Iranian Journal of Mathematical Sciences and Informatics
Vol. 21, No. 1 (2026), pp 197-216
DOI: 10.66224/ijmsi.21.1.197

Some Further Refinements of Hermite-Hadamard Type Inequalities for Harmonically Convex and P-Convex Functions via Fractional Integrals

Maamar BENBACHIR[a,b*], Farid CHABANE[c], İmdat İŞCAN[d]

[a]Faculty of Sciences, Saad Dahlab University, Blida1, Algeria
[b]National Higher School of Mathematics, Algiers, Algeria
[c]Laboratory of Mathematics and Applied Sciences, University of Ghardaia, Algeria
[d]Faculty of Arts and Sciences, Department of Mathematics, Giresun University, Giresun, Turkey

E-mail: mbenbachir2001@gmail.com
E-mail: chabane.farid@univ-ghardaia.dz
E-mail: imdat.iscan@giresun.edu.tr

Abstract. It is well known that Hermite-Hadamard inequality generates an estimate of the mean value of the convex function over a bounded interval, in this work we investigate some Hermite-Hadamard type integral inequalities for p-convex functions and harmonically convex functions in fractional integral forms. Precisely, we provide extensions better than those existing in earlier works.

Keywords: Hermite-Hadamard inequality, Harmonically convex function, P-Convex function, Fractional integral, Refinement.

2000 Mathematics subject classification: 26A51, 26A33, 26D10.

*Corresponding Author

Received 22 February 2022; Accepted 16 August 2022
©2026 Academic Center for Education, Culture and Research TMU

[Downloaded from ijmsi.ir on 2026-10-06]

[DOI: 10.66224/ijmsi.21.1.197]

1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on a real interval I and let a, b be two real constants in I such that $a < b$. The following inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \ \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

is well known in the literature as the inequality of Hermite-Hadamard (proved by Charles Hermite in 1883 and rediscovered ten years later by Jack Hadamard) [12, 13], it generates an estimate of the mean value of the convex function over a bounded interval. In this context, the Hermite-Hadamard inequality, which is, we can say, the first fundamental result for convex functions with a natural geometric interpretation and many applications, has attracted and continues to attract a great deal of interest in elementary mathematics. Many mathematicians have devoted their efforts to generalize, refine, counterbalance and extend to different classes of functions: quasi-convex functions, log-convex functions, p-convex functions, etc. or apply it to special means (p-logarithmic means, same average, etc.). In recent years there have been many extensions, generalizations and results similar to the Hermite Hadamard inequality for convex functions [1, 2, 3, 4, 5, 7, 9, 10, 14, 15, 17, 18, 19, 20, 22, 23, 24, 26, 28, 29, 30, 31, 32, 33, 34, 35, 37].

In [11], Fejér established the following Hermite-Hadamard-Fejér inequality which is the weighted generalization of the Hermite–Hadamard inequality (1.1):

Theorem 1.1. *Let $f : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function. Then the inequality*

$$f\left(\frac{a+b}{2}\right)\int_a^b w(x)\mathrm{d}x \leq \frac{1}{b-a}\int_a^b f(x)w(x)\,\mathrm{d}x \ \leq \frac{f(a)+f(b)}{2}\int_a^b w(x)\mathrm{d}x, \tag{1.2}$$

holds where $w : [a, b] \to \mathbb{R}$ is nonnegative, integrable and symmetric with respect to $(a + b)/2$.

Definition 1.2. Let $f \in L^1[a, b]$. The right-hand side and left-hand side Riemann-Liouville fractional integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha \in \mathbb{R}_+$, with $b > a \geq 0$ are defined by :

$$\begin{aligned} J^{\alpha}_{a+}f(x) &= \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)\mathrm{d}t \qquad x > a, \\ J^{\alpha}_{b-}f(x) &= \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)\mathrm{d}t \qquad b > x, \end{aligned}$$

respectively, where $\Gamma(\alpha)$ is the well known Gamma function.

In [31], Sarıkaya et. al. gave Hermite–Hadamard's inequalities in fractional integral forms as follows.

[Downloaded from ijmsi.ir on 2026-10-06]

[DOI: 10.66224/ijmsi.21.1.197]

Theorem 1.3 ([31]). *Let $f : [a, b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L^1[a, b]$. If f is a convex function on $[a, b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2\,(b-a)^{\alpha}}\left[J^{\alpha}_{a+}f(b) + J^{\alpha}_{b-}f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{1.3}$$

with $\alpha > 0$.

In [9], Farissi obtained the following refinement of the inequality (1.1) with a simple proof method.

Theorem 1.4. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on a real interval I and let a, b be two real constants in I such that $a < b$. The inequalities*

$$f\left(\frac{a+b}{2}\right) \leq s \leq \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leq S \ \leq \frac{f(a)+f(b)}{2}, \tag{1.4}$$

hold. Where

$$s = \frac{1}{2}f\left(\frac{b+3a}{4}\right) + \frac{1}{2}f\left(\frac{3b+a}{4}\right), \tag{1.5}$$

$$S = \frac{1}{2}\left[\frac{f(b)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right]. \tag{1.6}$$

In [9], the author also proved the result:

Theorem 1.5. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on a real interval I and let a, b be two real constants in I such that $a < b$. The inequalities*

$$f\left(\frac{a+b}{2}\right) \leq s(t) \leq \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leq S(t) \ \leq \frac{f(a)+f(b)}{2}, \tag{1.7}$$

hold. Where

$$s(t) = tf\left(\frac{(2-t)a+tb}{2}\right) + (1-t)f\left(\frac{(1-t)a+(1+t)b}{2}\right),$$

$$S(t) = \frac{1}{2}\left[f\left(tb+(1-t)a\right) + tf(a) + (1-t)f(b)\right],\ t \in [0,1].$$

Definition 1.6 ([18]). A function $f : I \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be harmonically convex, if

$$f\left(\frac{ab}{ta+(1-t)b}\right) \leq tf(a) + (1-t)f(b), \tag{1.8}$$

for all $a, b \in I$ and $t \in [0, 1]$. If the inequality in (1.8) is reversed, then f is said to be harmonically concave.

Theorem 1.7. [22] *Let $f : I \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$, if $f \in L^1([a, b])$. Then, we have:*

$$f\left(\frac{2ab}{a+b}\right) \leq \left(\frac{ab}{b-a}\right)\int_a^b \frac{f(x)}{x^2}\ \mathrm{d}x \ \leq \frac{f(a)+f(b)}{2}. \tag{1.9}$$

[Downloaded from ijmsi.ir on 2026-10-06]

[DOI: 10.66224/ijmsi.21.1.197]

Theorem 1.8. [24] *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a function such that $f \in L^1[a,b]$, where $a,b \in I$ with $a < b$. If f is a harmonically convex function on $[a;b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{2ab}{a+b}\right) \leq \left(\frac{ab}{b-a}\right)^{\alpha} \frac{\Gamma(\alpha+1)}{2^{1-\alpha}} \left[J^{\alpha}_{\frac{a+b}{2ab}+}(f\circ g)(\frac{1}{a}) + J^{\alpha}_{\frac{a+b}{2ab}-}(f\circ g)(\frac{1}{b})\right] \leq \frac{f(a)+f(b)}{2}, \tag{1.10}$$

with $\alpha > 0$ and $g(x) = \frac{1}{x},\ x \in [\frac{1}{a}, \frac{1}{b}]$.

Definition 1.9 ([27])**.** A function $w : [a,b] \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is said to be harmonically symmetric with respect to $\frac{2ab}{a+b}$ if $w(x) = w\left(\left[a^{-1}+b^{-1}-x^{-1}\right]^{-1}\right)$ holds for all $x \in [a,b]$.

In [6], Chen and Wu presented a Hermite–Hadamard–Fejer type inequality for harmonically convex functions as follows:

Theorem 1.10. *Let $f : I \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a harmonically convex function and $a,b \in I$ with $a < b$, if $f \in L^1([a,b])$ and $w : [a,b] \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is nonnegative, integrable and harmonically symmetric with respect to $\frac{2ab}{a+b}$, then*

$$f\left(\frac{2ab}{a+b}\right)\int_a^b \frac{w(x)}{x^2}\mathrm{d}x \leqslant \int_a^b \frac{f(x)w(x)}{x^2}\mathrm{d}x \ \leqslant \frac{f(a)+f(b)}{2}\int_a^b \frac{w(x)}{x^2}\mathrm{d}x. \tag{1.11}$$

In [36], Zhang and Wan gave the definition of a p-convex function on $I \subset \mathbb{R}$, in [16], İşcan gave a different definition of a p-convex function on $I \subset (0,\infty)$:

Definition 1.11. Let $I \subset (0,\infty)$ be a real interval and $p \in \mathbb{R}\setminus\{0\}$. A function $f : I \to \mathbb{R}$ is said to be a p-convex function, if

$$f([\alpha x^p + (1-\alpha)y^p]^{\frac{1}{p}}) \leq \alpha f(x) + (1-\alpha)f(y),$$

for all $x,y \in I$ and $\alpha \in [0,1]$.

