietz [21]. Throughout the study, $\mathcal{N}$ will denote the collection of all finite subsets of $\mathbb{N}$.

Lemma 2.6. *Consider* $G = (g_{\vartheta\varrho})$ *be an infinite matrix.*

$$\sup_{\mathcal{G}\in\mathcal{N}} \sum_{\varrho} \left| \sum_{\vartheta\in\mathcal{G}} g_{\vartheta\varrho} \right|^p < \infty, \tag{2.2}$$

$$\lim_{\vartheta\to\infty} g_{\vartheta\varrho} = g_{\varrho},\ \forall\, \varrho \in \mathbb{N}, \tag{2.3}$$

$$\sup_{\vartheta\in\mathbb{N}} \sum_{\varrho} |g_{\vartheta\varrho}|^p < \infty. \tag{2.4}$$

Then, the following holds:
(i) $G \in (l_p, l_1)$ *iff* (2.2) *hold,*
(ii) $G \in (l_p, c)$ *iff* (2.3) *and* (2.4) *hold,*
(iii) $G \in (l_p, l_\infty)$ *iff* (2.4) *hold.*

Theorem 2.7. *Consider the set* a_1 *defined by*

$$a_1 \quad = \quad \left\{ a = (a_{\vartheta}) \in \Omega : \sup_{\mathcal{G}\in\mathcal{N}} \sum_{\vartheta=0}^{\infty} \left| \sum_{\varrho\in\mathcal{G}} \frac{\binom{\varrho}{\vartheta}(-1)^{\varrho-\vartheta}2^{\vartheta}C_{\vartheta+1}H_{\varrho-\vartheta}Q_{\vartheta}a_{\vartheta}}{C_{\varrho}q_{\varrho}} \right|^p < \infty \right\}.$$

Then, $[\ell_p(\widehat{\mathcal{E}},\widehat{\mathcal{C}},\widehat{\mathcal{R}})]^{\alpha} = a_1$.

Proof. Suppose that $(a_\vartheta) \in \Omega$ and $x = (x_\vartheta)$ is defined in (2.2), then we have

$$\begin{aligned} a_\vartheta x_\vartheta &= \sum_{i=0}^{\vartheta} \frac{\binom{\vartheta}{i}(-1)^{\vartheta-i}2^i C_{i+1} H_{\vartheta-i} Q_i}{C_\vartheta q_\vartheta} a_\vartheta y_i \\ &= (By)_\vartheta, \end{aligned}$$

where $B = (b_{\vartheta i})$ is a matrix defined by

$$b_{\vartheta i} = \begin{cases} \frac{\binom{\vartheta}{i}(-1)^{\vartheta-i}2^i C_{i+1} H_{\vartheta-i} Q_i}{C_\vartheta q_\vartheta} a_\vartheta, & \text{for } 0 \leq i \leq \vartheta \\ 0, & \text{otherwise.} \end{cases}$$

Thus, we have $ax = (a_\vartheta x_\vartheta) \in l_1$ whenever $x \in \ell_p(\widehat{\mathcal{E}}, \widehat{\mathcal{C}}, \widehat{\mathcal{R}})$ iff $By \in l_1$ whenever $y \in l_p$ which gives $[\ell_p(\widehat{\mathcal{E}}, \widehat{\mathcal{C}}, \widehat{\mathcal{R}})]^\alpha = a_1$. □

Theorem 2.8. *The matrix $\tau = \kappa_{\vartheta\varrho}$ is given by*

$$\kappa_{\vartheta\varrho} = \begin{cases} \sum_{i=\varrho}^{\vartheta} \frac{\binom{\varrho}{i}(-1)^{\varrho-i}2^i C_{i+1} H_{\varrho-i} Q_i}{C_\varrho q_\varrho} a_i, & \textit{for } 0 \leq \varrho \leq \vartheta \\ 0, & \textit{otherwise.} \end{cases}$$

