

## Duality of $\mathbb{F}_q\mathbb{F}_q[u]$ -additive Skew Cyclic Codes

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ABSTRACT. Li et al. (2021) obtained the generator polynomials and the minimal generating sets of  $\mathbb{F}_q\mathbb{F}_q[u]$ -linear skew cyclic codes, where  $q$  is a power of a prime integer and  $u^2 = 0$ . In this paper, we determine the structure of dual of these codes in terms of their generating polynomials and we illustrate the dual of some special  $\mathbb{F}_q\mathbb{F}_q[u]$ -additive skew cyclic codes.

**Keywords:** Chain ring, Skew cyclic code, Additive skew cyclic code, Duality.

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### 1. INTRODUCTION

The class of constacyclic codes plays a very significant role in the theory of error-correcting codes. The most important class of these codes is the class of all cyclic codes. The  $\mathbb{Z}_2\mathbb{Z}_4$ -additive codes are subgroups of the group  $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ . These codes are observed as a generalization of quaternary and binary codes for  $\alpha = 0$  and  $\beta = 0$ , respectively. Abualrub et al. studied  $\mathbb{Z}_2\mathbb{Z}_4$ -additive cyclic codes in [1]. Inspired by this paper, Aydogdu et al. presented the structure

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of cyclic and constacyclic codes and their duals in [2]. Borges et al. carried the findings on  $\mathbb{Z}_2\mathbb{Z}_4$ -additive cyclic codes over  $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$ -additive cyclic codes for any prime  $p$ , in [4]. In order to generalize the concepts appeared in [4] and [2], Mahmoudi and Samei (in [12]) introduced and investigated the  $SR$ -additive codes as  $R$ -submodule of  $S^\alpha \times R^\beta$ , where  $R$  is an arbitrary finite commutative ring and  $S$  is a finite  $R$ -algebra.

The detailed structures of all constacyclic codes of length  $p^s$  over  $R_2 = \mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$  (with  $u^2 = 0$ ) has been classified in [7]. Dinh et al. [8] studied the structure of self-dual cyclic codes of length  $p^s$  over  $R_2$ .

One of the most applicable generalizations of cyclic codes is the class of skew cyclic codes which were introduced by Boucher in [5]. The algebraic structure and some properties of these codes over finite chain rings and their Euclidean and Hermitian dual codes have been established in [10]. Hesari et al. in [9] determined the structure of (Euclidean) dual of some special skew cyclic codes of length  $p^s$  over  $R_2$  and identified all self-dual codes in this category. Li et al. have studied the  $\mathbb{F}_q\mathbb{F}_q[u]$ -linear skew cyclic codes in [11].

The purpose of this paper is to continue the investigations done in [11] and to determine the structure of the (Euclidean) dual of  $\mathbb{F}_q\mathbb{F}_q[u]$ -additive skew cyclic codes obtained there.

This paper has been organized as follows. Section 2 contains some basic definitions, notations and specifics of the  $\mathbb{F}_q\mathbb{F}_q[u]$ -additive skew cyclic codes, where  $u^2 = 0$ . These codes have been classified into eight different types in terms of their explicit generator polynomials. Section 3 is divided into two parts. In Subsection 3.1, we calculate the dual of  $\mathbb{F}_q\mathbb{F}_q[u]$ -additive skew cyclic codes. Finally, in Subsection 3.2, as an application, we provide some examples of  $\mathbb{F}_q\mathbb{F}_q[u]$ -additive skew cyclic codes and compute their duals in terms of their generating polynomials.

## 2. PRELIMINARIES

In this section, we present some basic definitions, notations and previous results related to our work.

A ring  $R$  is a *principal left ideal ring* if it has unity and every left ideal is principally generated.  $R$  is called a *local ring* if  $R$  has a unique maximal right (left) ideal. Furthermore, a ring  $R$  is called a *chain ring* if the set of all ideals of  $R$  is linearly ordered under set-theoretic inclusion.

**Definition 2.1.** Let  $R$  be a finite commutative ring and  $\sigma$  be an automorphism of  $R$ . Consider the ring  $R[x; \sigma] = \{a_0 + a_1x + a_2x^2 + \dots + a_nx^n : a_i \in R \text{ and } n \in \mathbb{N}_0\}$ , where the addition is defined to be the usual addition of polynomials and the multiplication is defined by the basic rule  $xa = \sigma(a)x$  ( $a \in R$ ). The multiplication is extended to all elements in  $R[x; \sigma]$  by associativity

and distributivity. The ring  $R[x; \sigma]$  is called a *skew polynomial ring* over  $R$  and every element in  $R[x; \sigma]$  is called a *skew polynomial*. It is easily seen that the ring  $R[x; \sigma]$  is non-commutative unless  $\sigma$  is the identity automorphism on  $R$ .

**Proposition 2.2.** [10, Proposition 2.2] *Let  $R$  be a finite commutative ring,  $n$  be a positive integer and  $\lambda$  be a unit in  $R$ . Then the following statements are equivalent:*

- i)  $x^n - \lambda$  is central in  $R[x; \sigma]$ .
- ii)  $\langle x^n - \lambda \rangle$  is two-sided.
- iii)  $n$  is a multiple of the order of  $\sigma$  and  $\lambda$  is fixed by  $\sigma$ .

**Definition 2.3.** A code  $C$  of length  $n$  over  $R$  is a non-empty subset of  $R^n$  and the ring  $R$  is referred to as the alphabet of  $C$ . If  $C$  is also an  $R$ -submodule of  $R^n$ , then  $C$  is called a *linear code*.

For a given automorphism  $\sigma$  of  $R$ , a code  $C$  over  $R$  is called skew  $\sigma$ -cyclic, if  $C$  is closed under  $\sigma$ -cyclic shift  $\rho_\sigma : R^n \rightarrow R^n$  which is defined by

$$\rho_\sigma((a_0, a_1, \dots, a_{n-1})) = (\sigma(a_{n-1}), \sigma(a_0), \dots, \sigma(a_{n-2})).$$

Each codeword  $c = (c_0, c_1, \dots, c_{n-1})$  is customarily identified with its polynomial representation  $c(x) = c_0 + c_1x + \dots + c_{n-1}x^{n-1}$ . In particular, if  $o(\sigma) \mid n$ , then  $\frac{R[x; \sigma]}{\langle x^n - 1 \rangle}$  is a ring and the polynomial  $xc(x)$  corresponds to the  $\sigma$ -cyclic shift of  $c = (c_0, c_1, \dots, c_{n-1})$ . In this way,  $C$  is a skew cyclic code of length  $n$  over  $R$  if and only if  $C$  is a left ideal of  $\frac{R[x; \sigma]}{\langle x^n - 1 \rangle}$ . When there is no ambiguity, we say “skew cyclic” instead of “skew  $\sigma$ -cyclic”.