In [8, Theorem 5], if we take $I \subset (0,\infty)$, $p \in \mathbb{R}\setminus\{0\}$ and $h(t) = t$, then we have the following theorem.

Theorem 1.12. [16] *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a p-convex function, $p \in \mathbb{R}\setminus\{0\}$ and $a,b \in I$ with $a < b$. If $f \in L^1[a,b]$, then the following inequalities hold:*

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}\,dx \ \leq \frac{f(a)+f(b)}{2}. \tag{1.12}$$

Theorem 1.13. [22] *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a p-convex function, $p \in \mathbb{R}\setminus\{0\}$, $\alpha > 0$ and $a,b \in I$ with $a < b$. If $f \in L^1[a,b]$, then the following inequalities hold:*
(i) if $p > 0$

$$\begin{aligned} f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) &\leq \frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b^p-a^p)^{\alpha}}\left[J^{\alpha}_{\frac{a^p+b^p}{2}+}(f\circ g)(b^p) + J^{\alpha}_{\frac{a^p+b^p}{2}-}(f\circ g)(a^p)\right] \\ &\leq \frac{f(a)+f(b)}{2}, \end{aligned} \tag{1.13}$$

[Downloaded from ijmsi.ir on 2026-10-06]

[DOI: 10.66224/ijmsi.21.1.197]

with :

$$g(x) = x^{\frac{1}{p}}, x \in [a^p, b^p],$$

(ii) if $p < 0$

$$\begin{aligned} f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) &\leq \frac{\Gamma(\alpha+1)}{2^{1-\alpha}(a^p-b^p)^{\alpha}}\left[J^{\alpha}_{\frac{a^p+b^p}{2}+}(f\circ g)(a^p) + J^{\alpha}_{\frac{a^p+b^p}{2}-}(f\circ g)(b^p)\right] \\ &\leq \frac{f(a)+f(b)}{2}, \end{aligned} \tag{1.14}$$

with:

$$g(x) = x^{\frac{1}{p}}, x \in [b^p, a^p].$$

Definition 1.14. Let $p \in \mathbb{R} \setminus \{0\}$. A function $w : [a,b] \subset (0,\infty) \to \mathbb{R}$ is said to be p-symmetric with respect to $\left[\frac{a^p+b^p}{2}\right]^{1/p}$ if $w(x) = w\left(\left[a^p+b^p-x^p\right]^{1/p}\right)$ holds for all $x \in [a,b]$.

In [25], Kunt and İşcan presented a Hermite–Hadamard–Fejer type inequality for p-convex functions as follows:

Theorem 1.15. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a p-convex function, $p \in \mathbb{R} \setminus \{0\}$ and $a, b \in I$ with $a < b$. If $f \in L^1([a,b])$ and $w : [a,b] \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is nonnegative, integrable and p-symmetric with respect to $\left(\frac{a^p+b^p}{2}\right)^{1/p}$, then the following inequalities hold:*

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\int_a^b \frac{w(x)}{x^{1-p}}\mathrm{d}x \leqslant \int_a^b \frac{f(x)w(x)}{x^{1-p}}\mathrm{d}x \leqslant \frac{f(a)+f(b)}{2}\int_a^b \frac{w(x)}{x^{1-p}}\mathrm{d}x. \tag{1.15}$$

The aim of this paper is to give some refinements for the inequalities (1.9), (1.10), (1.11), (1.12), (1.13), (1.14) and (1.15).

2. Refinements of some inequalities for harmonically convex functions

The main aim of this section is to prove estimations better than those of (1.9), (1.10) and (1.11).

Theorem 2.1. *Let $f : I \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$, if $f \in L^1([a,b])$ and $w : [a,b] \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is nonnegative, integrable and harmonically symmetric with respect to $\frac{2ab}{a+b}$, then*

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right)\int_a^b \frac{w(x)}{x^2}\mathrm{d}x &\leqslant \ell\int_a^b \frac{w(x)}{x^2}\mathrm{d}x \leqslant \int_a^b \frac{f(x)w(x)}{x^2}\,\mathrm{d}x \\ &\leqslant L\int_a^b \frac{w(x)}{x^2}\mathrm{d}x \leqslant \frac{f(a)+f(b)}{2}\int_a^b \frac{w(x)}{x^2}\mathrm{d}x, \end{aligned} \tag{2.1}$$

[Downloaded from ijmsi.ir on 2026-10-06]

[DOI: 10.66224/ijmsi.21.1.197]

where

$$\ell = \frac{1}{2}f\left(\frac{4ab}{b+3a}\right)+\frac{1}{2}f\left(\frac{4ab}{3b+a}\right), \tag{2.2}$$

$$L = \frac{1}{2}\left[\frac{f(b)+f(b)}{2}+f\left(\frac{2ab}{a+b}\right)\right]. \tag{2.3}$$

Proof. Applying (1.11) on each of the subintervals $[a,H]$ and $[H,b]$ with $H=\frac{2ab}{a+b}$
Firstly, applying (1.11) on $[a,H]$, we get:

$$f\left(\frac{2aH}{a+H}\right)\int_a^H \frac{w(x)}{x^2}\mathrm{d}x \leqslant \int_a^H \frac{f(x)w(x)}{x^2}\,\mathrm{d}x \leqslant \frac{f(a)+f(H)}{2}\int_a^H \frac{w(x)}{x^2}\mathrm{d}x,$$

which implies

$$f\left(\frac{4ab}{3b+a}\right)\int_a^H \frac{w(x)}{x^2}\mathrm{d}x \leqslant \int_a^H \frac{f(x)w(x)}{x^2}\,\mathrm{d}x \leqslant \frac{f(a)+f(H)}{2}\int_a^H \frac{w(x)}{x^2}\mathrm{d}x. \tag{2.4}$$

Secondly, applying (1.9) on $[H,b]$, we obtain

$$f\left(\frac{2bH}{b+H}\right)\int_H^b \frac{w(x)}{x^2}\mathrm{d}x \leqslant \int_H^b \frac{f(x)w(x)}{x^2}\,\mathrm{d}x \leqslant \frac{f(b)+f(H)}{2}\int_H^b \frac{w(x)}{x^2}\mathrm{d}x,$$

thus

$$f\left(\frac{4ab}{b+3a}\right)\int_H^b \frac{w(x)}{x^2}\mathrm{d}x \leqslant \int_H^b \frac{f(x)w(x)}{x^2}\,\mathrm{d}x \leqslant \frac{f(b)+f(H)}{2}\int_H^b \frac{w(x)}{x^2}\mathrm{d}x. \tag{2.5}$$

Since g is harmonically symmetric with respect to $\frac{2ab}{a+b}$, we have

$$\int_a^H \frac{w(x)}{x^2}\mathrm{d}x = \int_H^b \frac{w(x)}{x^2}\mathrm{d}x = \frac{1}{2}\int_a^b \frac{w(x)}{x^2}\mathrm{d}x.$$

So, summing up (2.5) and (2.4) side by side we get:

$$\begin{aligned}\frac{1}{2}\left[f\left(\frac{4ab}{b+3a}\right)+f\left(\frac{4ab}{3b+a}\right)\right]\int_a^b \frac{w(x)}{x^2}\mathrm{d}x &\leqslant \int_a^b \frac{f(x)w(x)}{x^2}\,\mathrm{d}x \\ &\leqslant \frac{1}{2}\left[\frac{f(b)+f(H)}{2}+\frac{f(a)+f(H)}{2}\right]\int_a^b \frac{w(x)}{x^2}\mathrm{d}x \\ &= \frac{1}{2}\left[\frac{f(b)+f(b)}{2}+f(H)\right]\int_a^b \frac{w(x)}{x^2}\mathrm{d}x.\end{aligned}$$

As f is a harmonically convex function, so

$$\begin{aligned}f\left(\frac{2ab}{a+b}\right) &= f\left[\frac{2\frac{4ab}{3a+b}\frac{4ab}{3b+a}}{\frac{4ab}{3a+b}+\frac{4ab}{3b+a}}\right] \\ &\leqslant \frac{1}{2}f\left(\frac{4ab}{b+3a}\right)+\frac{1}{2}f\left(\frac{4ab}{3b+a}\right) \\ &= \ell,\end{aligned} \tag{2.6}$$

[Downloaded from ijmsi.ir on 2026-10-06]

