

## Some Properties of a Special Subalgebra of Central Derivations

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ABSTRACT. Let  $L$  be a Lie algebra, and  $\text{Der}(L)$  and  $\text{IDer}(L)$  be the set of all derivations and inner derivations of  $L$ , respectively. Let  $\mathcal{D}$  be a subalgebra of  $\text{Der}(L)$  such that it contains  $\text{IDer}(L)$  and  $H = \bigcap_{\alpha \in \mathcal{D}} \text{Ker}\alpha$ . If  $\text{Der}^H(L)$  denotes the set of all derivations of  $L$  whose images are in  $H$ , then we give necessary and sufficient conditions under which  $\text{Der}^H(L)$  is equal to some subalgebras of  $\text{Der}(L)$  for finite dimensional nilpotent Lie algebras.

**Keywords:** Derivation, Central derivation, Inner derivation.

**2000 Mathematics subject classification:** Primary 16W25, Secondary 17B40.

### 1. INTRODUCTION AND PRELIMINARIES

Let  $L$  be a Lie algebra over an arbitrary field  $F$ . We denote by  $L^2$  and  $Z(L)$  the derived algebra and the center of  $L$ , respectively. The linear map  $\alpha : L \rightarrow L$  is said to be a derivation if

$$\alpha([x, y]) = [\alpha(x), y] + [x, \alpha(y)],$$

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for all  $x, y \in L$ . The vector space of all derivations of  $L$  is denoted by  $\text{Der}(L)$ . It is well-known that  $\text{Der}(L)$  is a Lie algebra through the following bracket

$$[\alpha_1, \alpha_2] := \alpha_1 \circ \alpha_2 - \alpha_2 \circ \alpha_1,$$

for all  $\alpha_1, \alpha_2 \in \text{Der}(L)$ . This Lie algebra is called the *derivation algebra* of  $L$ . The map  $\text{ad}_x : L \rightarrow L$  given by  $y \mapsto [x, y]$  is a derivation called *inner derivation* for all  $x \in L$ . The set  $\text{IDer}(L) = \{\text{ad}_x \mid x \in L\}$  of inner derivations is an ideal of  $\text{Der}(L)$ .

Study of the necessary and sufficient conditions under which subalgebras of  $\text{Der}(L)$  coincide gose back to sixties when Jacobson [2], initiated study the complete Lie algebra. A Lie  $L$  is said to be *complete*, if its center is zero and  $\text{Der}(L) = \text{IDer}(L)$ . Later these Lie algebras are studied in [3, 4, 5, 6, 7, 8, 9, 10, 15]. Also, Tôgô [17] characterized Lie algebras  $L$  over a field of characteristic zero with  $Z(L) \neq 0$  for which  $\text{Der}_{Z(L)}(L) = \text{IDer}(L)$  and  $\text{Der}_{Z(L)}(L) = \text{Der}(L)$ , where  $\text{Der}_{Z(L)}(L)$  is a subalgebra of  $\text{Der}(L)$  consisting of all central derivations of  $L$ , that is, the set of all derivations of  $L$  whose images lie in the center of  $L$ .

Sheikh-Mohseni et al. [11, 14] introduced the new subalgebra  $\text{Der}_J^I(L)$ , the set of all derivations of  $L$  whose images are in  $I$  and send  $J$  to zero, and  $\text{Der}_c(L)$ , the set of all derivations  $\alpha$  for which  $\alpha(x) \in [x, L] := \{[x, y] \mid y \in L\}$  and investigated the equalities  $\text{Der}_J^I(L) = \text{IDer}(L)$ ,  $\text{Der}_c(L) = \text{IDer}(L)$  and  $\text{Der}_c(L) = \text{Der}_{Z(L)}(L)$ . Also, they characterized those Lie algebras for which  $\text{ID}(L) = \text{IDer}(L)$ ,  $\text{ID}^*(L) = \text{Der}_{Z(L)}(L)$ , where  $\text{ID}(L) = \text{Der}^{L^2}(L)$  and  $\text{ID}^*(L) = \text{Der}_{Z(L)}^{L^2}(L)$  (See [12, 13]).