**Proposition 2.4.** [10, Proposition 2.3] *Let  $h(x), g(x) \in R[x; \sigma]$ . If  $h(x)g(x)$  is a monic central skew polynomial, then  $h(x)g(x) = g(x)h(x)$ .*

Let  $\mathbb{F}_q$  denote the finite field with  $q$  elements and  $\delta$  be a primitive  $(q - 1)$ -th root of unity in  $\mathbb{F}_q$ , i.e.,

$$\mathbb{F}_q = \{0, \delta, \dots, \delta^{q-2}, \delta^{q-1} = 1\}.$$

Suppose that  $R_2 = \mathbb{F}_q + u\mathbb{F}_q$ , with  $u^2 = 0$ . It is known that  $R_2$  is a chain ring with the unique maximal ideal  $u\mathbb{F}_q$ .

**Lemma 2.5.** [10, Corollary 2.1] *For  $\theta \in \text{Aut}(\mathbb{F}_q)$  and  $\eta \in \mathbb{F}_q^*$ , let*

$$\Theta_{\theta, \eta} : \mathbb{F}_q + u\mathbb{F}_q \rightarrow \mathbb{F}_q + u\mathbb{F}_q$$

*be defined by*

$$\Theta_{\theta, \eta}(a + bu) = \theta(a) + \eta\theta(b)u.$$

*Then  $\text{Aut}(\mathbb{F}_q + u\mathbb{F}_q) = \{\Theta_{\theta, \eta} : \theta \in \text{Aut}(\mathbb{F}_q) \text{ and } \eta \in \mathbb{F}_q^*\}$ .*

As we saw in the above lemma, every automorphism of  $\mathbb{F}_q + u\mathbb{F}_q$  is of the form  $\Theta_{\theta,\eta}$ . From now on, let  $\eta = 1$  and  $\Theta_{\theta,1}$  will be denoted by  $\Theta$ .

We say that  $f(x)$  is a *right divisor* (*left divisor*) of  $g(x)$  in  $\mathbb{F}_q[x, \theta]$  and we write  $f(x) \mid_r g(x)$  ( $f(x) \mid_l g(x)$ ) if there exists a skew polynomial  $h(x)$  such that  $g(x) = h(x)f(x)$  ( $g(x) = f(x)h(x)$ ).

**Definition 2.6.** Suppose  $f(x), g(x)$  are skew polynomials in  $\mathbb{F}_q[x; \theta]$ . The *greatest common right divisor* of  $f(x)$  and  $g(x)$  is the monic polynomial  $d_r(x) \in \mathbb{F}_q[x; \theta]$ , where  $d_r(x) \mid_r f(x), d_r(x) \mid_r g(x)$  and for any  $d'_r(x) \in \mathbb{F}_q[x; \theta]$  such that  $d'_r(x) \mid_r f(x)$  and  $d'_r(x) \mid_r g(x)$ , we have  $d'_r(x) \mid_r d_r(x)$ . We denote  $d_r(x)$  by  $\gcd_r(f(x), g(x))$ .

The *greatest common left divisor*  $d_l(x)$  of  $f(x)$  and  $g(x)$ , written

$$d_l(x) = \gcd_l(f(x), g(x))$$

is defined in the same manner using left division.

**Definition 2.7.** The *least common left multiple* ( $\text{lcm}_l$ ) of  $f(x)$  and  $g(x)$  is the unique monic polynomial  $m_l(x) = \text{lcm}_l(f(x), g(x))$  such that  $f(x) \mid_r m_l(x), g(x) \mid_r m_l(x)$  and for any  $m'(x) \in \mathbb{F}_q[x; \theta]$  such that  $f(x) \mid_r m'(x)$  and  $g(x) \mid_r m'(x)$ , then we have  $m_l(x) \mid_r m'(x)$ .

**Lemma 2.8.** [13, Page 486] *Let  $f(x)$  and  $g(x)$  be skew polynomials in  $\mathbb{F}_q[x; \theta]$ . Then,*

$$\deg(\text{lcm}_l(f(x), g(x))) = \deg(f(x)) + \deg(g(x)) - \deg(\gcd_r(f(x), g(x))).$$

**Lemma 2.9.** *Let  $f(x), g(x), h(x)$  and  $c(x)$  be skew polynomials in the ring  $\mathbb{F}_q[x; \theta]$  such that  $c(x)$  is a central element. If*

$$f(x)g(x) \equiv 0 \pmod{c(x)}$$

and

$$f(x)h(x) \equiv 0 \pmod{c(x)}.$$

Then,

$$f(x)\gcd_l(g(x), h(x)) \equiv 0 \pmod{c(x)}.$$

*Proof.* By the hypothesis,  $f(x)g(x) = k_1(x)c(x)$  and  $f(x)h(x) = k_2(x)c(x)$ , for some  $k_1(x), k_2(x) \in \mathbb{F}_q[x; \theta]$ . In other hand, there exist  $a(x)$  and  $b(x)$  in  $\mathbb{F}_q[x; \theta]$  such that

$$g(x)a(x) + h(x)b(x) = \gcd_l(g(x), h(x)).$$

Hence,  $f(x)g(x)a(x) + f(x)h(x)b(x) = f(x)\gcd_l(g(x), h(x))$ . We have

$$k_1(x)c(x)a(x) + k_2(x)c(x)b(x) = f(x)\gcd_l(g(x), h(x)).$$

Since  $c(x)$  is central element, then  $c(x) \mid f(x)\gcd_l(g(x), h(x))$ . □

Throughout this paper, we use the following symbols for simplicity:

- $R_2 = \mathbb{F}_q[u] = \mathbb{F}_q + u\mathbb{F}_q$ .
- $m = \text{lcm}(\alpha, \beta)$ .
- $\mathcal{R}_{1,k} = \frac{\mathbb{F}_q[x;\theta]}{\langle x^k - 1 \rangle}$ , for  $k = \alpha, \beta, m$ .
- $\mathcal{R}_\tau = \frac{R_2[x;\Theta]}{\langle x^\tau - 1 \rangle}$ , for  $\tau = \beta, n, m$ .
- $\mathcal{R} = \mathcal{R}_{1,\alpha} \times \mathcal{R}_\beta$ .

In this paper, we assume that  $o(\Theta) = o(\theta) \mid \text{gcd}(\alpha, \beta)$ , where  $\alpha$  and  $\beta$  are positive integers and  $o(\theta)$  is the order of  $\theta$ . By Proposition 2.2,  $\langle x^\alpha - 1 \rangle$  and  $\langle x^\beta - 1 \rangle$  are two-sided ideals of  $R_2[x; \Theta]$ .

Since  $x^\alpha - 1$  and  $x^\beta - 1$  are monic central skew polynomials, by Proposition 2.4, right divisors of  $x^\alpha - 1$  and  $x^\beta - 1$  are two-sided divisors.