Then, $[\ell_p(\widehat{\mathcal{E}}, \widehat{\mathcal{C}}, \widehat{\mathcal{R}})]^\beta = a_2 \cap a_3$, *where*

$$a_2 = \left\{ a = (a_\varrho) \in \Omega : \lim_{\vartheta \to \infty} \kappa_{\vartheta\varrho} \textit{ exists for each } \varrho \in \mathbb{N} \right\}$$

and

$$a_3 = \left\{ a = (a_\varrho) \in \Omega : \sup_{\vartheta \in \mathbb{N}} \sum_{\varrho} \left| \kappa_{\vartheta\varrho} \right|^p < \infty \right\}.$$

Proof. Consider the equation

$$\begin{aligned} \sum_{\varrho=0}^{\vartheta} a_\varrho x_\varrho &= \sum_{\varrho=0}^{\vartheta} a_\varrho \sum_{i=0}^{\varrho} \frac{\binom{\varrho}{i}(-1)^{\varrho-i}2^i C_{i+1} H_{\varrho-i} Q_i}{C_\varrho q_\varrho} y_i \\ &= \sum_{\varrho=0}^{\vartheta} \left(\sum_{i=\varrho}^{\vartheta} \frac{\binom{\varrho}{i}(-1)^{\varrho-i}2^i C_{i+1} H_{\varrho-i} Q_i}{C_\varrho q_\varrho} a_i \right) y_\varrho \\ &= (\tau y)_\vartheta, \end{aligned}$$

$\forall\, \vartheta \in \mathbb{N}$. Hence $ax = (a_\vartheta x_\vartheta) \in cs$ whenever $x_\vartheta \in \ell_p(\widehat{\mathcal{E}}, \widehat{\mathcal{C}}, \widehat{\mathcal{R}})$ iff $\tau y \in c$ whenever $(y_\vartheta) \in l_p$. Thus, we have

$$\lim_{\vartheta \to \infty} \kappa_{\vartheta\varrho} \text{ exists for each } \varrho \in \mathbb{N}$$

and

$$\sup_{\vartheta \in \mathbb{N}} \sum_{n} \left| \kappa_{\vartheta\varrho} \right|^p < \infty, \quad (1 < p < \infty),$$

which gives $[\ell_p(\widehat{\mathcal{E}}, \widehat{\mathcal{C}}, \widehat{\mathcal{R}})]^\beta = a_2 \cap a_3$. □

Theorem 2.9. $[\ell_p(\widehat{\mathcal{E}},\widehat{\mathcal{C}},\widehat{\mathcal{R}})]^\gamma = a_3$.

Proof. In a similar manner as in the above theorem one can easily get the proof of this theorem. □

3. Matrix transformations related to the sequence space $\ell_p(\widehat{\mathcal{E}},\widehat{\mathcal{C}},\widehat{\mathcal{R}})$

In this section, we characterize the matrix transformations from $\ell_p(\widehat{\mathcal{E}},\widehat{\mathcal{C}},\widehat{\mathcal{R}})$ into any given sequence space $\mathcal{V}$.
Also, since $\ell_p(\widehat{\mathcal{E}},\widehat{\mathcal{C}},\widehat{\mathcal{R}}) \cong l_p$ it is trivial that the equivalence $x \in \ell_p(\widehat{\mathcal{E}},\widehat{\mathcal{C}},\widehat{\mathcal{R}})$ if and only if $y \in \ell_p$ holds.
Now, we give the following two theorems to determine matrix classes on the space $\ell_p(\widehat{\mathcal{E}},\widehat{\mathcal{C}},\widehat{\mathcal{R}})$.