[DOI: 10.66224/ijmsi.21.1.197]

and

$$\begin{aligned}\frac{f(a)+f(b)}{2} &= \frac{1}{2}[f(a)+f(b)] \\ &\geqslant \frac{1}{2}\left[\frac{f(b)+f\left(b\right)}{2}+f\left(H\right)\right] \\ &= L,\end{aligned} \tag{2.7}$$

which ends the proof. □

Corollary 2.2. *If we choose $w(x)=1$ in (2.1), then we get*

$$f\left(\frac{2ab}{a+b}\right) \leqslant \ell \leqslant \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,\mathrm{d}x \leqslant L \leqslant \frac{f(a)+f(b)}{2}.$$

Corollary 2.3. *If we choose $w(x)=\frac{1}{\Gamma(\alpha)}\left[\left(\frac{1}{a}-\frac{1}{x}\right)^{\alpha-1}+\left(\frac{1}{x}-\frac{1}{b}\right)^{\alpha-1}\right]$ in (2.1), then by using the equalities*

$$\begin{aligned}&J^{\alpha}_{\frac{1}{b}+}(f\circ g)(\frac{1}{a})+J^{\alpha}_{\frac{1}{a}-}(f\circ g)(\frac{1}{b}) \\ =\;& \frac{1}{\Gamma(\alpha)}\int_a^b\left[\left(\frac{1}{a}-\frac{1}{x}\right)^{\alpha-1}+\left(\frac{1}{x}-\frac{1}{b}\right)^{\alpha-1}\right]\frac{f(x)}{x^2}dx,\ g(x)=1/x,\end{aligned}$$

and

$$\frac{1}{\Gamma(\alpha)}\int_a^b\left[\left(\frac{1}{a}-\frac{1}{x}\right)^{\alpha-1}+\left(\frac{1}{x}-\frac{1}{b}\right)^{\alpha-1}\right]\frac{1}{x^2}dx=\frac{2}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha},$$

we get

$$\begin{aligned}f\left(\frac{2ab}{a+b}\right) &\leqslant \ell \leqslant \frac{\Gamma(\alpha+1)}{2}\left(\frac{ab}{b-a}\right)^{\alpha}\left[J^{\alpha}_{\frac{1}{b}+}(f\circ g)(\frac{1}{a})+J^{\alpha}_{\frac{1}{a}-}(f\circ g)(\frac{1}{b})\right] \\ &\leqslant L \leqslant \frac{f(a)+f(b)}{2}.\end{aligned}$$

Corollary 2.4. *If we choose*

$$w(x)=\begin{cases}\frac{1}{\Gamma(\alpha)}\left(\frac{1}{a}-\frac{1}{x}\right)^{\alpha-1}, & x\in[a,H] \\ \frac{1}{\Gamma(\alpha)}\left(\frac{1}{x}-\frac{1}{b}\right)^{\alpha-1}, & x\in[H,b]\end{cases}$$

in (2.1), then by using the equalities

$$\begin{aligned}&J^{\alpha}_{\frac{a+b}{2ab}+}(f\circ g)(\frac{1}{a})+J^{\alpha}_{\frac{a+b}{2ab}-}(f\circ g)(\frac{1}{b}) \\ =\;& \frac{1}{\Gamma(\alpha)}\left\{\int_a^H\left(\frac{1}{a}-\frac{1}{x}\right)^{\alpha-1}\frac{f(x)}{x^2}dx+\int_H^b\left(\frac{1}{x}-\frac{1}{b}\right)^{\alpha-1}\frac{f(x)}{x^2}dx\right\},\end{aligned}$$

[Downloaded from ijmsi.ir on 2026-10-06]

[DOI: 10.66224/ijmsi.21.1.197]

and

$$\begin{aligned} & \frac{1}{\Gamma(\alpha)}\left\{\int_a^H\left(\frac{1}{a}-\frac{1}{x}\right)^{\alpha-1}\frac{1}{x^2}dx+\int_H^b\left(\frac{1}{x}-\frac{1}{b}\right)^{\alpha-1}\frac{1}{x^2}dx\right\} \\ = & \frac{2^{1-\alpha}}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha}, \end{aligned}$$

we get

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right) &\leqslant \ell \leqslant \frac{\Gamma(\alpha+1)}{2^{1-\alpha}}\left(\frac{ab}{b-a}\right)^{\alpha}\left[J^{\alpha}_{\frac{a+b}{2ab}+}(f\circ g)(\frac{1}{a})+J^{\alpha}_{\frac{a+b}{2ab}-}(f\circ g)(\frac{1}{b})\right] \\ &\leqslant L \leqslant \frac{f(a)+f(b)}{2}. \end{aligned} \tag{2.8}$$

So, we obtain a refinement of the inequality (1.10).

Theorem 2.5. *Let $f : I \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$, if $f \in L^1([a, b])$. Then there exist two real functions l, L such that* (1.9) *takes the following form:*

$$f\left(\frac{2ab}{a+b}\right) \leqslant \ell(t) \leqslant \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x \ \leqslant L(t) \leqslant \frac{f(a)+f(b)}{2}, \tag{2.9}$$

where

$$\begin{aligned} \ell(t) &= (1-t)f\left(\frac{2ab}{(1-t)b+(1+t)a}\right)+tf\left(\frac{2ab}{(2-t)b+ta}\right), \\ L(t) &= \frac{1}{2}\left[f\left(\frac{ab}{tb+(1-t)a}\right)+tf(a)+(1-t)f(b)\right],\ t\in[0,1]. \end{aligned}$$

Proof. Applying (1.9) on the subinterval $[a, H_t]$ with $H_t = \frac{ab}{(1-t)b+ta}$and $t \neq 0$ we get

$$f\left[\frac{2aH_t}{a+H_t}\right] \leqslant \frac{aH_t}{H_t-a}\int_a^{H_t}\frac{f(x)}{x^2}\mathrm{d}x \leqslant \frac{f(a)+f\left(H_t\right)}{2},$$

so

$$f\left(\frac{2ab}{(2-t)b+ta}\right) \leqslant \frac{ab}{t\left[b-a\right]}\int_a^{H_t}\frac{f(x)}{x^2}\mathrm{d}x \leqslant \frac{f(a)+f\left(H_t\right)}{2}. \tag{2.10}$$

Applying (1.9) on the subinterval $[H_t, b]$ with $t \neq 1$
we obtain 6

$$f\left[\frac{2H_tb}{H_t+b}\right] \leqslant \frac{H_tb}{b-H_t}\int_{H_t}^b\frac{f(x)}{x^2}\mathrm{d}x \leqslant \frac{f(b)+f\left(H_t\right)}{2},$$

thus

$$f\left(\frac{2ab}{(1-t)b+(1+t)a}\right) \leqslant \frac{ab}{(1-t)\left[b-a\right]}\int_{H_t}^b\frac{f(x)}{x^2}\mathrm{d}x \leqslant \frac{f(b)+f\left(H_t\right)}{2}. \tag{2.11}$$

[Downloaded from ijmsi.ir on 2026-10-06]

[DOI: 10.66224/ijmsi.21.1.197]

Multiplying (2.10) by t and (2.11) by $(1-t)$ then by summing up the obtained inequalities side to side, we can deduce that

$$\underbrace{(1-t)f\left(\frac{2ab}{(1-t)b+(1+t)a}\right)+tf\left(\frac{2ab}{(2-t)b+ta}\right)}_{\ell(t)} \leqslant \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x \leqslant \underbrace{[f(H_t)+(1-t)f(b)+tf(a)]}_{L(t)}.$$

As f is a harmonically convex function on $[a,b]$, we get:

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right) &= f\left[\frac{\frac{2ab}{(1-t)b+(1+t)a}\frac{2ab}{(2-t)b+ta}}{t\left(\frac{2ab}{(1-t)b+(1+t)a}\right)+(1-t)\left(\frac{2ab}{(2-t)b+ta}\right)}\right] \\ &\leqslant (1-t)f\left(\frac{2ab}{(1-t)b+(1+t)a}\right)+tf\left(\frac{2ab}{(2-t)b+ta}\right) \\ &\leqslant \ell(t), \end{aligned}$$

and

$$\begin{aligned} \frac{f(a)+f(b)}{2} &= \frac{1}{2}[f(a)+f(b)] \\ &\geqslant \frac{1}{2}\left[f(H_t)+(1-t)f(b)+tf(a)\right], \end{aligned}$$

this achieves the proof. □

Lemma 2.6. *Let f and g be two integrable functions on I. Then one has*

$$\left(\frac{ab}{b-a}\right)^{\alpha}\Gamma(\alpha)\left[J^{\alpha}_{\frac{a+b}{2ab}+}(f\circ g)(\frac{1}{a})+J^{\alpha}_{\frac{a+b}{2ab}-}(f\circ g)(\frac{1}{b})\right]$$

$$= \left(\frac{ab}{b-a}\right)^{\alpha}\left\{\int_{\frac{a+b}{2ab}}^{\frac{1}{a}}(\frac{1}{a}-x)^{\alpha-1}f\left(\frac{1}{x}\right)\,dx+\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}(x-\frac{1}{b})^{\alpha-1}f\left(\frac{1}{x}\right)\mathrm{d}x\right\} \tag{2.12}$$

$$= \left(\frac{ab}{b-a}\right)^{\alpha}\left\{\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}(x-\frac{1}{b})^{\alpha-1}f\left(\frac{1}{\frac{1}{b}+\frac{1}{a}-x}\right)\,dx+\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}(x-\frac{1}{b})^{\alpha-1}f\left(\frac{1}{x}\right)\mathrm{d}x\right\}$$

$$= \int_0^{\frac{1}{2}} t^{\alpha-1}\left[\frac{f\left(\frac{ab}{ta+(1-t)b}\right)+\left(\frac{ab}{tb+(1-t)a}\right)}{2}\right]\mathrm{d}t, \tag{2.13}$$

with $\alpha>0$ and $g(x)=\frac{1}{x}, x\in[\frac{1}{a},\frac{1}{b}], \quad a,b\in I \quad a<b.$

Proof. We use the change of variables
$y=\frac{1}{a}+\frac{1}{b}-x$ to prove (2.12) and $x=\dfrac{ta+(1-t)b}{ab}$ to prove (2.13).
For $x=\frac{1}{b}$, $y=\frac{1}{b}+\frac{1}{a}-\frac{1}{b} \implies y=\frac{1}{a}$ for $x=\frac{a+b}{2ab}$, $y=\frac{1}{b}+\frac{1}{a}-\frac{a+b}{2ab} \implies y=\frac{a+b}{2ab}$ which gives (2.12), for $x=\frac{1}{a}$, $a=\dfrac{ab}{ta+(1-t)b} \implies t=0$ for $x=\frac{1}{b}$, $b=\dfrac{ab}{ta+(1-t)b} \implies t=1$

[Downloaded from ijmsi.ir on 2026-10-06]

[DOI: 10.66224/ijmsi.21.1.197]

for $x=\frac{a+b}{2ab}$, $\frac{a+b}{2ab}=\frac{ta+(1-t)b}{ab} \implies t=\frac{1}{2}$
for $x=\dfrac{ta+(1-t)b}{ab} \implies \mathrm{d}t=-\frac{ab}{b-a}\mathrm{d}x$. Using (2.12) we get (2.13). □

Theorem 2.7. *Let $f: I\subset(0,\infty)\to\mathbb{R}$ be a harmonically convex function and $a,b\in I$ with $a<b$, if $f\in L^1([a,b])$. Then there exists a real function which depends on α such that the inequalities* (1.10) *take the following form:*

$$f\left(\frac{2ab}{a+b}\right)\leqslant K(\alpha)\leqslant\left(\frac{ab}{b-a}\right)^{\alpha}\Gamma(\alpha)\left[J^{\alpha}_{\frac{a+b}{ab}+}(f\circ g)(\frac{1}{a})+J^{\alpha}_{\frac{a+b}{ab}-}(f\circ g)(\frac{1}{b})\right]\leqslant\frac{f(a)+f(b)}{2}, \tag{2.14}$$

with $\alpha>0$ and $g(x)=\frac{1}{x}, x\in[\frac{1}{a},\frac{1}{b}]$,
with $K(\alpha)=\frac{\alpha}{2^{1-\alpha}}\int_0^{\frac{1}{2}}t^{\alpha-1}\left[f\left(\frac{4ab}{(1+2t)b+(3-2t)a}\right)+f\left(\frac{4ab}{(1+2t)a+(3-2t)b}\right)\right]\mathrm{d}t$.

Proof. Let $H_t=\frac{ab}{(1-t)a+tb},\ t\in[0,1]$ .As f is a harmonically convex function on $I=[a,b]$, we have for all $x,y\in I$ (by taking $t=\frac{1}{2}$ in (1.8)) we get

$$f\left(\frac{2xy}{x+y}\right)=f\left[\frac{2\frac{4xy}{3x+y}\frac{4xy}{3y+x}}{\frac{4xy}{3x+y}+\frac{4xy}{3y+x}}\right]\leqslant\frac{1}{2}f\left(\frac{4xy}{y+3x}\right)+\frac{1}{2}f\left(\frac{4xy}{3y+x}\right)\leqslant\frac{f(x)+f(y)}{2}, \tag{2.15}$$

for $x=H_{1-t},\ y=H_t$ and for all $t\in[0,1]$, we obtain from the inequality (2.15), the following result

$$\begin{aligned}f\left(\frac{2ab}{a+b}\right)&\leqslant\frac{1}{2}f\left(\frac{4H_{1-t}H_t}{H_t+3H_{1-t}}\right)+\frac{1}{2}f\left(\frac{4H_{1-t}H_t}{H_t+H_{1-t}}\right)\\&\leqslant\frac{f\left(H_{1-t}\right)+f\left(H_t\right)}{2},\end{aligned}$$

which allows us to write

$$\begin{aligned}f\left(\frac{2ab}{a+b}\right)&\leqslant\frac{f\left(\frac{4ab}{(1+2t)b+(3-2t)a}\right)+f\left(\frac{4ab}{(1+2t)a+(3-2t)b}\right)}{2}\\&\leqslant\frac{f\left(H_{1-t}\right)+f\left(H_t\right)}{2}\leqslant\frac{f(a)+f(b)}{2}.\end{aligned}$$

By multiplying the above inequality by $2t^{\alpha-1}$, then integrating the obtained inequality with respect to t on $[0,\frac{1}{2}]$, we get :

$$\begin{aligned}f\left(\frac{2ab}{a+b}\right)&\leqslant\frac{\alpha}{2^{1-\alpha}}\int_0^{\frac{1}{2}}t^{\alpha-1}\left[f\left(\frac{4ab}{(1+2t)b+(3-2t)a}\right)+f\left(\frac{4ab}{(1+2t)a+(3-2t)b}\right)\right]\mathrm{d}t\\&\leqslant\frac{\alpha}{2^{1-\alpha}}\int_0^{\frac{1}{2}}t^{\alpha-1}\left[f\left(H_{1-t}\right)+f\left(H_t\right)\right]\mathrm{d}t\leqslant\frac{[f(a)+f(b)]}{2}.\end{aligned}$$

Applying Lemma (2.6) we obtain:

$$\begin{aligned}f\left(\frac{2ab}{a+b}\right)&\leqslant\underbrace{\frac{\alpha}{2^{1-\alpha}}\int_0^{\frac{1}{2}}t^{\alpha-1}\left[f\left(\frac{4ab}{(1+2t)b+(3-2t)a}\right)+f\left(\frac{4ab}{(1+2t)a+(3-2t)b}\right)\right]\mathrm{d}t}_{K(\alpha)}\\&\leqslant\frac{1}{2^{1-\alpha}}\left(\frac{ab}{b-a}\right)^{\alpha}\Gamma(\alpha+1)\left[J^{\alpha}_{\frac{a+b}{ab}+}(f\circ g)(\frac{1}{a})+J^{\alpha}_{\frac{a+b}{ab}-}(f\circ g)(\frac{1}{b})\right]\leqslant\frac{[f(a)+f(b)]}{2}.\end{aligned}$$

[Downloaded from ijmsi.ir on 2026-10-06]
[DOI: 10.66224/ijmsi.21.1.197]

□

3. Refinements of some inequalities for p-convex functions

The main aim of this section is to prove estimations better than those of (1.12), (1.13), (1.14) and (1.15). Also, some of the results obtained in this section are reduced to the results obtained in the previous section in special cases, as well as the improvements of the inequalities (1.1), (1.2) and (1.3) in special cases.

Theorem 3.1. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a p-convex function $p \in \mathbb{R}\setminus\{0\}$. and $a, b \in I$ with $a < b$. If $f \in L^1[a,b]$, then there exist two real numbers $\mathcal{M}$ and $\mathcal{F}$ such that* (1.12) *takes the following*

$$\begin{aligned} f\left(\frac{[a^p+b^p]^{\frac{1}{p}}}{2}\right)\int_a^b \frac{w(x)}{x^{1-p}}\mathrm{d}x &\leqslant \mathcal{M}\int_a^b \frac{w(x)}{x^{1-p}}\mathrm{d}x \leqslant \int_a^b \frac{f(x)w(x)}{x^{1-p}}\mathrm{d}x \\ &\leqslant \mathcal{F}\int_a^b \frac{w(x)}{x^{1-p}}\mathrm{d}x \leqslant \frac{f(a)+f(b)}{2}\int_a^b \frac{w(x)}{x^{1-p}}\mathrm{d}x, \quad (3.1) \end{aligned}$$

where

$$\mathcal{M} = \frac{1}{2}\left[f\left(\left[\frac{3a^p+b^p}{4}\right]^{1/p}\right) + f\left(\left[\frac{a^p+3b^p}{4}\right]^{1/p}\right)\right],$$

$$\mathcal{F} = \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\right].$$

Proof. Applying (1.15) on each of the subintervals $[a, A_p]$ and $[A_p, b]$ with $A_p = \left[\frac{a^p+b^p}{2}\right]^{1/p}$.