Let  $\mu : R_2 \rightarrow \mathbb{F}_q$ , be the natural ring morphism, defined by  $\mu(a_0 + ua_1) = a_0$ . We consider the set

$$\mathbb{F}_q \times R_2 = \{(a|b) : a \in \mathbb{F}_q, b \in R_2\}.$$

By the following scalar multiplication,  $\mathbb{F}_q \times R_2$  is a left  $R_2$ -module,

$$\begin{aligned} * : R_2 \times (\mathbb{F}_q \times R_2) &\longrightarrow \mathbb{F}_q \times R_2, \\ \nu * (a|b) &= (\mu(\nu)a|\nu b). \end{aligned}$$

This multiplication can be extended to the set  $\mathbb{F}_q^\alpha \times R_2^\beta$  in the following way. For any  $\nu \in R_2$  and  $(a_0, a_1, \dots, a_{\alpha-1}|b_0, \dots, b_{\beta-1}) \in \mathbb{F}_q^\alpha \times R_2^\beta$  define

$$\nu * (a_0, a_1, \dots, a_{\alpha-1}|b_0, \dots, b_{\beta-1}) = (\mu(\nu)a_0, \mu(\nu)a_1, \dots, \mu(\nu)a_{\alpha-1}|\nu b_0, \dots, \nu b_{\beta-1}).$$

**Definition 2.10.** A non-empty subset  $C$  of  $\mathbb{F}_q^\alpha \times R_2^\beta$  is called an  $\mathbb{F}_q\mathbb{F}_q[u]$ -additive skew cyclic code of length  $n = \alpha + \beta$ , if  $C$  is a left  $R_2$ -submodule of  $\mathbb{F}_q^\alpha \times R_2^\beta$ .

There is a bijective map between  $\mathbb{F}_q^\alpha \times R_2^\beta$  and  $\mathcal{R} = \mathcal{R}_{1,\alpha} \times \mathcal{R}_\beta$  given by  $(a_0, \dots, a_{\alpha-1}|b_0, \dots, b_{\beta-1}) \mapsto (a_0 + \dots + a_{\alpha-1}x^{\alpha-1}|b_0 + \dots + b_{\beta-1}x^{\beta-1}) = (a(x)|b(x))$ .

Suppose  $(f(x)|g(x)) \in \mathcal{R}$  and  $\nu(x) \in R_2[x; \Theta]$ , we have

$$\begin{aligned} * : R_2[x; \Theta] \times \mathcal{R} &\longrightarrow \mathcal{R}, \\ \nu(x) * (f(x)|g(x)) &= (\mu(\nu(x))f(x)|\nu(x)g(x)), \end{aligned}$$

where

$$\mu(\nu(x)) = \mu\left(\sum_{j=0}^{\alpha-1} \nu_j x^j\right) = \sum_{j=0}^{\alpha-1} \mu(\nu_j) x^j$$

and  $\nu_j \in R_2$ .

Note that the skew polynomial ring  $\mathbb{F}_q[x; \theta]$  is not a unique factorization domain. In fact, many different factorizations may be possible in this domain. The ring  $R_2[x; \Theta]$  is neither left nor right Euclidean. However, left and right divisions can be defined for some suitable elements. Let  $f(x), g(x)$  be skew polynomials in  $R_2[x; \Theta]$ , where leading coefficient of  $f(x)$  is a unit in  $R_2$ . Then there exist  $q(x), r(x) \in R_2[x, \Theta]$  such that  $g(x) = q(x)f(x) + r(x)$ , where  $r(x) = 0$  or  $\deg(r(x)) < \deg(f(x))$ . Note that  $q(x)$  and  $r(x)$  are unique.

Set  $\mathcal{F}_k := \{a(x) \in \mathbb{F}_q[x; \theta] : a(x) \text{ is a monic factor of } x^k - 1\}$ , for  $k = \alpha, \beta, m$ .

**Proposition 2.11.** [5, Theorem 1] *The ring  $\mathcal{R}_{1,k}$  is a principal left ideal ring, in which left ideals are generated by  $a(x) + \langle x^k - 1 \rangle$ , where  $a(x)$  is a monic divisor of  $x^k - 1$  in  $\mathbb{F}_q[x; \theta]$ .*

It is well known that  $R_2 = \mathbb{F}_q + u\mathbb{F}_q$  is a finite chain ring of nilpotency index 2 and the unique maximal ideal  $u\mathbb{F}_q$ . In [11], Li et al. determined the  $\mathbb{F}_q\mathbb{F}_q[u]$ -additive skew cyclic codes of length  $\alpha + \beta$ , where  $\alpha$  and  $\beta$  are multiples of the order of  $\Theta$ .

**Proposition 2.12.** [10, Theorem 2.2] *A linear code  $\mathcal{C}$  of length  $\beta$  is a skew cyclic code over  $R_2$  if and only if  $\mathcal{C}$  is a left ideal of  $\mathcal{R}_\beta$ .*

For each left ideal  $\mathcal{I}$  in  $\mathcal{R}_\beta$ , the image  $\mu(\mathcal{I} :_{\mathcal{R}_\beta} u) = \mu(\{v \in \mathcal{R}_\beta : vu \in \mathcal{I}\})$  is a left ideal in  $\mathcal{R}_{1,\beta}$ . Inasmuch as every skew cyclic code  $\mathcal{C}$  over the ring  $R_2$  is a left ideal of  $\mathcal{R}_\beta$ , the image  $\mu(\mathcal{C} :_{\mathcal{R}_\beta} u)$  is in fact a skew cyclic code over  $\mathbb{F}_q$  which is called the *torsion code* associated to  $\mathcal{C}$  and it is denoted by  $\text{Tor}(\mathcal{C})$ .

**Lemma 2.13.** [11, Lemma 3] *A code  $\mathcal{C}$  is an  $\mathbb{F}_q\mathbb{F}_q[u]$ -additive skew cyclic code of length  $\alpha + \beta$  if and only if  $\mathcal{C}$  is a left  $R_2[x; \Theta]$ -submodule of  $\mathcal{R}$ .*

**Theorem 2.14.** [11, Theorem 1] *Every left  $R_2[x; \Theta]$ -submodule of  $\mathcal{R}$  is of the form*

$$\mathcal{R}_n((a(x)|0)) + \mathcal{R}_n((k_1(x)|a_1(x) + ug_1(x))) + \mathcal{R}_n((k_2(x)|ua_2(x))),$$

where  $n := \alpha + \beta$ ,  $a(x) \in \mathcal{F}_\alpha$ ,  $a_i(x) \in \mathcal{F}_\beta$ ,  $a_2(x) \mid_r a_1(x)$ ,  $k_i(x) \in \mathcal{R}_{1,\alpha}$ ,  $\deg(k_i(x)) < \deg(a(x))$  and  $\deg(g_1(x)) < \deg(a_2(x))$ .

We can list all  $\mathbb{F}_q\mathbb{F}_q[u]$ -additive skew cyclic codes of length  $\alpha + \beta$  as follows:

**Theorem 2.15.** [11, Theorem 2] *Every  $\mathbb{F}_q\mathbb{F}_q[u]$ -additive skew cyclic code of length  $n = \alpha + \beta$  can be presented in one of the following forms:*

- Type 1 :  $0, \mathcal{R}$ .
- Type 2 :  $\mathcal{R}_n((a(x)|0))$ , where  $a(x) \in \mathcal{F}_\alpha$  and  $0 \leq \deg(a(x)) \leq \alpha - 1$ .
- Type 3 :  $\mathcal{R}_n((k_1(x)|a_1(x) + ug_1(x)))$ , where  $k_1(x) \in \mathcal{R}_{1,\alpha}$ ,  $a_1(x) \in \mathcal{F}_\beta$ ,

$0 \leq \deg(a_1(x)) \leq \beta - 1$ ,  $g_1(x) \in \mathcal{R}_{1,\beta}$  and  $\deg(g_1(x)) < \deg(a_1(x))$ .