Theorem 3.1. *Suppose that $\mathcal{V}$ be any given sequence space and the entries of an infinite matrix $B = (b_{\vartheta\varrho})$ and $D = (d_{\vartheta\varrho})$ are connected with the relation*

$$d_{\vartheta\varrho} = \sum_{\varrho=0}^{\vartheta} \frac{\binom{\vartheta}{\varrho}(-1)^{\vartheta-\varrho}2^\varrho C_{\varrho+1}H_{\vartheta-\varrho}Q_\varrho}{C_\vartheta q_\vartheta} b_{\vartheta\varrho}, \tag{3.1}$$

for all $\vartheta, \varrho \in \mathbb{N}$. Then, $B \in (\ell_p(\widehat{\mathcal{E}},\widehat{\mathcal{C}},\widehat{\mathcal{R}}) : \mathcal{V})$ iff $\{b_{\vartheta\varrho}\}_{\varrho\in\mathbb{N}} \in \{\ell_p(\widehat{\mathcal{E}},\widehat{\mathcal{C}},\widehat{\mathcal{R}})\}^\beta$ for all $\vartheta \in \mathbb{N}$ and $D \in (\ell_p : \mathcal{V})$.

Proof. Suppose that $\mathcal{V}$ be any given sequence space. Consider that (3.1) holds between $B = (b_{\vartheta\varrho})$ and $D = (d_{\vartheta\varrho})$ and take into account that the space $\ell_p(\widehat{\mathcal{E}},\widehat{\mathcal{C}},\widehat{\mathcal{R}})$ and ℓ_p are linearly isomorphic.
Suppose $B \in (\ell_p(\widehat{\mathcal{E}},\widehat{\mathcal{C}},\widehat{\mathcal{R}}) : \mathcal{V})$ and take any $y = (y_\varrho) \in \ell_p$. Then, $D\widetilde{C}$ exists and $\{b_{\vartheta\varrho}\}_{\varrho\in\mathbb{N}} \in a_2 \cap a_3$ which implies that $\{b_{\vartheta\varrho}\}_{\varrho\in\mathbb{N}} \in \ell_1$, $\forall\vartheta \in \mathbb{N}$. Hence Dy exists and thus

$$\sum_\varrho d_{\vartheta\varrho}y_\varrho = \sum_\varrho b_{\vartheta\varrho}x_\varrho, \ \forall\vartheta \in \mathbb{N}.$$

Therefore, we have $Dy = Bx$ which gives $D \in (\ell_p : \mathcal{V})$.
Conversely, suppose $\{b_{\vartheta\varrho}\}_{\varrho\in\mathbb{N}} \in \{\ell_p(\widehat{\mathcal{E}},\widehat{\mathcal{C}},\widehat{\mathcal{R}})\}^\beta$ for all $\vartheta \in \mathbb{N}$ and $D \in (\ell_p : \mathcal{V})$ hold. Consider any $x = (x_\varrho) \in \ell_p(\widehat{\mathcal{E}},\widehat{\mathcal{C}},\widehat{\mathcal{R}})$. Then Bx exists. Thus, we have

$$\sum_{\varrho=0}^\infty b_{\vartheta\varrho}x_\varrho = \sum_{\varrho=0}^\infty \left[\sum_{i=0}^{\varrho} \frac{\binom{\varrho}{i}(-1)^{\varrho-i}2^iC_{i+1}H_{\varrho-i}Q_i}{C_\varrho q_\varrho} b_{\varrho i}\right] y_\varrho$$

$\forall\ \vartheta \in \mathbb{N}$, that $Dy = Bx$ and therefore $B \in (\ell_p(\widehat{\mathcal{E}},\widehat{\mathcal{C}},\widehat{\mathcal{R}}) : \mathcal{V})$. □

Theorem 3.2. *Suppose that $\mathcal{V}$ be any given sequence space and the entries of an infinite matrix $A = (a_{\vartheta\varrho})$ and $B = (b_{\vartheta\varrho})$ are connected with the relation*

$$b_{\vartheta\varrho} = \sum_{i=0}^{\varrho} \frac{\binom{\varrho}{i}C_iC_{\varrho-i}q_i}{2^\varrho C_{\varrho+1}Q_\varrho} a_{i\varrho},$$

$\forall\ \vartheta, \varrho \in \mathbb{N}$. Then, $A \in (\mathcal{V} : \ell_p(\widehat{\mathcal{E}},\widehat{\mathcal{C}},\widehat{\mathcal{R}}))$ iff $B \in (\mathcal{V} : \ell_p)$.