Let  $\mathcal{D}$  be a subalgebra of  $\text{Der}(L)$  and it contains  $\text{IDer}(L)$ . Put  $H = \bigcap_{\alpha \in \mathcal{D}} \text{Ker} \alpha$ . Clearly,  $H$  is a central subalgebra of  $L$ . In fact, if  $\mathcal{D} = \text{IDer}(L)$ , then  $H = Z(L)$ . Also, if  $\mathcal{D} = \text{Der}(L)$ , then  $H$  is a absolut center of  $L$  was introduced by stitzinger [16] in 1944. The aim of this paper is characterized the Lie algebras  $L$  such that  $\text{Der}^H(L)$  is equal to  $\text{Der}_c(L)$ . Also, we give necessary and sufficient conditions under which  $\text{Der}_{Z(L)}^H(L) = \text{ID}^*(L)$ . In final, as an application of our results, we find those nilpotent Lie algebras  $L$  for which  $\dim L \leq 6$  and  $\text{Der}^H(L) = \text{Der}_c(L)$  or  $\text{Der}_{Z(L)}^H(L) = \text{ID}^*(L)$ .

Recall that for a Lie algebra  $L$ , the *lower central series* of  $L$  is defined as follows:

$$L = L^1 \supseteq L^2 \supseteq \cdots \supseteq L^n \supseteq \cdots,$$

where  $L^2$  is the derived algebra of  $L$  and  $L^n = [L^{n-1}, L]$ .

Also, the *upper central series* of  $L$  is defined as

$$\{0\} = Z_0(L) \subseteq Z_1(L) \subseteq \cdots \subseteq Z_n(L) \subseteq \cdots,$$

where  $Z_1(L) = Z(L)$  is the center of  $L$  and  $Z_{n+1}(L)/Z_n(L) = Z(L/Z_n(L))$ .

A Lie algebra  $L$  is *nilpotent* if there exists a non-negative integer  $k$  such that  $L^k = 0$  (or  $Z_k(L) = L$ ). The smallest integer  $k$  for which  $L^{k+1} = 0$  (or  $Z_k(L) = L$ ) is called the *nilpotency class* of  $L$ . A Lie algebra  $L$  of dimension  $n$

is *fiform* if it has maximal nilpotency class  $n - 1$ ,  $\dim Z(L) = 1$  and  $\dim L^2 = n - 2$ . de Graaf [1] classified all nilpotent Lie algebras of dimension at most 6.

We recall that the Heisenberg Lie algebras are nilpotent of class 2. A Lie algebra  $L$  is called Heisenberg Lie algebra if  $L^2 = Z(L)$  and  $\dim L^2 = 1$ . Heisenberg Lie algebras of finite dimension are odd dimensional with basis  $x_1, \dots, x_{2k}, x_{2k+1}$  and the only non-zero multiplication between basis elements are  $[x_{2i-1}, x_{2i}] = x_{2i+1}$  for  $i = 1, \dots, k$ . The symbol  $H(k)$  denotes the Heisenberg Lie algebra of dimension  $2k + 1$ .

The following results are of our interest and we use them to characterize all finite dimensional nilpotent Lie algebras of dimension  $n \leq 6$ , for which  $\text{Der}^H(L) = \text{Der}_{Z(L)}(L)$ ,  $\text{Der}_{Z(L)}^H(L) = \text{Der}_{Z(L)}(L)$  and  $\text{Der}_{Z(L)}^H(L) = \text{IDer}(L)$ .

**Theorem 1.1.** [11, Corollary 3] *Let  $L$  be a finite dimensional Lie algebra and  $I, J$  be two ideals of  $L$  such that  $I \subseteq Z(L)$ . Then  $\text{Der}_J^I(L) = \text{Der}_{Z(L)}(L)$  if and only if  $I = Z(L)$  and  $J \subseteq L^2$ .*

If  $I = H$  and  $J = Z(L)$ , then we have  $\text{Der}_{Z(L)}^H(L) = \text{Der}_{Z(L)}(L)$  if and only if  $H = Z(L)$  and  $Z(L) \subseteq L^2$ . Also if  $I = Z(L)$ , since every element of  $L^2$  is sent to zero by every element of  $\text{Der}^H(L)$ , by Theorem 1.1,  $\text{Der}^H(L) = \text{Der}_{Z(L)}(L)$  if and only if  $H = Z(L)$ .