- *Type 4* :  $\mathcal{R}_n((k_2(x)|ua_2(x)))$ , where  $k_2(x) \in \mathcal{R}_{1,\alpha}$ ,  $a_2(x) \in \mathcal{F}_\beta$  and  $0 \leq \deg(a_2(x)) \leq \beta - 1$ .
- *Type 5* :  $\mathcal{R}_n((a(x)|0)) + \mathcal{R}_n((k_1(x)|a_1(x) + ug_1(x)))$ , where  $a(x) \in \mathcal{F}_\alpha$ ,  $0 \leq \deg(a(x)) \leq \alpha - 1$ ,  $k_1(x) \in \mathcal{R}_{1,\alpha}$ ,  $a_1(x) \in \mathcal{F}_\beta$ ,  $0 \leq \deg(a_1(x)) \leq \beta - 1$ ,  $g_1(x) \in \mathcal{R}_{1,\beta}$ ,  $\deg(k_1(x)) < \deg(a(x))$  and  $\deg(g_1(x)) < \deg(a_1(x))$ .
- *Type 6* :  $\mathcal{R}_n((a(x)|0)) + \mathcal{R}_n((k_2(x)|ua_2(x)))$ , where  $a(x) \in \mathcal{F}_\alpha$ ,  $0 \leq \deg(a(x)) \leq \alpha - 1$ ,  $k_2(x) \in \mathcal{R}_{1,\alpha}$ ,  $a_2(x) \in \mathcal{F}_\beta$ ,  $0 \leq \deg(a_2(x)) \leq \beta - 1$  and  $\deg(k_2(x)) < \deg(a(x))$ .
- *Type 7* :  $\mathcal{R}_n((k_1(x)|a_1(x) + ug_1(x))) + \mathcal{R}_n((k_2(x)|ua_2(x)))$ , where  $k_i(x) \in \mathcal{R}_{1,\alpha}$ ,  $a_i(x) \in \mathcal{F}_\beta$ ,  $a_2(x) \mid_r a_1(x)$ ,  $0 \leq \deg(a_1(x)) \leq \beta - 1$ ,  $g_1(x) \in \mathcal{R}_{1,\beta}$  and  $\deg(g_1(x)) < \deg(a_2(x))$ .
- *Type 8* :  $\mathcal{R}_n((a(x)|0)) + \mathcal{R}_n((k_1(x)|a_1(x) + ug_1(x))) + \mathcal{R}_n((k_2(x)|ua_2(x)))$ , where  $a(x) \in \mathcal{F}_\alpha$ ,  $0 \leq \deg(a(x)) \leq \alpha - 1$ ,  $a_i(x) \in \mathcal{F}_\beta$ ,  $0 \leq \deg(a_i(x)) \leq \beta - 1$ ,  $a_2(x) \mid_r a_1(x)$ ,  $k_i(x) \in \mathcal{R}_{1,\alpha}$ ,  $\deg(k_i(x)) < \deg(a(x))$  and  $\deg(g_1(x)) < \deg(a_2(x))$ .

**Lemma 2.16.** *Let  $C = \mathcal{R}_n((a(x)|0)) + \mathcal{R}_n((k_1(x)|a_1(x) + ug_1(x))) + \mathcal{R}_n((k_2(x)|ua_2(x)))$ , be an  $\mathbb{F}_q\mathbb{F}_q[u]$ -additive skew cyclic codes of length  $\alpha + \beta$  such that  $\deg(g_1(x)) < \deg(a_2(x))$ . Then,  $g_1(x)$  with the above condition is unique.*

*Proof.* Let

$$\begin{aligned} \pi : C &\longrightarrow \mathcal{R}_\beta, \\ (\lambda(x)|\lambda_1(x)) &\longmapsto \lambda_1(x) \end{aligned}$$

be a homomorphism between two left  $R_2[x; \Theta]$ -modules and assume that  $g'(x)$  is a polynomial satisfying

$$C = \mathcal{R}_n((a(x)|0)) + \mathcal{R}_n((k_1(x)|a_1(x) + ug'(x))) + \mathcal{R}_n((k_2(x)|ua_2(x))).$$

Hence  $(a_1(x) + ug_1(x)) - (a_1(x) + ug'(x)) \in \text{Im}(\pi)$ . So  $u(g_1(x) - g'(x)) \in \text{Im}(\pi)$ , implies that  $g_1(x) - g'(x) \in \text{Tor}(\text{Im}(\pi)) = \mathcal{R}_{1,\beta}(a_2(x))$ . If  $g_1(x) \neq g'(x)$ , then  $\deg(a_2(x)) \leq \deg(g_1(x) - g'(x))$ . But by the hypothesis,  $\deg(g_1(x) - g'(x)) < \deg(a_2(x))$ , a contradiction. Thus  $g_1(x) = g'(x)$ .  $\square$

### 3. DUALITY OF $\mathbb{F}_q\mathbb{F}_q[u]$ -ADDITIVE SKEW CYCLIC CODES

This section is devoted to discussing the structural properties of the dual codes of  $\mathbb{F}_q\mathbb{F}_q[u]$ -additive skew cyclic codes.

### 3.1. Dual of $\mathbb{F}_q\mathbb{F}_q[u]$ -additive skew cyclic codes of length $\alpha + \beta$ .

In this subsection, we determine the dual of  $\mathbb{F}_q\mathbb{F}_q[u]$ -additive skew cyclic codes of length  $\alpha + \beta$ .

Let

$$x = (x_0, x_1, \dots, x_{\alpha-2}, x_{\alpha-1} | x'_0, x'_1, \dots, x'_{\beta-2}, x'_{\beta-1})$$

and

$$y = (y_0, y_1, \dots, y_{\alpha-2}, y_{\alpha-1} | y'_0, y'_1, \dots, y'_{\beta-2}, y'_{\beta-1})$$

be elements of  $\mathbb{F}_q^\alpha \times R_2^\beta$ . The inner product is defined as

$$x \cdot y = u \sum_{i=0}^{\alpha-1} x_i y_i + \sum_{j=0}^{\beta-1} x'_j y'_j.$$

The dual of an  $\mathbb{F}_q\mathbb{F}_q[u]$ -additive skew cyclic code of length  $\alpha + \beta$  is an  $\mathbb{F}_q\mathbb{F}_q[u]$ -additive skew cyclic code of length  $\alpha + \beta$ , defined by

$$C^\perp = \{x \in \mathbb{F}_q^\alpha \times R_2^\beta : x \cdot y = 0, \text{ for all } y \in C\}.$$

A code  $C$  is called *self-orthogonal* if  $C \subseteq C^\perp$ , and it is called *self-dual* if  $C = C^\perp$ .

**Definition 3.1.** [6, Definition 3] Let  $f(x) = a_0 + a_1x + \dots + a_t x^t \in \mathbb{F}_q[x; \theta]$ , where  $a_t \neq 0$ . Then the polynomial  $f^*(x) = a_t + \theta(a_{t-1})x + \dots + \theta^t(a_0)x^t$  is called the *reciprocal polynomial* of  $f(x)$ . Equivalently,  $f^*(x)$  can be expressed by  $f^*(x) = \sum_{i=0}^t \theta^i(a_{t-i})x^i$ .