Proof. Suppose $z = (z_\varrho) \in \mathcal{V}$ and consider the following equality

$$\sum_{\varrho=0}^{\upsilon} b_{\vartheta\varrho} z_\varrho = \sum_{i=0}^{\varrho} \frac{\binom{\varrho}{i} C_i C_{\varrho-i} q_i}{2^\varrho C_{\varrho+1} Q_\varrho} \left(\sum_{\varrho=0}^{\upsilon} a_{i\varrho} z_\varrho \right), \ \forall \upsilon, \varrho \in \mathbb{N},$$

which yields as $\upsilon \to \infty$ that $(Bz)_\vartheta = \{\widetilde{C}(Az)\}_\vartheta$, $\forall \vartheta \in \mathbb{N}$. Since $Az \in (\ell_p(\widehat{\mathcal{E}}, \widehat{\mathcal{C}}, \widehat{\mathcal{R}}))$ whenever $z \in \mathcal{V}$ iff $Bz \in \ell_p$ whenever $z \in \mathcal{V}$. □

4. Geometric properties of $\ell_p(\widehat{\mathcal{E}}, \widehat{\mathcal{C}}, \widehat{\mathcal{R}})$

Here, we study some geometric properties of the space $\ell_p(\widehat{\mathcal{E}}, \widehat{\mathcal{C}}, \widehat{\mathcal{R}})$, $(1 < p < \infty)$.
Let $(\mathcal{U}, ||.||_\mathcal{U})$ be a Banach space. Then $\mathcal{U}$ admits the Banach-Saks property if every bounded sequence $x = (x_\vartheta)$ contains a subsequence $z = (z_\vartheta)$ such that the Cesàro mean which is defined as

$$t_i(z) = \frac{1}{\vartheta + 1} \sum_{i=0}^{\vartheta} z_i$$

is norm convergent [7].

Further, $\mathcal{U}$ is said to have the weak Banach-Saks property whenever for any weakly null sequence $x = (x_\vartheta) \subset \mathcal{U}$ and there exists a subsequence $z = (z_\vartheta)$ whose Cesàro means sequence $\frac{1}{\vartheta+1} \sum_{j=0}^{\vartheta} z_j$ is norm convergent. A Banach space $\mathcal{U}$ has Banach-Saks type p, if every weakly null sequence $x = (x_\vartheta)$ in $\mathcal{U}$ admits $z = (z_\vartheta)$ such that, for some $G > 0$,

$$\left\| \sum_{i=0}^{\vartheta} z_\vartheta \right\|_\mathcal{U} \leq G(\vartheta + 1)^{1/p},$$

$\forall\ \vartheta \in \mathbb{N}$ [15]. In [9] Gracia-Falset introduce the following coefficient:

$$R(\mathcal{U}) = \sup\{\lim_{\vartheta\to\infty} \inf ||x_\vartheta - x|| : (x_\vartheta) \subset B(\mathcal{U}),\ x_\vartheta \to 0 \text{ weakly},\ x \in B(\mathcal{U})\} < 2, \tag{4.1}$$

where $B(\mathcal{U})$ is the unit ball of $\mathcal{U}$.

Remark 4.1. Suppose a Banach space $\mathcal{U}$ with $R(\mathcal{U}) < 2$. Then $\mathcal{U}$ is said to contain the weak fixed point property [9].

For a detailed study about Banach-Saks type p one can refer to ([8, 18, 26]).