Firstly, on the subinterval $[a, A_p]$ inequality (1.15) transform to :

$$f\left(\left[\frac{a^p+A_p^p}{2}\right]^{\frac{1}{p}}\right)\int_a^{A_p} \frac{w(x)}{x^{1-p}}\mathrm{d}x \leqslant \int_a^{A_p} \frac{f(x)w(x)}{x^{1-p}}\mathrm{d}x \leqslant \frac{f(a)+f(A_p)}{2}\int_a^{A_p} \frac{w(x)}{x^{1-p}}\mathrm{d}x,$$

which can be written

$$f\left(\left[\frac{3a^p+b^p}{4}\right]^{1/p}\right)\int_a^{A_p} \frac{w(x)}{x^{1-p}}\mathrm{d}x \leqslant \int_a^{A_p} \frac{f(x)w(x)}{x^{1-p}}\mathrm{d}x \leqslant \frac{f(a)+f(A_p)}{2}\int_a^{A_p} \frac{w(x)}{x^{1-p}}\mathrm{d}x. \quad (3.2)$$

Secondly on the subinterval $[A_p, b]$ the inequality (1.15) leads to :

$$f\left(\left[\frac{A_p^p+b^p}{2}\right]^{\frac{1}{p}}\right)\int_{A_p}^b \frac{w(x)}{x^{1-p}}\mathrm{d}x \leqslant \int_{A_p}^b \frac{f(x)w(x)}{x^{1-p}}\mathrm{d}x \leqslant \frac{f(A_p)+f(b)}{2}\int_{A_p}^b \frac{w(x)}{x^{1-p}}\mathrm{d}x,$$

which takes the form:

$$f\left(\left[\frac{a^p+3b^p}{4}\right]^{1/p}\right)\int_{A_p}^b \frac{w(x)}{x^{1-p}}\mathrm{d}x \leqslant \int_{A_p}^b \frac{f(x)w(x)}{x^{1-p}}\mathrm{d}x \leqslant \frac{f(A_p)+f(b)}{2}\int_{A_p}^b \frac{w(x)}{x^{1-p}}\mathrm{d}x. \quad (3.3)$$

Since g is p-symmetric with respect to A_p, we have

$$\int_a^{A_p} \frac{w(x)}{x^{1-p}}\mathrm{d}x = \int_{A_p}^b \frac{w(x)}{x^{1-p}}\mathrm{d}x = \frac{1}{2}\int_a^b \frac{w(x)}{x^{1-p}}\mathrm{d}x. \quad (3.4)$$

[Downloaded from ijmsi.ir on 2026-10-06]

[DOI: 10.66224/ijmsi.21.1.197]

By summing (3.2) and (3.3), also via (3.4), we get :

$$\mathcal{M}\int_a^b \frac{w(x)}{x^{1-p}}\mathrm{d}x \leqslant \int_{A_p}^b \frac{f(x)w(x)}{x^{1-p}}\mathrm{d}x \leqslant \mathcal{F}\int_a^b \frac{w(x)}{x^{1-p}}\mathrm{d}x.$$

As f is p-convex function on $[a,b]$, so :

$$\begin{aligned} f\left(A_p\right) &= f\left(\left[\frac{\frac{3a^p}{4}+\frac{b^p}{4}+\frac{a^p}{4}+\frac{3b^p}{4}}{2}\right]^{1/p}\right) \\ &\leqslant \frac{1}{2}\left[f\left(\left[\frac{a^p+3b^p}{4}\right]^{1/p}\right)+f\left(\left[\frac{3a^p+b^p}{4}\right]^{1/p}\right)\right]=\mathcal{M}, \end{aligned}$$

and

$$\frac{f(a)+f(b)}{2} \geqslant \frac{1}{2}\left[\frac{f(a)+f(b)}{2}+2f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\right]=\mathcal{F}.$$

This ends the proof. □

Corollary 3.2. *If we choose $w(x)=1$ in (3.1), then we get*

$$f\left(\frac{\left[a^p+b^p\right]^{\frac{1}{p}}}{2}\right) \leqslant \mathcal{M} \leqslant \frac{p}{b^p-a^p}\int_a^b \frac{f(x)w(x)}{x^{1-p}}\mathrm{d}x \leqslant \mathcal{F} \leqslant \frac{f(a)+f(b)}{2}. \quad (3.5)$$

Remark 3.3. (i) For $p=1$, the inequality (3.5) reduces to the inequality (1.4).
(ii) For $p=-1$, the inequality (3.5) reduces to the inequality (2.1).

Corollary 3.4. *If we choose*

$$w(x)=\begin{cases} \frac{1}{\Gamma(\alpha)}\left[(x^p-a^p)^{\alpha-1}+(b^p-x^p)^{\alpha-1}\right] & p>0, \\ \frac{1}{\Gamma(\alpha)}\left[(a^p-x^p)^{\alpha-1}+(x^p-b^p)^{\alpha-1}\right] & p<0, \end{cases}$$

in (3.1), then by using the equalities

$$\begin{aligned} & J^{\alpha}_{a^p+}(f\circ g)(b^p)+J^{\alpha}_{b^p-}(f\circ g)(a^p) \\ = \;& \frac{p}{\Gamma(\alpha)}\int_a^b \left[(x^p-a^p)^{\alpha-1}+(b^p-x^p)^{\alpha-1}\right]\frac{f(x)}{x^{1-p}}dx,\ g(x)=x^{1/p}, p>0, \end{aligned}$$

$$\begin{aligned} & J^{\alpha}_{b^p+}(f\circ g)(a^p)+J^{\alpha}_{a^p-}(f\circ g)(b^p) \\ = \;& \frac{p}{\Gamma(\alpha+1)}\int_a^b \left[(a^p-x^p)^{\alpha-1}+(x^p-b^p)^{\alpha-1}\right]\frac{f(x)}{x^{1-p}}dx,\ g(x)=x^{1/p}, p<0, \end{aligned}$$

and

$$\begin{aligned} \frac{p}{\Gamma(\alpha)}\int_a^b \left[(x^p-a^p)^{\alpha-1}+(b^p-x^p)^{\alpha-1}\right]\frac{1}{x^{1-p}}dx &= \frac{2}{\Gamma(\alpha+1)}\left(b^p-a^p\right)^{\alpha}, p>0, \\ \frac{p}{\Gamma(\alpha)}\int_a^b \left[(a^p-x^p)^{\alpha-1}+(x^p-b^p)^{\alpha-1}\right]\frac{1}{x^{1-p}}dx &= \frac{2}{\Gamma(\alpha+1)}\left(b^p-a^p\right)^{\alpha}, p<0, \end{aligned}$$

[Downloaded from ijmsi.ir on 2026-10-06]

[DOI: 10.66224/ijmsi.21.1.197]

we get

$$\begin{aligned} f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) &\leqslant \mathcal{M} \leqslant \frac{\Gamma(\alpha+1)}{2\left(b^p-a^p\right)^{\alpha}}\left[J^{\alpha}_{a^p+}(f\circ g)(b^p)+J^{\alpha}_{b^p-}(f\circ g)(a^p)\right] \\ &\leqslant \mathcal{F} \leqslant \frac{f(a)+f(b)}{2}, p>0, \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) &\leqslant \mathcal{M} \leqslant \frac{\Gamma(\alpha+1)}{2\left(a^p-b^p\right)^{\alpha}}\left[J^{\alpha}_{a^p-}(f\circ g)(b^p)+J^{\alpha}_{b^p+}(f\circ g)(a^p)\right] \\ &\leqslant \mathcal{F} \leqslant \frac{f(a)+f(b)}{2}, p<0. \end{aligned} \tag{3.7}$$

Remark 3.5. (i) For $p=1$, the inequality (3.6) reduces to a refinement of the inequality (1.3).