**Lemma 3.2.** [6, Section 3] *Let  $\psi$  be the map*

$$\begin{aligned} \psi : R_2[x; \Theta] &\longrightarrow R_2[x; \Theta], \\ \sum_{i=0}^n a_i x^i &\longmapsto \sum_{i=0}^n \Theta(a_i) x^i, \end{aligned}$$

where  $a_i \in R_2$ . Then  $\psi$  is a ring homomorphism.

**Lemma 3.3.** *Let  $f(x), g(x)$  be skew polynomials in  $R_2[x; \Theta]$ .*

- (1) *If  $\deg(f) \geq \deg(g)$ , then  $(f(x) + g(x))^* = f^*(x) + x^{\deg f - \deg g} g^*(x)$ .*
- (2)  *$(fg)^* = \psi^{\deg f} (g^*) f^*$ .*
- (3)  *$(f^*)^* = \psi^n(f)$ , where  $\deg(f) = n$ .*

*Proof.* It is similar to the proof of [9, Lemma 2.8]. □

Set  $\Gamma_\nu(x) = \sum_{j=0}^{\nu-1} x^j$ . As the unit element 1 remains fixed under automorphism  $\theta$ , we have the following lemma whose proof is evident.

**Lemma 3.4.** *Let  $n, n' \in \mathbb{N}$ . Then  $x^{nn'} - 1 = (x^n - 1)\Gamma_{n'}(x^n) = \Gamma_{n'}(x^n)(x^n - 1)$ .*

**Definition 3.5.** Let  $\mathbf{v}(x) = (v(x)|v'(x))$  and  $\mathbf{w}(x) = (w(x)|w'(x))$  be any two elements in  $\mathcal{R}$  and  $m = \text{lcm}(\alpha, \beta)$ . Define the map

$$\circ : \mathcal{R} \times \mathcal{R} \longrightarrow \mathcal{R}_m$$

such that

$$\begin{aligned} \circ(\mathbf{v}(x), \mathbf{w}(x)) &= uv(x)\psi^{m-\text{deg}(w(x))}(w^*(x))x^{m-1-\text{deg}(w(x))}\Gamma_{\frac{m}{\alpha}}(x^\alpha) \\ &\quad + v'(x)\psi^{m-\text{deg}(w'(x))}(w'^*(x))x^{m-1-\text{deg}(w'(x))}\Gamma_{\frac{m}{\beta}}(x^\beta). \end{aligned}$$

The map  $\circ$  is bilinear between left  $R_2[x; \Theta]$ -modules. We denote  $\circ(\mathbf{v}(x), \mathbf{w}(x))$  by  $\mathbf{v}(x) \circ \mathbf{w}(x)$ .

**Proposition 3.6.** *Let  $\mathbf{v}$  and  $\mathbf{w}$  be elements in  $\mathbb{F}_q^\alpha \times R_2^\beta$  with associated polynomials  $\mathbf{v}(x) = (v(x)|v'(x))$  and  $\mathbf{w}(x) = (w(x)|w'(x))$ , respectively. Then,  $\mathbf{w}$  is orthogonal to  $\mathbf{v}$  and all its  $\Theta$ -shifts if and only if*

$$\mathbf{v}(x) \circ \mathbf{w}(x) = 0.$$

*Proof.* Let

$$\mathbf{v} = (v_0, v_1, \dots, v_{\alpha-1} | v'_0, v'_1, \dots, v'_{\beta-1})$$

and

$$\mathbf{w} = (w_0, w_1, \dots, w_{\alpha-1} | w'_0, w'_1, \dots, w'_{\beta-1}).$$

Let

$$\mathbf{v}^{(i)} = (\theta^i(v_{0-i}), \theta^i(v_{1-i}), \dots, \theta^i(v_{\alpha-1-i}) | \Theta^i(v'_{0-i}), \Theta^i(v'_{1-i}), \dots, \Theta^i(v'_{\beta-1-i}))$$

be the  $i$ -th cyclic  $\Theta$ -shift of  $\mathbf{v}$  such that  $0 \leq i \leq m - 1$ . For  $0 \leq j_1 \leq \alpha - 1$  and  $0 \leq j_2 \leq \beta - 1$ , indices  $j_1 - i$  and  $j_2 - i$  are computed modulo  $\alpha$  and  $\beta$ , respectively. Hence

$$\mathbf{v}^{(i)} \cdot \mathbf{w} = 0 \quad \text{if and only if} \quad u \sum_{j=0}^{\alpha-1} \theta^i(v_{j-i})w_j + \sum_{\nu=0}^{\beta-1} \Theta^i(v'_{\nu-i})w'_\nu = 0.$$

Let  $S_i = u \sum_{j=0}^{\alpha-1} \theta^i(v_{j-i})w_j + \sum_{\nu=0}^{\beta-1} \Theta^i(v'_{\nu-i})w'_\nu$ . We can get

$$\begin{aligned} \mathbf{v}(x) \circ \mathbf{w}(x) &= u \left[ \sum_{\mu=0}^{\alpha-1} \sum_{j=\mu}^{\alpha-1} v_{j-\mu} \theta^{m-\mu}(w_j) x^{m-1-\mu} \right. \\ &\quad + \sum_{\mu=1}^{\alpha-1} \sum_{j=\mu}^{\alpha-1} v_j \theta^\mu(w_{j-\mu}) x^{m-1+\mu} \left. \right] \Gamma_{\frac{m}{\alpha}}(x^\alpha) \\ &\quad + \left[ \sum_{\eta=0}^{\beta-1} \sum_{\nu=\eta}^{\beta-1} v'_{\nu-\eta} \Theta^{m-\eta}(w'_\nu) x^{m-1-\eta} \right. \\ &\quad + \sum_{\eta=1}^{\beta-1} \sum_{\nu=\eta}^{\beta-1} v'_\eta \Theta^\eta(w'_{\nu-\eta}) x^{m-1+\eta} \left. \right] \Gamma_{\frac{m}{\beta}}(x^\beta) \\ &= \sum_{i=0}^{m-1} \Theta^{m-i}(S_i) x^{m-1-i}. \end{aligned}$$

So,  $\mathbf{v}(x) \circ \mathbf{w}(x) = 0$  if and only if  $S_i = 0$  for  $0 \leq i \leq m-1$ . □

**Lemma 3.7.** *Let  $\mathbf{v}(x) = (v(x)|v'(x))$  and  $\mathbf{w}(x) = (w(x)|w'(x))$  be elements in  $\mathcal{R}$  such that  $\mathbf{v}(x) \circ \mathbf{w}(x) = 0$ . If  $v'(x) = 0$  or  $w'(x) = 0$ , then*

$$v(x)\psi^{m-\deg(w(x))}(w^*(x)) \equiv 0 \pmod{(x^\alpha - 1)}.$$

Respectively, if  $v(x) = 0$  or  $w(x) = 0$ , then

$$v'(x)\psi^{m-\deg(w'(x))}(w'^*(x)) \equiv 0 \pmod{(x^\beta - 1)}.$$

*Proof.* Suppose that  $v'(x) = 0$  or  $w'(x) = 0$ . Therefore

$$\mathbf{v}(x) \circ \mathbf{w}(x) = uv(x)\psi^{m-\deg(w(x))}(w^*(x))x^{m-1-\deg(w(x))} \Gamma_{\frac{m}{\alpha}}(x^\alpha) + 0 \pmod{(x^m - 1)}.$$