**Theorem 1.2.** [11, Corollary 4] *Let  $I$  and  $J$  be two ideals of a finitely generated non-abelian Lie algebra  $L$  with  $Z(L) \neq 0$  such that  $I \subseteq Z(L) \subseteq J$ . Then  $\text{Der}_J^I(L) = \text{IDer}(L)$  if and only if  $L$  is a finite dimensional nilpotent Lie algebra of class 2,  $J = Z(L)$ ,  $L^2 \subseteq I$  and  $\dim I = 1$*

If  $I = H$  and  $J = Z(L)$ , then  $\text{Der}_{Z(L)}^H(L) = \text{IDer}(L)$  if and only if  $L^2 = H$  and  $\dim H = 1$ .

## 2. MAIN RESULTS

In this section, we give necessary and sufficient conditions under which the Lie algebra  $\text{Der}^H(L) = \text{Der}_c(L)$  and  $\text{Der}_{Z(L)}^H(L) = \text{ID}^*(L)$  for finite dimensional nilpotent Lie algebras.

Let  $A, B$  be two Lie algebras and  $\text{Hom}(A, B)$  be the set of all homomorphism from  $A$  to  $B$ . Clearly, if  $B$  is an abelian Lie algebra, then  $\text{Hom}(A, B)$  equipped with Lie bracket  $[f, g](x) = [f(x), g(x)]$  for all  $x \in A$  and  $f, g \in \text{Hom}(A, B)$  is an abelian Lie algebra.

The following lemma is useful for the proof of our main results.

**Lemma 2.1.** *If  $U, V$  and  $W$  are finite dimensional Lie algebras and  $W$  be Lie algebra abelian, then*

$$\text{Hom}(U \oplus V, W) \cong \text{Hom}(U, W) \oplus \text{Hom}(V, W).$$

**Lemma 2.2.** *Let  $I, J$  be two ideals of Lie algebra  $L$  such that  $I \subseteq J$  and  $I \subseteq Z(L)$ . Then*

$$\text{Der}_J^I(L) \cong \text{Hom}(L/J, I).$$

*Proof.* For any  $\alpha \in \text{Der}_J^I(L)$ , the map  $\psi_\alpha : L/J \rightarrow I$  defined by  $\psi_\alpha(x + J) = \alpha(x)$  for all  $x \in L$  is a Lie homomorphism. Since  $\text{Der}_J^I(L)$  and  $\text{Hom}(L/J, I)$  are abelian, it is easy to see that the map  $\psi : \text{Der}_J^I(L) \rightarrow \text{Hom}(L/J, I)$  defined by  $\psi(\alpha) = \psi_\alpha$  is a Lie isomorphism.  $\square$

**Lemma 2.3.** *Let  $L$  be a non-abelian nilpotent Lie algebra of finite dimension and  $\text{Der}^H(L) = \text{Der}_c(L)$ . Then*

- (i)  $\text{Der}_c(L) \cong \text{Hom}(L/Z(L), L^2) \cong \text{Hom}(L/Z(L), H) \cong \text{Hom}(L/L^2, H)$ .
- (ii)  $\dim \text{Der}_c(L) = \sum_{i=1}^d \dim[x, L]$ , where  $\{x_1, \dots, x_d\}$  is a minimal generating set for  $L$ .

*Proof.* (i) Since  $\text{IDer}(L) \subseteq \text{Der}_c(L) = \text{Der}^H(L)$ ,  $L^2 \subseteq H$ . Therefore

$$\text{Der}_c(L) \subseteq \text{ID}^*(L) \subseteq \text{Der}_{Z(L)}^H(L) \subseteq \text{Der}^H(L) = \text{Der}_c(L).$$

Now, by Lemma 2.2,  $\text{Der}_c(L) = \text{ID}^*(L) \cong \text{Hom}(L/Z(L), L^2)$  and  $\text{Der}_c(L) = \text{Der}_{Z(L)}^H(L) \cong \text{Hom}(L/Z(L), H)$ . Therefore  $L^2 = H$ . Since every element of  $L^2$  is sent to zero by every element of  $\text{Der}^H(L)$ , by Lemma 2.2,

$$\text{Der}_c(L) = \text{Der}_{Z(L)}^H(L) = \text{Der}_{L^2}^H(L) \cong \text{Hom}(L/L^2, H).$$