(ii) For $p=-1$, the inequality (3.7) reduces to the inequality

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right) &\leqslant \ell \leqslant \left(\frac{ab}{b-a}\right)^{\alpha}\frac{\Gamma(\alpha+1)}{2}\left[J^{\alpha}_{a^{-1}-}(f\circ g)(b^{-1})+J^{\alpha}_{b^{-1}+}(f\circ g)(a^{-1})\right] \\ &\leqslant L \leqslant \frac{f(a)+f(b)}{2}, \end{aligned}$$

where ℓ and L are as in (2.2) and (2.3).

Corollary 3.6. *If we choose*

$$w(x)=\begin{cases} \frac{1}{\Gamma(\alpha)}\left(x^p-a^p\right)^{\alpha-1}, & x\in[a,A_p] \\ \frac{1}{\Gamma(\alpha)}\left(b^p-x^p\right)^{\alpha-1}, & x\in[A_p,b] \end{cases}, A_p=\frac{a^p+b^p}{2}, p>0,$$

in (3.1), then for $p>0$, by using the equalities

$$\begin{aligned} & J^{\alpha}_{A_p^p+}(f\circ g)(b^p)+J^{\alpha}_{A_p^p-}(f\circ g)(a^p) \\ = & \frac{1}{\Gamma(\alpha)}\left\{\int_{A_p}^{b}\left(b^p-x^p\right)^{\alpha-1}\frac{f(x)}{x^{1-p}}dx+\int_{a}^{A_p}\left(x^p-a^p\right)^{\alpha-1}\frac{f(x)}{x^{1-p}}dx\right\} \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{\Gamma(\alpha)}\left\{\int_{A_p}^{b}\left(b^p-x^p\right)^{\alpha-1}\frac{1}{x^{1-p}}dx+\int_{a}^{A_p}\left(x^p-a^p\right)^{\alpha-1}\frac{1}{x^{1-p}}dx\right\} \\ = & \frac{2^{1-\alpha}}{\Gamma(\alpha+1)}\left(b^p-a^p\right)^{\alpha}, \end{aligned}$$

we get

$$\begin{aligned} f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) &\leqslant \mathcal{M} \leqslant \frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b^p-a^p)^{\alpha}}\left[J^{\alpha}_{\frac{a^p+b^p}{2}+}(f\circ g)(b^p)+J^{\alpha}_{\frac{a^p+b^p}{2}-}(f\circ g)(a^p)\right] \\ &\leqslant \mathcal{F} \leqslant \frac{f(a)+f(b)}{2}. \end{aligned} \tag{3.8}$$

So, we obtain a refinement of the inequality (1.13).

[Downloaded from ijmsi.ir on 2026-10-06]

[DOI: 10.66224/ijmsi.21.1.197]

Similarly, for $p<0$, if we choose

$$w(x)=\begin{cases}\frac{1}{\Gamma(\alpha)}\left(a^p-x^p\right)^{\alpha-1}, & x\in[a,A_p]\\ \frac{1}{\Gamma(\alpha)}\left(x^p-b^p\right)^{\alpha-1}, & x\in[A_p,b]\end{cases},A_p=\frac{a^p+b^p}{2},$$

in (3.1), then we get

$$\begin{aligned}f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) &\leqslant \mathcal{M}\leqslant\frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b^p-a^p)^{\alpha}}\left[J^{\alpha}_{\frac{a^p+b^p}{2}+}(f\circ g)(a^p)+J^{\alpha}_{\frac{a^p+b^p}{2}-}(f\circ g)(b^p)\right]\\ &\leqslant \mathcal{F}\leqslant\frac{f(a)+f(b)}{2}.\end{aligned}\tag{3.9}$$

So, we obtain a refinement of the inequality (1.14).

Remark 3.7. (i) For $p=1$, the inequality (3.8) reduces to the inequality

$$\begin{aligned}f\left(\frac{a+b}{2}\right) &\leqslant s\leqslant\frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b-a)}\left[J^{\alpha}_{\frac{a+b}{2}+}f(b)+J^{\alpha}_{\frac{a+b}{2}-}f(a)\right]\\ &\leqslant S\leqslant\frac{f(a)+f(b)}{2},\end{aligned}$$

where s and S are given in (1.5) and (1.6).

(ii) For $p=-1$, the inequality (3.9) reduces to the inequality (2.8).

Theorem 3.8. *Let $f:I\subset(0,\infty)\to\mathbb{R}$ be a p-convex function $p\in\mathbb{R}\setminus\{0\}$ and $a,b\in I$ with $a<b$. If $f\in L^1[a,b]$, then there exist two real functions $\mathcal{M}(t)$ and $\mathcal{F}(t)$ such that* (1.12) *takes the following*

$$f\left(\frac{[a^p+b^p]^{\frac{1}{p}}}{2}\right)\leqslant\mathcal{M}(t)\leqslant\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}\mathrm{d}x\leqslant\mathcal{F}(t)\leqslant\frac{f(a)+f(b)}{2},\tag{3.10}$$

where

$$\mathcal{M}(t)=(1-t)f\left(\left[\frac{(1-t)a^p+(1+t)b^p}{2}\right]^{1/p}\right)+tf\left(\left[\frac{(2-t)a^p+tb^p}{2}\right]^{1/p}\right),$$

$$\mathcal{F}(t)=\frac{1}{2}\left[tf(a)+(1-t)f(b)+f([(1-t)a^p+tb^p]^{1/p})\right],\ t\in[0,1].$$

Proof. Let $A_p(t)=[(1-t)a^p+tb^p]^{1/p},\ t\in[0,1]$. Applying inequality (1.12) on each of the subintervals $[a,A_p(t)]$; $[A_p(t),b]$ for all $t\in[0;1]$.
Firstly on the subinterval $[a,A_p(t)]$ with $t\neq1$ inequality (1.12) becomes :

$$\begin{aligned}f\left(\frac{[a^p+A_p(t)]^{\frac{1}{p}}}{2}\right)&\leqslant\frac{p}{A_p^p(t)-a^p}\int_a^{A_p(t)}\frac{f(x)}{x^{1-p}}\mathrm{d}x\\ &\leqslant\frac{f(a)+f(A_p(t))}{2},\end{aligned}$$

which gives

$$f\left(\left[\frac{(2-t)a^p+tb^p}{2}\right]^{1/p}\right)\leqslant\frac{p}{t(b^p-a^p)}\int_a^{A_p(t)}\frac{f(x)}{x^{1-p}}\mathrm{d}x\leqslant\frac{f(a)+f(A_p(t))}{2}.\tag{3.11}$$

[Downloaded from ijmsi.ir on 2026-10-06]

[DOI: 10.66224/ijmsi.21.1.197]

Secondly on the subinterval $[A_p(t), b]$ inequality (1.12) becomes :

$$f\left(\frac{[b^p+[(1-t)a^p+tb^p]]^{\frac{1}{p}}}{2}\right) \leqslant \frac{p}{[b^p-A_p^p(t)]}\int_{A_p(t)}^{b}\frac{f(x)}{x^{1-p}}\mathrm{d}x \leqslant \frac{f(b)+f(A_p(t))}{2},$$

which allows us to write

$$f\left(\left[\frac{(1-t)a^p+(1+t)b^p}{2}\right]^{1/p}\right) \leqslant \frac{p}{(1-t)b^p-a^p}\int_{A_p(t)}^{b}\frac{f(x)}{x^{1-p}}\mathrm{d}x \leqslant \frac{f(b)+f(A_p(t))}{2}. \tag{3.12}$$

Multiplying (3.12) by $(1-t)$ and (3.11) by t and summing up side to side, we get :

$$\underbrace{(1-t)f\left(\left[\frac{(1-t)a^p+(1+t)b^p}{2}\right]^{1/p}\right)+tf\left(\left[\frac{(2-t)a^p+tb^p}{2}\right]^{1/p}\right)}\leqslant \frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}\mathrm{d}x$$
$$\leqslant \underbrace{\frac{1}{2}\left[tf(a)+(1-t)f(b)+f(A_p(t))\right]}_{\mathcal{F}(t)}.$$

As f is a p-convex function on $[a,b]$, so we can easily write :

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right) \leqslant \mathcal{M}(t),$$

and

$$\frac{f(a)+f(b)}{2} \geqslant \mathcal{F}(t),$$

the proof is ended. □

Remark 3.9. (i) For $p=1$, the inequality (3.10) reduces to the inequality (1.7).
(ii) For $p=-1$, the inequality (3.10) reduces to the inequality (2.9).