This imply that there exists  $f(x) \in R_2[x, \Theta]$  such that

$$uv(x)\psi^{m-\deg(w(x))}(w^*(x))x^{m-1-\deg(w(x))} \Gamma_{\frac{m}{\alpha}}(x^\alpha) = uf(x)(x^m - 1).$$

Since  $\Gamma_{\frac{m}{\alpha}}(x^\alpha) = \frac{x^m - 1}{x^\alpha - 1}$  and  $(x^m - 1)(x^\alpha - 1) = (x^\alpha - 1)(x^m - 1)$ , we have

$$v(x)\psi^{m-\deg(w(x))}(w^*(x))x^m = f(x)x^{\deg(w(x))+1}(x^\alpha - 1).$$

Hence

$$v(x)\psi^{m-\deg(w(x))}(w^*(x)) \equiv 0 \pmod{(x^\alpha - 1)}.$$

The same argument can be used to prove the other case. □

**Notation 3.8.** *Let  $C$  be an  $\mathbb{F}_q\mathbb{F}_q[u]$ -additive skew cyclic code and  $\alpha$  (resp.  $\beta$ ) be the set of  $\mathbb{F}_q$  (resp.  $R_2$ ) coordinate positions. Denote  $C_\alpha$  (resp.  $C_\beta$ ) the punctured code of  $C$  by deleting the coordinates  $\beta$  (resp.  $\alpha$ ).*

**Proposition 3.9.** [11, Lemma 5] *Let  $C$  be an  $\mathbb{F}_q\mathbb{F}_q[u]$ -additive skew cyclic code of length  $\alpha + \beta$ , then  $C^\perp$  is also an additive skew cyclic code.*

If  $C = \mathcal{R}_n((a(x)|0)) + \mathcal{R}_n((k_1(x)|a_1(x) + ug_1(x))) + \mathcal{R}_n((k_2(x)|ua_2(x)))$  is an  $\mathbb{F}_q\mathbb{F}_q[u]$ -additive skew cyclic code of length  $\alpha + \beta$ . For simplicity, we denote the polynomial  $a_1(x) + ug_1(x)$  by  $A(x)$ . The dual code  $C^\perp$  is also an  $\mathbb{F}_q\mathbb{F}_q[u]$ -additive skew cyclic code of length  $\alpha + \beta$ . We denote

$$C^\perp = \mathcal{R}_n((\bar{a}(x)|0)) + \mathcal{R}_n((\bar{k}_1(x)|\bar{A}(x))) + \mathcal{R}_n((\bar{k}_2(x)|u\bar{a}_2(x))),$$

where  $\bar{a}(x) \in \mathcal{F}_\alpha$ ,  $\bar{a}_2(x) \in \mathcal{F}_\beta$ ,  $\bar{A}(x) \in \mathcal{R}_\beta$ ,  $0 \leq \deg(\bar{a}(x)) \leq \alpha$ ,  $\bar{k}_i(x) \in \mathcal{R}_{1,\alpha}$  and  $\deg(\bar{k}_i(x)) < \deg(\bar{a}(x))$ , for  $i = 1, 2$ .

**Theorem 3.10.** *Let*

$$C = \mathcal{R}_n((a(x)|0)) + \mathcal{R}_n((k_1(x)|A(x))) + \mathcal{R}_n((k_2(x)|ua_2(x)))$$

*be an  $\mathbb{F}_q\mathbb{F}_q[u]$ -additive skew cyclic code of length  $\alpha + \beta$ . Let*

$$C^\perp = \mathcal{R}_n((\bar{a}(x)|0)) + \mathcal{R}_n((\bar{k}_1(x)|\bar{A}(x))) + \mathcal{R}_n((\bar{k}_2(x)|u\bar{a}_2(x)))$$

*be its dual code. Then we have*

- (1)  $\bar{a}(x) = \frac{x^\alpha - 1}{\gcd_1(\psi^{m-\deg(a(x))}(a^*(x)), \psi^{m-\deg(k_1(x))}(k_1^*(x)), \psi^{m-\deg(k_2(x))}(k_2^*(x)))}$ .
- (2)  $\bar{k}_1(x)\psi^{m-\deg(a(x))}(a^*(x)) = f(x)(x^\alpha - 1)$ , for some  $f(x) \in \mathbb{F}_q[x, \theta]$ .
- (3)  $\bar{A}(x)\psi^{m-\deg(\frac{\text{lcm}_1(a(x), k_1(x))}{k_1(x)}A(x))}((\frac{\text{lcm}_1(a(x), k_1(x))}{k_1(x)}A(x))^*) = \mu(x)(x^\beta - 1)$ , for some  $\mu(x) \in R_2[x, \Theta]$ .
- (4)  $\bar{k}_2(x)\psi^{m-\deg(a(x))}(a^*(x)) = \nu(x)(x^\alpha - 1)$ , for some  $\nu(x) \in \mathbb{F}_q[x, \theta]$ .
- (5)  $\bar{a}_2(x)\psi^{m-\deg(\frac{\text{lcm}_1(k_1(x), k_2(x))}{k_1(x)}A(x))}((\frac{\text{lcm}_1(k_1(x), k_2(x))}{k_1(x)}A(x))^*) = \lambda(x)(x^\beta - 1)$ , for some  $\lambda(x) \in R_2[x, \Theta]$ .

*Proof.* (1) Since  $(\bar{a}(x)|0)$  is an element in  $C^\perp$ , it follows that

$$\begin{aligned} (\bar{a}(x)|0) \circ (a(x)|0) &\equiv 0 \pmod{(x^m - 1)}, \\ (\bar{a}(x)|0) \circ (k_1(x)|A(x)) &\equiv 0 \pmod{(x^m - 1)} \end{aligned}$$

and

$$(\bar{a}(x)|0) \circ (k_2(x)|ua_2(x)) \equiv 0 \pmod{(x^m - 1)}.$$

By Lemma 3.7,

$$\begin{aligned} \bar{a}(x)\psi^{m-\deg(a(x))}(a^*(x)) &\equiv 0 \pmod{(x^\alpha - 1)}, \\ \bar{a}(x)\psi^{m-\deg(k_1(x))}(k_1^*(x)) &\equiv 0 \pmod{(x^\alpha - 1)} \end{aligned}$$

and

$$\bar{a}(x)\psi^{m-\deg(k_2(x))}(k_2^*(x)) \equiv 0 \pmod{(x^\alpha - 1)}.$$

Using Lemma 2.9,

$$\bar{a}(x)\gcd_l(\psi^{m-\deg(a(x))}(a^*(x)), \psi^{m-\deg(k_1(x))}(k_1^*(x)), \psi^{m-\deg(k_2(x))}(k_2^*(x))) \equiv 0 \pmod{(x^\alpha - 1)}.$$

Therefore, there exists  $h(x) \in \mathbb{F}_q[x, \theta]$  such that

$$\bar{a}(x)\gcd_l(\psi^{m-\deg(a(x))}(a^*(x)), \psi^{m-\deg(k_1(x))}(k_1^*(x)), \psi^{m-\deg(k_1(x))}(k_1^*(x))) = h(x)(x^\alpha - 1).$$