Theorem 4.2. *The sequence space $\ell_p(\widehat{\mathcal{E}}, \widehat{\mathcal{C}}, \widehat{\mathcal{R}})$ is of Banach-Saks type p.*

Proof. Consider a sequence (ϵ_ϑ) of positive numbers such that $\sum_\vartheta \epsilon_\vartheta \leq 1/2$ and (x_ϑ) is the weak null sequence in $B(\ell_p(\widehat{\mathcal{E}}, \widehat{\mathcal{C}}, \widehat{\mathcal{R}}))$. Fix $k_0 = x_0 = 0$ and $k_1 = x_{\vartheta_1} = x_1$. Then $\exists\, g_1 \in \mathbb{N}$ such that

$$\Big\| \sum_{i=g_1+1}^{\infty} k_1(i) e^{(i)} \Big\|_{\ell_p(\widehat{\mathcal{E}}, \widehat{\mathcal{C}}, \widehat{\mathcal{R}})} \quad < \quad \varepsilon_1.$$

Since (x_ϑ) is weakly null sequence implying $x_\vartheta \to 0$ coordinatewise, then $\exists\, \vartheta_2 \in \mathbb{N}$ such that

$$\Big\| \sum_{i=0}^{g_1} x_\vartheta(i) e^{(i)} \Big\|_{\ell_p(\widehat{\mathcal{E}}, \widehat{\mathcal{C}}, \widehat{\mathcal{R}})} \quad < \quad \varepsilon_1,$$

where $\vartheta \geq \vartheta_2$. Set $k_2 = x_{\vartheta_2}$, so there exists $g_2 > g_1$ such that

$$\Big\| \sum_{i=g_2+1}^{\infty} k_2(i) e^{(i)} \Big\|_{\ell_p(\widehat{\mathcal{E}}, \widehat{\mathcal{C}}, \widehat{\mathcal{R}})} \quad < \quad \varepsilon_2.$$

Since $x_\vartheta \to 0$ coordinatewise, then $\exists\, \vartheta_3 > \vartheta_2$ such that

$$\Big\| \sum_{i=0}^{g_2} x_\vartheta(i) e^{(i)} \Big\|_{\ell_p(\widehat{\mathcal{E}}, \widehat{\mathcal{C}}, \widehat{\mathcal{R}})} \quad < \quad \varepsilon_2,$$

for $\vartheta \geq \vartheta_3$. Proceeding similarly, we can find two subsequences (g_j) and (ϑ_j) such that

$$\Big\| \sum_{i=0}^{g_j} x_\vartheta(i) e^{(i)} \Big\|_{\ell_p(\widehat{\mathcal{E}}, \widehat{\mathcal{C}}, \widehat{\mathcal{R}})} \quad < \quad \varepsilon_j,$$

for $\vartheta \geq \vartheta_{j+1}$ and

$$\Big\| \sum_{i=g_j+1}^{\infty} k_j(i) e^{(i)} \Big\|_{\ell_p(\widehat{\mathcal{E}}, \widehat{\mathcal{C}}, \widehat{\mathcal{R}})} \quad < \quad \varepsilon_j$$