Theorem 3.10. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a p-convex function $p \in \mathbb{R}\setminus\{0\}$ and $a,b \in I$ with $a<b$. If $f \in L^1[a,b]$, then there exists a real function $S(\alpha)$ such that Theorem (1.13) takes the following form :*
(i) if $p>0$

$$f\left(\frac{[a^p+b^p]^{\frac{1}{p}}}{2}\right) \leq S(\alpha) \leq \frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b^p-a^p)^\alpha}\left[J^\alpha_{\frac{a^p+b^p}{2}+}(f\circ g)(b^p)+J^\alpha_{\frac{a^p+b^p}{2}-}(f\circ g)(a^p)\right] \leq \frac{f(a)+f(b)}{2}, \tag{3.13}$$

with :

$$g(x)=x^{\frac{1}{p}}, x\in[a^p,b^p]$$

[Downloaded from ijmsi.ir on 2026-10-06]
[DOI: 10.66224/ijmsi.21.1.197]

where

$$S(\alpha)=\frac{\alpha}{2^{1-\alpha}}\int_0^{1/2}t^{\alpha-1}\left[f\left(\left[\frac{(1+2t)a^p+(1-2t)b^p}{4}\right]^{1/p}\right)+f\left(\left[\frac{(1+2t)b^p+(1-2t)a^p}{4}\right]^{1/p}\right)\right]\mathrm{d}t,$$

(*ii*) *if* $p<0$

$$\begin{aligned} f\left(\frac{[a^p+b^p]^{\frac{1}{p}}}{2}\right) &\leq S(\alpha)\leq\frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b^p-a^p)^\alpha}\left[J^\alpha_{\frac{a^p+b^p}{2}+}(f\circ g)(a^p)+J^\alpha_{\frac{a^p+b^p}{2}-}(f\circ g)(b^p)\right] \\ &\leq \frac{f(a)+f(b)}{2}, \end{aligned} \tag{3.14}$$

with

$$g(x)=x^{\frac{1}{p}}, x\in[b^p,a^p].$$

Proof. Let $p>0$, and f be p-convex function, so we have :

$$f\left(\left[\frac{x^p+y^p}{2}\right]^{1/p}\right)=f\left(\left[\frac{\frac{3x^p}{4}+\frac{y^p}{4}+\frac{x^p}{4}+\frac{3y^p}{4}}{2}\right]^{1/p}\right) \tag{3.15}$$

$$\leqslant\frac{f\left(\left[\frac{x^p+3y^p}{4}\right]^{1/p}\right)+f\left(\left[\frac{3x^p+y^p}{4}\right]^{1/p}\right)}{2}$$

and

$$\frac{f(x)+f(y)}{2}\geq\frac{f\left(\left[\frac{x^p+3y^p}{4}\right]^{1/p}\right)+f\left(\left[\frac{3x^p+y^p}{4}\right]^{1/p}\right)}{2}. \tag{3.16}$$

If we choose $x=A_p(1-t)=(ta^p+(1-t)b^p)^{\frac{1}{p}}$ and $y=A_p(t)=(tb^p+(1-t)a^p)^{\frac{1}{p}}$ in (3.15) and (3.16), then we get

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\leq\frac{f\left(\left[\frac{(1+2t)a^p+(1-2t)b^p}{4}\right]^{1/p}\right)+f\left(\left[\frac{(1+2t)b^p+(1-2t)a^p}{4}\right]^{1/p}\right)}{2}$$

and

$$\frac{f(A_p(1-t))+f(A_p(t))}{2}\geq\frac{f\left(\left[\frac{(1+2t)a^p+(1-2t)b^p}{4}\right]^{1/p}\right)+f\left(\left[\frac{(1+2t)b^p+(1-2t)a^p}{4}\right]^{1/p}\right)}{2}$$

but, we have $\frac{f(a)+f(b)}{2}\geq\frac{f(A_p(1-t))+f(A_p(t))}{2}$ i.e

$$\begin{aligned} f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right) &\leq\frac{f\left(\left[\frac{(1+2t)a^p+(1-2t)b^p}{4}\right]^{1/p}\right)+f\left(\left[\frac{(1+2t)b^p+(1-2t)a^p}{4}\right]^{1/p}\right)}{2} \\ &\leq\frac{f(A_p(1-t))+f(A_p(t))}{2}\leq\frac{f(a)+f(b)}{2}. \end{aligned}$$

[Downloaded from ijmsi.ir on 2026-10-06]

[DOI: 10.66224/ijmsi.21.1.197]

Multiplying the above inequalities by $2t^{\alpha-1}$, then integrating the result inequalities with respect to t on $[0,1/2)$, we obtain :

$$\begin{aligned} f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right) &\leq \frac{\alpha}{2^{1-\alpha}}\int_0^{1/2} t^{\alpha-1}\left[f\left(\left[\frac{(1+2t)a^p+(1-2t)b^p}{4}\right]^{1/p}\right)\right. \\ &\quad \left. +f\left(\left[\frac{(1+2t)b^p+(1-2t)a^p}{4}\right]^{1/p}\right)\right]\mathrm{d}t \\ &\leq \frac{\alpha}{2^{1-\alpha}}\int_0^{1/2} t^{\alpha-1}\left[f(A_p(1-t))+f(A_p(t))\right]\mathrm{d}t \leq \frac{f(a)+f(b)}{2}. \end{aligned}$$

As:

$$\begin{aligned} &\frac{\alpha}{2^{1-\alpha}}\int_0^{1/2} t^{\alpha-1}\left[f(A_p(1-t))+f(A_p(t))\right]\mathrm{d}t \\ = \quad &\frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b^p-a^p)^\alpha}\left[J^\alpha_{\frac{a^p+b^p}{2}+}(f\circ g)(b^p)+J^\alpha_{\frac{a^p+b^p}{2}-}(f\circ g)(a^p)\right] \end{aligned}$$

and

$$\frac{\alpha}{2^{1-\alpha}}\int_0^{1/2} t^{\alpha-1}\left[f\left(\left[\frac{(1+2t)a^p+(1-2t)b^p}{4}\right]^{1/p}\right)+f\left(\left[\frac{(1+2t)b^p+(1-2t)a^p}{4}\right]^{1/p}\right)\right]\mathrm{d}t = S(\alpha).$$

So, we get

$$\begin{aligned} f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right) &\leq S(\alpha) \leq \frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b^p-a^p)^\alpha}\left[J^\alpha_{\frac{a^p+b^p}{2}+}(f\circ g)(b^p)+J^\alpha_{\frac{a^p+b^p}{2}-}(f\circ g)(a^p)\right] \\ &\leq \frac{f(a)+f(b)}{2}. \end{aligned}$$

We prove (ii) as (i) . □

Remark 3.11. (i) For $p=1$, the inequality (3.13) reduces to the inequality

$$\begin{aligned} f\left(\frac{[a^p+b^p]^{\frac{1}{p}}}{2}\right) &\leq S(\alpha) \leq \frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b^p-a^p)^\alpha}\left[J^\alpha_{\frac{a^p+b^p}{2}+}(f\circ g)(b^p)+J^\alpha_{\frac{a^p+b^p}{2}-}(f\circ g)(a^p)\right] \\ &\leq \frac{f(a)+f(b)}{2}, \end{aligned}$$

where $S(\alpha)=\frac{\alpha}{2^{1-\alpha}}\int_0^{1/2} t^{\alpha-1}\left[f\left(\frac{(1+2t)a+(1-2t)b}{4}\right)+f\left(\frac{(1+2t)b+(1-2t)a}{4}\right)\right]\mathrm{d}t$

(ii) For $p=-1$, the inequality (3.14) reduces to the inequality

$$\begin{aligned} f\left(\frac{[a^p+b^p]^{\frac{1}{p}}}{2}\right) &\leq S(\alpha) \leq \frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b^p-a^p)^\alpha}\left[J^\alpha_{\frac{a^p+b^p}{2}+}(f\circ g)(a^p)+J^\alpha_{\frac{a^p+b^p}{2}-}(f\circ g)(b^p)\right] \\ &\leq \frac{f(a)+f(b)}{2}, \end{aligned}$$

where

$$S(\alpha)=\frac{\alpha}{2^{1-\alpha}}\int_0^{1/2} t^{\alpha-1}\left[f\left(\left[\frac{(1+2t)a^{-1}+(1-2t)b^{-1}}{4}\right]^{-1}\right)+f\left(\left[\frac{(1+2t)b^{-1}+(1-2t)a^{-1}}{4}\right]^{-1}\right)\right]\mathrm{d}t.$$

[Downloaded from ijmsi.ir on 2026-10-06]

[DOI: 10.66224/ijmsi.21.1.197]

4. Conclusion

In the article, we provide refinements of the Hermite-Hadamard inequality, harmonically convex and p-convex functions. Our proved results can be presented as improvements and extensions of some exiting results. We can obtain more results with respect to the type of convexity.

Acknowledgments

The authors express their gratitude to the referee for very helpful and detailed comments and suggestions, which have significantly improved the presentation of this paper.