(ii) By using Proposition 2.3 in [14], we have  $\dim \text{Der}_c(L) \leq \sum_{i=1}^d \dim[x, L]$ . On the other hand,

$$\begin{aligned} \dim \text{Der}_c(L) &= \dim \text{Hom}(L/Z(L), L^2) \\ &= \dim \text{Hom}(\langle \bar{x}_1 \rangle \oplus \dots \oplus \langle \bar{x}_d \rangle, L^2) \\ &= \sum_{i=1}^d \dim \text{Hom}(\langle \bar{x}_i \rangle, L^2) \\ &= \sum_{i=1}^d \dim L^2 \geq \sum_{i=1}^d \dim[x_i, L]. \end{aligned}$$

This completes the proof.  $\square$

**Definition.** Let  $L$  be a finite dimensional Lie algebra and  $I \neq 0$  be an ideal of  $L$ . Then  $(L, I)$  is called a Camina pair if  $I \subseteq [x, L]$  for all  $x \in L \setminus I$ .  $L$  is called a Camina Lie algebra if  $(L, L^2)$  is a Camina pair.

**Theorem 2.4.** *Let  $L$  be a finite dimensional non-abelian nilpotent Lie algebra. Then  $\text{Der}^H(L) = \text{Der}_c(L)$  if and only if  $L^2 = H = Z(L)$  and  $L$  is a Camina Lie algebra.*

*Proof.* First suppose that  $L$  is a Camina Lie algebra and  $L^2 = H = Z(L)$ . Then  $\text{Der}^H(L) = \text{Der}_{Z(L)}(L)$  and  $\text{Der}_c(L) \subseteq \text{Der}_{Z(L)}(L)$ . Now, let  $\alpha \in \text{Der}_{Z(L)}(L)$  and  $x \in L^2$ . Therefore  $\alpha(x) = 0 \in [x, L]$ . If  $x \in L \setminus L^2$ , then  $\alpha(x) \in [x, L]$ , because  $L$  is a Camina Lie algebra. Therefore  $\alpha \in \text{Der}_c(L)$  and the result follows.

Conversely suppose that  $\text{Der}^H(L) = \text{Der}_c(L)$ . Then by Lemma 2.1,  $L^2 = H \subseteq Z(L)$ . On the other hand, by Lemma 2.3(i),

$$\dim \text{Der}_c(L) = \dim \text{Hom}(L/Z(L), H) = \dim \text{Hom}(L/L^2, H).$$

Therefore  $L^2 = Z(L)$  and  $L^2 = H = Z(L)$ . If there exists  $x \in L \setminus L^2$  such that  $[x, L] \subsetneq L^2$ , then we can assume without loss of generality that  $\{x = x_1, x_2, \dots, x_d\}$  is a minimal generating set for  $L$ . Now, by Lemma 2.3(ii),

$$\dim \text{Der}_c(L) = \sum_{i=1}^d \dim [x_i, L] \not\geq d \dim L^2.$$

On the other hand by Lemma 2.3(i), we have

$$\dim \text{Der}_c(L) = \dim \text{Hom}(L/Z(L), L^2) = d \dim L^2,$$

which is a contradiction. Therefore,  $L$  is a Camina Lie algebra.  $\square$

As an immediate consequence of Theorem 2.4, we obtain the following result.

**Corollary 2.5.** [14, Corollary 4.4] *Let  $L$  be a finite dimensional non-abelian nilpotent Lie algebra. Then  $\text{Der}_c(L) = \text{Der}_{Z(L)}(L)$  if and only if  $L$  is a Camina Lie algebra of class 2.*

*Proof.* By putting  $\mathcal{D} = \text{IDer}(L)$ , we have  $H = Z(L)$  and the result is clear.  $\square$

**Theorem 2.6.** *Let  $L$  be a finite dimensional non-abelian nilpotent Lie algebra. Then  $\text{Der}_{Z(L)}^H(L) = \text{ID}^*(L)$  if and only if  $L^2 = H$ .*

*Proof.* First, suppose that  $\text{Der}_{Z(L)}^H(L) = \text{ID}^*(L)$ . Then, since  $\text{IDer}(L) \subseteq \text{ID}^*(L) = \text{Der}_{Z(L)}^H(L)$ , we have  $L^2 \subseteq H$ . Also, by Lemma 2.2,

$$\text{Der}_{Z(L)}^H(L) \cong \text{Hom}(L/Z(L), H), \text{ID}^*(L) \cong \text{Hom}(L/Z(L), L^2).$$

Therefore

$$\dim \text{Der}_{Z(L)}^H(L) = \dim(L/Z(L)) \times \dim H, \dim \text{ID}^*(L) = \dim L/Z(L) \times \dim L^2.$$

Since  $L^2 \subseteq H$  and  $\dim \text{ID}^*(L) = \dim \text{Der}_{Z(L)}^H(L)$ , we have  $L^2 = H$ .