Since  $C_\alpha = \mathcal{R}_{1,\alpha}(\gcd_r(a(x), k_1(x), k_2(x)))$ , we have

$$\bar{a}(x) = \frac{x^\alpha - 1}{\gcd_l(\psi^{m-\deg(a(x))}(a^*(x)), \psi^{m-\deg(k_1(x))}(k_1^*(x)), \psi^{m-\deg(k_2(x))}(k_2^*(x)))}.$$

(2) Inasmuch as  $(\bar{k}_1(x)|\bar{A}(x)) \in C^\perp$ , hence

$$(\bar{k}_1(x)|\bar{A}(x)) \circ (a(x)|0) \equiv 0 \pmod{(x^m - 1)}.$$

By Lemma 3.7,  $\bar{k}_1(x)\psi^{m-\deg(a(x))}(a^*(x)) \equiv 0 \pmod{(x^\alpha - 1)}$ . Thus there exists  $f(x) \in \mathbb{F}_q[x, \theta]$  such that  $\bar{k}_1(x)\psi^{m-\deg(a(x))}(a^*(x)) = f(x)(x^\alpha - 1)$ .

(3) Let

$$\begin{aligned} c(x) &= \frac{\text{lcm}_l(a(x), k_1(x))}{k_1(x)} \cdot (k_1(x)|A(x)) - \frac{\text{lcm}_l(a(x), k_1(x))}{a(x)} \cdot (a(x)|0) \\ &= (0|\frac{\text{lcm}_l(a(x), k_1(x))}{k_1(x)}A(x)). \end{aligned}$$

Hence  $c(x) \in C$ , which implies that

$$(\bar{k}_1(x)|\bar{A}(x)) \circ (0|\frac{\text{lcm}_l(a(x), k_1(x))}{k_1(x)}A(x)) = 0.$$

Using Lemma 3.7, there exists  $\mu(x) \in R_2[x; \Theta]$  such that

$$\bar{A}(x)\psi^{m-\deg(\frac{\text{lcm}_l(a(x), k_1(x))}{k_1(x)}A(x))}((\frac{\text{lcm}_l(a(x), k_1(x))}{k_1(x)}A(x))^*) = \mu(x)(x^\beta - 1).$$

(4) Similarly to the proof of (2).

(5) Let

$$\begin{aligned} c(x) &= \frac{\text{lcm}_l(k_1(x), k_2(x))}{k_1(x)} \cdot (k_1(x)|A(x)) - \frac{\text{lcm}_l(k_1(x), k_2(x))}{k_2(x)} \cdot (k_2(x)|ua_2(x)) \\ &= (0|\frac{\text{lcm}_l(k_1(x), k_2(x))}{k_1(x)}A(x) - u\frac{\text{lcm}_l(k_1(x), k_2(x))}{k_2(x)}a_2(x)). \end{aligned}$$

Hence  $c(x) \in C$ , which implies that

$$(\bar{k}_2(x)|u\bar{a}_2(x)) \circ (0|\frac{\text{lcm}_l(k_1(x), k_2(x))}{k_1(x)}A(x) - u\frac{\text{lcm}_l(k_1(x), k_2(x))}{k_2(x)}a_2(x)) = 0.$$

Using Lemma 3.7, there exists  $f(x) \in R_2[x; \theta]$  such that

$$u\bar{a}_2(x)\psi^{m-\deg(\frac{\text{lcm}_l(k_1(x), k_2(x))}{k_1(x)}A(x))}((\frac{\text{lcm}_l(k_1(x), k_2(x))}{k_1(x)}A(x))^*) = uf(x)(x^\beta - 1).$$

□

When  $\theta$  is the identity automorphism, we have the following corollary.

**Corollary 3.11.** *Let  $C = \langle (a(x)|0), (k_1(x)|A(x)), (k_2(x)|ua_2(x)) \rangle$  be an  $\mathbb{F}_q\mathbb{F}_q[u]$ -additive cyclic code of length  $\alpha + \beta$ . Let*

$$C^\perp = \langle (\bar{a}(x)|0), (\bar{k}_1(x)|\bar{A}(x)), (\bar{k}_2(x)|u\bar{a}_2(x)) \rangle$$

be its dual code. Then we have

$$(1) \bar{a}(x) = \frac{x^\alpha - 1}{\gcd(a^*(x), k_1^*(x), k_2^*(x))}.$$

$$(2) \bar{k}_1(x)a^*(x) = f(x)(x^\alpha - 1), \text{ for some } f(x) \in \mathbb{F}_q[x].$$

(3) In the ring  $\mathbb{F}_q[x]$ , we have

$$\text{lcm}(a(x), k_1(x)) \cdot \gcd(a(x), k_1(x)) = a(x)k_1(x).$$

Hence,  $\bar{A}(x)a^*(x)A^*(x) = \mu(x)\gcd(a^*(x), k_1^*(x))(x^\beta - 1)$ , for some  $\mu(x) \in R_2[x]$ .

$$(4) \bar{k}_2(x)a^*(x) = \nu(x)(x^\alpha - 1), \text{ for some } \nu(x) \in \mathbb{F}_q[x].$$

$$(5) \bar{a}_2(x)k_1^*(x)A^*(x) = \lambda(x)\gcd(k_1^*(x), k_2^*(x))(x^\beta - 1), \text{ for some } \lambda(x) \in R_2[x].$$

### 3.2. Examples.

In this subsection, we provide some examples of  $\mathbb{F}_q\mathbb{F}_q[u]$ -additive skew cyclic codes of length  $\alpha + \beta$ , which are generated by different factors of  $x^\alpha - 1$  and  $x^\beta - 1$  and we specify their dual.

EXAMPLE 3.12. Consider  $\mathcal{R}_{1,3} = \frac{\mathbb{F}_{27}[x;\theta]}{\langle x^3-1 \rangle}$  and  $\mathcal{R}_9 = \frac{(\mathbb{F}_{27}+u\mathbb{F}_{27})[x;\Theta]}{\langle x^9-1 \rangle}$ , with  $\Theta$  be an automorphism of  $\mathbb{F}_{27}+u\mathbb{F}_{27}$  which is defined by  $\Theta(a+ub) = \theta(a)+u\theta(b)$ , where  $\theta$  is the Frobenius map,  $\theta(\alpha) = \alpha^3$ , for all  $\alpha \in \mathbb{F}_{27}$ . Clearly,  $o(\Theta) = o(\theta) = 3$ . Let  $\delta$  be a primitive 26th root of unity in  $\mathbb{F}_{27}$ , i.e,  $\mathbb{F}_{27} = \{0, \delta, \dots, \delta^{25}, \delta^{26} = 1\}$ . A factorization of  $x^3 - 1$  in  $\mathbb{F}_{27}[x; \theta]$  is

$$x^3 - 1 = (x - \delta^2)(x - \delta^{18})(x - \delta^6).$$

- Let  $C = \mathcal{R}_{12}(((x - \delta^{18})(x - \delta^6)|0))$ . Then  $C^\perp = \mathcal{R}_{12}((\delta^6x - 1|0))$ .