where $k_j = x_{\vartheta_j}$. Thus,

$$\begin{aligned}
\Big\| \sum_{j=0}^{\vartheta} k_j \Big\|_{\ell_p(\widehat{\mathcal{E}}, \widehat{\mathcal{C}}, \widehat{\mathcal{R}})} &= \Big\| \sum_{j=0}^{\vartheta} \Big(\sum_{i=0}^{g_{j-1}} k_j(i) e^{(i)} + \sum_{i=g_{j-1}+1}^{g_j} k_j(i) e^{(i)} + \sum_{i=g_j+1}^{\infty} k_j(i) e^{(i)} \Big) \Big\|_{\ell_p(\widehat{\mathcal{E}}, \widehat{\mathcal{C}}, \widehat{\mathcal{R}})} \\
&\leq \Big\| \sum_{j=0}^{\vartheta} \Big(\sum_{i=0}^{g_{j-1}} k_j(i) e^{(i)} \Big) \Big\|_{\ell_p(\widehat{\mathcal{E}}, \widehat{\mathcal{C}}, \widehat{\mathcal{R}})} + \Big\| \sum_{j=0}^{\vartheta} \Big(\sum_{i=g_{j-1}+1}^{g_j} k_j(i) e^{(i)} \Big) \Big\|_{\ell_p(\widehat{\mathcal{E}}, \widehat{\mathcal{C}}, \widehat{\mathcal{R}})} \\
&+ \Big\| \sum_{j=0}^{\vartheta} \Big(\sum_{i=g_j+1}^{\infty} k_j(i) e^{(i)} \Big) \Big\|_{\ell_p(\widehat{\mathcal{E}}, \widehat{\mathcal{C}}, \widehat{\mathcal{R}})} \\
&\leq \Big\| \sum_{j=0}^{\vartheta} \Big(\sum_{i=g_{j-1}+1}^{g_j} k_j(i) e^{(i)} \Big) \Big\|_{\ell_p(\widehat{\mathcal{E}}, \widehat{\mathcal{C}}, \widehat{\mathcal{R}})} + 2 \sum_{j=0}^{\vartheta} \varepsilon_j.
\end{aligned}$$

Clearly, $||x_\vartheta||_{\ell_p(\widehat{\mathcal{E}},\widehat{\mathcal{C}},\widehat{\mathcal{R}})} \leq 1$. Thus,

$$\begin{aligned}\left\|\sum_{j=0}^{\vartheta}\left(\sum_{i=g_{j-1}+1}^{g_j} k_j(i)e^{(i)}\right)\right\|^p_{\ell_p(\widehat{\mathcal{E}},\widehat{\mathcal{C}},\widehat{\mathcal{R}})} &= \sum_{j=0}^{\vartheta}\sum_{i=g_{j-1}+1}^{g_j}\left|\sum_{u=0}^{i}\frac{\binom{i}{u}C_uC_{i-u}q_u}{2^iC_{i+1}Q_i}k_j(u)\right|^p \\ &\leq \sum_{j=0}^{\vartheta}\sum_{i=0}^{\infty}\left|\sum_{u=0}^{i}\frac{\binom{i}{u}C_uC_{i-u}q_u}{2^iC_{i+1}Q_i}k_j(u)\right|^p \\ &\leq \vartheta+1.\end{aligned}$$

Therefore,

$$\left\|\sum_{j=0}^{\vartheta}\left(\sum_{i=g_{j-1}+1}^{g_j} k_j(i)e^{(i)}\right)\right\|_{\ell_p(\widehat{\mathcal{E}},\widehat{\mathcal{C}},\widehat{\mathcal{R}})} \leq (\vartheta+1)^{1/p}.$$

Since $1 \leq (\vartheta+1)^{1/p}$, $\forall\, \vartheta \in \mathbb{N}$, we get

$$\begin{aligned}\left\|\sum_{j=0}^{\vartheta} k_j\right\|_{\ell_p(\widehat{\mathcal{E}},\widehat{\mathcal{C}},\widehat{\mathcal{R}})} &\leq 1+(\vartheta+1)^{1/p} \\ &\leq 2(\vartheta+1)^{1/p}.\end{aligned}$$

Hence, $\ell_p(\widehat{\mathcal{E}},\widehat{\mathcal{C}},\widehat{\mathcal{R}})$ contains Banach-saks type p. □

Remark 4.3. $R(\ell_p(\widehat{\mathcal{E}},\widehat{\mathcal{C}},\widehat{\mathcal{R}})) = R(l_p) = 2^{1/p}$, as $\ell_p(\widehat{\mathcal{E}},\widehat{\mathcal{C}},\widehat{\mathcal{R}})$ is linearly isomorphic to l_p.

By using (4.1) and remark (4.3), we have

Theorem 4.4. *$\ell_p(\widehat{\mathcal{E}},\widehat{\mathcal{C}},\widehat{\mathcal{R}})$ has a weak fixed point property, where $1 < p < \infty$.*

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