References

1. M. Adil Khan, Y. Chu, T. Ullah Khan, J. Khan, Some New Inequalities of Hermite-Hadamard Type for s-convex Functions with Applications, *Open Math.*, **15**(1), (2017), 1414–1430.
2. M. Amer Latif, S. Hussain, Y.-M. Chu, Generalized Hermite–Hadamard Type Inequalities for Differentiable Harmonically–convex and Harmonically Quasi–convex Functions, *J. Math. Ineq.,* **15**(2), (2021), 755–766.
3. M. Amer Latif, S. Rashid, S. S. Dragomir, Y.-M. Chu, Hermite–Hadamard Type Inequalities for Co-ordinated Convex and Qausi-convex Functions and their Applications, *J. Inequa. and Appl.*, **2019**(1), (2019), 317.
4. M. U. Awan, N. Akhtar, S. Iftikhar, M. A. Noor, Y.-M. Chu, New Hermite–Hadamard Type Inequalities for n-polynomial Harmonically Convex Functions, *J. Ineq. Appl.*, **2020**(1), (2020), 125.
5. G. D. Anderson, M. K. Vamanamurthy, M. Vuorinen, Generalized Convexity and Inequalities, *J. Math. Anal. Appl.*, **335**(2), (2007), 1294-1308.
6. F. Chen, S. Wu, Fejér and Hermite-Hadamard Type Inqequalities for Harmonically Convex Functions, *J. Appl. Math.*, **2014**(1), (2014), Article Id:386806.
7. Y. -M. Chu, S. Rashid, T. Abdeljawad, A. Khalid, H. Kalsoom, On New Generalized Unified Bounds via Generalized Exponentially Harmonically s-convex Functions on Fractal Sets, *Adv. Diff. Equ.*, **2021**(1), (2021), 218.
8. Z.B. Fang, R. Shi, On the (p,h)-convex Function and Some Integral Inequalities, *J. Inequal. Appl.*, **45**, (2014), 1-16.
9. A. EL Farissi, Simple Proof and Refinement of Hermite-Hadamard Inequality, *J. Math. Inequal.*, **4**, (2010), 365-369.
10. A. El Farissi, M. Benbachir, M. Dahmane, An Extension of the Hermite-Hadamard Inequality for Convex Symmetrized Functions, *Real Analysis Exchange*, **38**(2), 467-474. DOI: 10.14321/realanalexch.38.2.0467
11. L. Fejér, Uber die Fourierreihen, ii, *Math. Naturwise. Anz Ungar. Akad.*, **24**, (1906), 369-390.
12. J. Hadamard, Étude Sur Les Propriétés des Fonctions Entières et en Particulier D'une Fonction Considérée par Riemann, *J. Math. Pures Appl.*, **58**, (1893), 171-215.
13. C. Hermite, Sur Deux Limites D'une Intégrale Définie, *Mathesis*, **3**, (1883), 82-83.
14. İ. İscan, Hermite-Hadamard Type Inequalities for GA-s-convex Functions, *Le Matematiche*, **LXIX**, (2014), Fasc. II, pp. 129146.

[Downloaded from ijmsi.ir on 2026-10-06]

[DOI: 10.66224/ijmsi.21.1.197]

15. İ. İscan, M. Kunt, Hermite-Hadamard-Feér Type Inequalities for Harmonically s-convex Functions via Fractional Integrals, *The Australian Journal of Mathematical Analysis and Applications*, **12**(1), Article 10, (2015), 1-16.
16. İ. İşcan, Ostrowski Type Inequalities for p-convex Functions, *New Trends Math. Sci.*, **4**(3), (2016), 140-150.
17. İ. İscan, Hermite-Hadamard Type Inequalities for p-convex Functions, *International Journal of Analysis and Applications*, **11**(2), (2016), 137-145.
18. İ. İscan, Hermite-Hadamard Type Inequalities for Harmonically Convex Functions, *Hacettepe Journal of Mathematics and Statistics*, **43**(6), (2014), 935-942.
19. C. Y. Jung, M. Yussouf, Y. M. Chu, G. Farid, S. M. Kang, Generalized Fractional Hadamard and Fejér–Hadamard Inequalities for Generalized Harmonically Convex Functions, *Hindawi Journal of Mathematics*, **2020**(1), (2020), Article ID 8245324, 13 pages.
20. U. S. Kirmaci, M. K. Bakula, M. E. Özdemir, J. Peçarić, Hadamard-type Inequalities for s-convex Functions, *Applied Mathematics and Computation*, **193**(1), (2007), 26-35.
21. Y. Khurshid, M. Adil Khan, Y. M. Chu, Conformable Integral Version of Hermite-Hadamard-Fejér Inequalities via η-convex Functions, *AIMS Mathematics*, **5**(5), 5106–5120.
22. M. Kunt, İ. İscan, On Hermite-Hadamard Type Inequalities for p-convex Functions via Fractional Integrals ,Di:10.1515/, *mjpaa*, 2017-0003, **3**(1), (2017).
23. M. Kunt, S. Turhan, İ. İscan, On Hermite-Hadamard Type Inequalities with Respect to the Generalization of Some Types of S-Convexity, *Konuralp Journal of Mathematics*, **8**(1), (2020), 165-174.
24. M. Kunt, İ. İscan, N. Yazici, U. Gozutok, On New Inequalities of Hermite-Hadamard-Fejér Type for Harmonically Convex Functions via Fractional Integrals, *Springer plus*, **5**(635), (2016), 1-19.
25. M. Kunt, İ. İscan, Hermite–Hadamard–Fejér Type Inequalities for p-convex Functions, *Arab J Math. Sci.*, **23**, (2017), 215-230.
26. M. Kunt, İ. İşcan, Hermite-Hadamard-Fejér Type Inequalities for p-convex Functions via Fractional Integrals, *Iran J. Sci. Technol. Trans. Sci*, **42**(4), (2018), 2079-2089.
27. M. A. Latif, S. S. Dragomir, E. Momoniat, Some Fejér Type Inequalities for Harmonically-convex Functions with Applications to Special Means, *International Journal of Analysis and Applications*, **13**(1), (2017), 1-14.
28. C. Park, Y. -M. Chu, M. S. Saleem, S. Mukhtar, N. Rehman, Hermite–Hadamard-type Inequalities for η_h-convex Functions via ψ-Riemann–Liouville Fractional Integrals, *Ad. Diff. Eq.*, **2020**(1), (2020), 602.
29. C. Park, Y. -M. Chu, M. S. Saleem, N. Jahangir, N. Rehman, On n-polynomial p-convex Functions and Some Related Inequalities, *Ad. Diff. Eq.*, **2020**(1), (2020), 666.
30. S. Rashid , M. A. Noor, K. I. Noor, F. Safdar, Y. -M. Chu, Hermite-Hadamard Type Inequalities for the Class of Convex Functions on Time Scale, *AIMS Mathematics*, **7**(10), (2019), 956.
31. M. Z. Sarikaya, E. Set, H. Yaldiz, N. Başak, Hermite-Hadamard Inequalities for Fractional Integrals and Related Fractional Inequalities, *Mathematical and Computer Modeling*, **57**, (2013), 2403-2470.
32. J. Wang, X. Li, M. Fečkan, Y. Zhou, Hermite-Hadamard-type Inequalities for Riemann Liouville Fractional Integrals via Two Kinds of Convexity, *Appl. Anal.*, **92**(11), (2012), 2241-2253. doi:10.1080/00036811.2012.727986.
33. R. Xiang, Refinements of Hermite-Hadamard Type Inequalities for Convex Functions via Fractional Integrals, *J. Appl. Math. and Informatics*, **33**(1-2), (2015), 119-125.
34. G. S. Yang, K. L. Tseng, On Certain Integral Inequalities Related to Hermite-Hadamard Inequalities, *J. Math. Anal. Appl.*, **239**, (1999), 180-187.

[Downloaded from ijmsi.ir on 2026-10-06]

[DOI: 10.66224/ijmsi.21.1.197]

35. X. -X. You, M. Aamir Ali, H. Budak, P. Agarwal, Y. -M. Chu, Extensions of Hermite–Hadamard Inequalities for Harmonically Convex Functions via Generalized Fractional Integrals, *J. Ineq. Appl.*, **2021**, (2021), 102.
36. K. S. Zhang, J. P. Wan, p-convex Functions and their Properties, *Pure Appl. Math.*, **23**(1), (2007), 130-133.
37. S. Zaheer Ullah, M. Adil Khan, Z. A. Khan, Y.-M. Chu, Coordinate Strongly s–convex Functions and Related Results, *J. Math. Ineq.*, **14**(3), (2020), 829–843.

[Downloaded from ijmsi.ir on 2026-10-06]

[DOI: 10.66224/ijmsi.21.1.197]