Consider the following code:

- $C = \mathcal{R}_{12}((x - \delta^2|0)) + \mathcal{R}_{12}((\delta|x^3 - 1)) + \mathcal{R}_{12}((\delta^2 + 1|u(x - \delta^{18})(x - \delta^6)))$ .

By Theorem 3.10, we have

$$(1) \bar{a}(x) = \frac{x^3-1}{\gcd_t(\psi^8(1-\delta^6x), \psi^9(\delta), \psi^9(\delta^2+1))} = \delta^{25}(x^3-1).$$

$$(2) \bar{k}_1(x)(1-\delta^2x) = f(x)(x^3-1), \text{ for some } f(x) \in \mathbb{F}_{27}[x, \theta].$$

$$(3) \bar{A}(x)(\delta^{25}-\delta x-\delta^{25}x^3+\delta x^4) = \mu(x)(x^9-1), \text{ for some } \mu(x) \in R_2[x, \Theta].$$

$$(4) \bar{k}_2(x)(1-\delta^2x) = \nu(x)(x^3-1), \text{ for some } \nu(x) \in \mathbb{F}_{27}[x, \theta].$$

$$(5) \bar{a}_2(x)((\delta^{25}+\delta)x^3-\delta^{25}-\delta) = -\lambda(x)(x^9-1), \text{ for some } \lambda(x) \in R_2[x, \Theta].$$

EXAMPLE 3.13. Consider  $\mathcal{R}_{1,2} = \frac{\mathbb{F}_{16}[x;\theta]}{\langle x^2-1 \rangle}$  and  $\mathcal{R}_4 = \frac{(\mathbb{F}_{16}+u\mathbb{F}_{16})[x;\Theta]}{\langle x^4-1 \rangle}$  with  $\Theta$  be an automorphism of  $\mathbb{F}_{16}+u\mathbb{F}_{16}$  which is defined by  $\Theta(a+ub) = \theta(a)+u\theta(b)$ , where  $\theta$  is the Frobenius map,  $\theta(\nu) = \nu^2$ , for all  $\nu \in \mathbb{F}_{16}$ . Then  $o(\Theta) = o(\theta) = 4$ . Let  $\delta$  be a primitive 15th root of unity in  $\mathbb{F}_{16}$ , i.e.,  $\mathbb{F}_{16} = \{0, \delta, \dots, \delta^{14}, \delta^{15} = 1\}$ . Consider a factorization of  $x^4-1$  in  $\mathbb{F}_{16}[x; \theta]$

$$x^4-1 = (x-\delta^{10})(x-\delta^5)(x-\delta^{10})(x-\delta^5).$$

• Suppose  $C = \mathcal{R}_6((x-\delta^5|0))$ . We have  $\bar{a}(x) = \frac{x^2-1}{\psi^3(1-\delta^{10}x)} = \delta^5(x-\delta^{10})$  and

$$C^\perp = \mathcal{R}_6((\delta^5(x-\delta^{10})|0)).$$

Consider the following code:

•  $C = \mathcal{R}_6((x-\delta^{10}|0)) + \mathcal{R}_6((\delta^2|(x-1)(x-\delta^{10}))) + \mathcal{R}_6((\delta^2+1|u(x-\delta^{10})))$ .

Using Theorem 3.10, we have

$$(1) \bar{a}(x) = \frac{x^2-1}{\gcd_t(\psi(1-\delta^5x), \psi^4(\delta^2), \psi^4(\delta^2+1))} = \delta^{13}(x^2-1).$$

$$(2) \bar{k}_1(x)(1-\delta^{10}x) = f(x)(x^2-1), \text{ for some } f(x) \in \mathbb{F}_{16}[x, \theta].$$

$$(3) \bar{A}(x)(\delta^7+\delta^{10}x+\delta^{12}x^2+\delta^3x^3) = \mu(x)(x^4-1), \text{ for some } \mu(x) \in R_2[x, \Theta].$$

$$(4) \bar{k}_2(x)(1-\delta^{10}x) = \nu(x)(x^2-1), \text{ for some } \nu(x) \in \mathbb{F}_{16}[x, \theta].$$

$$(5) \bar{a}_2(x)(\delta^9+\delta^8x+\delta x^2) = \lambda(x)(x^4-1), \text{ for some } \lambda(x) \in R_2[x, \Theta].$$

EXAMPLE 3.14. Consider  $\mathcal{R}_{1,4} = \frac{\mathbb{F}_{16}[x;\theta]}{\langle x^4-1 \rangle}$  and  $\mathcal{R}_8 = \frac{(\mathbb{F}_{16}+u\mathbb{F}_{16})[x;\Theta]}{\langle x^8-1 \rangle}$  with  $\Theta$  be an automorphism of  $\mathbb{F}_{16}+u\mathbb{F}_{16}$  which is defined by  $\Theta(a+ub) = \theta(a)+u\theta(b)$ , where  $\theta$  is the Frobenius map,  $\theta(\nu) = \nu^4$ , for all  $\nu \in \mathbb{F}_{16}$ . Then  $o(\Theta) = o(\theta) = 2$ . A factorization of  $x^8-1$  in  $\mathbb{F}_{16}[x; \theta]$  is

$$x^8-1 = (x-\delta)(x-\delta^3)(x-\delta^5)(x-\delta^6)(x-\delta)(x-\delta^3)(x-\delta^5)(x-\delta^6).$$

Consider:

$$C = \mathcal{R}_{12}(((x-\delta)(x-\delta^3)|0)) + \mathcal{R}_{12}((\delta^5|(x^4-1)(x-\delta)(x-\delta^3))) + \mathcal{R}_{12}((\delta+1|u(x-\delta^3))).$$

By Theorem 3.10, we can get

$$(1) \bar{a}(x) = \frac{x^4-1}{\gcd_l(\psi^6(1+\delta^7x+\delta^4x^2), \psi^8(\delta^5), \psi^8(\delta+1))} = \delta^{11}(x^4-1).$$

$$(2) \bar{k}_1(x)(1 + \delta^7x + \delta^4x^2) = f(x)(x^4-1), \text{ for some } f(x) \in \mathbb{F}_{16}[x, \theta].$$

$$(3) \bar{A}(x)(1 + \delta^2x + \delta^{12}x^2 + x^4 + \delta^2x^5 + \delta^{12}x^6) = \mu(x)(x^8-1), \text{ for some } \mu(x) \in R_2[x, \Theta].$$

$$(4) \bar{k}_2(x)(1 + \delta^7x + \delta^4x^2) = \nu(x)(x^4-1), \text{ for some } \nu(x) \in \mathbb{F}_{16}[x, \theta].$$

$$(5) \bar{a}_2(x)(1 - \delta^7x + \delta^4x^2 + x^4 + \delta^7x^5 + \delta^4x^6) = \lambda(x)(x^8-1), \text{ for some } \lambda(x) \in R_2[x, \Theta].$$

#### 4. CONCLUSIONS

In this paper, we studied the structure of the (Euclidean) dual of  $\mathbb{F}_q\mathbb{F}_q[u]$ -additive skew cyclic codes in terms of their generating polynomials, where  $q$  is a power of prime integer and  $u^2 = 0$ .

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