The converse is clear.  $\square$

**Corollary 2.7.** *If  $L$  is a non-abelian filiform Lie algebra of finite dimension, then  $\text{Der}_{Z(L)}^H(L) = \text{ID}^*(L)$  if and only if  $L^2 = H = Z(L)$ .*

Let  $L$  be an  $n$ -dimensional Lie algebra with basis  $x_1, \dots, x_n$  and  $\alpha \in \text{Der}(L)$ . Using this basis, we may define scalars  $\alpha_{i,j}$  by  $\alpha(x_i) = \sum_{j=1}^n \alpha_{i,j} x_j$ . We say that the following matrix is the matrix of  $\alpha$  with respect to our chosen basis and  $\alpha$  have the following matrix forms.

$$\begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} & \cdots & \alpha_{1,n} \\ \alpha_{2,1} & \alpha_{2,2} & \cdots & \alpha_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n,1} & \alpha_{n,2} & \cdots & \alpha_{n,n} \end{bmatrix}.$$

The following example shows that the conditions given in Theorems 2.4 and 2.6 are necessary.

**Example.** Let  $L$  be a Lie algebra with basis  $x_1, \dots, x_4$  and the only non-zero multiplication between basis elements is  $[x_1, x_2] = x_3$ . Then  $L^2 = \langle x_3 \rangle$ ,  $Z(L) = \langle x_3, x_4 \rangle$  and  $L^3 = 0$ . It is obvious that elements of  $\text{Der}_c(L)$  has the following matrix form

$$\begin{bmatrix} 0 & 0 & \alpha_{1,3} & 0 \\ 0 & 0 & \alpha_{2,3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore  $\dim \text{Der}_c(L) = 2$ . If  $\mathcal{D} = \text{Der}_c(L)$ , then  $H = Z(L) = \langle x_3, x_4 \rangle$  and  $\text{Der}^H(L)$  has the following matrix form

$$\begin{bmatrix} 0 & 0 & \alpha_{1,3} & \alpha_{1,4} \\ 0 & 0 & \alpha_{2,3} & \alpha_{2,4} \\ 0 & 0 & 0 & \alpha_{3,4} \\ 0 & 0 & \alpha_{4,3} & \alpha_{4,4} \end{bmatrix}$$

Thus  $\dim \text{Der}^H(L) = 7$  and  $\text{Der}^H(L) \neq \text{Der}_c(L)$ . Also, it is easy to see that  $\text{ID}^*(L)$  and  $\text{Der}_{Z(L)}^H(L)$  have the following matrix forms

$$\begin{bmatrix} 0 & 0 & \alpha_{1,3} & 0 \\ 0 & 0 & \alpha_{2,3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in \text{ID}^*(L),$$

$$\begin{bmatrix} 0 & 0 & \alpha_{1,3} & \alpha_{1,4} \\ 0 & 0 & \alpha_{2,3} & \alpha_{2,4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in \text{Der}_{Z(L)}^H(L).$$

Therefore  $\text{Der}_{Z(L)}^H(L) \neq \text{ID}^*(L)$ .

## 3. APPLICATIONS

In this section, we use the classification of all finite dimensional nilpotent Lie algebras of dimension  $n$ , where  $n \leq 6$  given by de Graaf [1]. As an application of our results, we find those nilpotent Lie algebras  $L$  of dimension  $n \leq 6$  for which  $\text{Der}^H(L) = \text{Der}_c(L)$  or  $\text{Der}_{Z(L)}^H(L) = \text{ID}^*(L)$ . Also, we use theorem 1.1 and theorem 1.2 to characterize all nilpotent Lie algebras of dimension  $n \leq 6$ , for which  $\text{Der}^H(L) = \text{Der}_{Z(L)}(L)$ ,  $\text{Der}_{Z(L)}^H(L) = \text{Der}_{Z(L)}(L)$  and  $\text{Der}_{Z(L)}^H(L) = \text{IDer}(L)$ .

Clearly, if  $L$  is an abelian Lie algebra, then

$$\text{IDer}(L) = \text{Der}_c(L) = \text{ID}^*(L) = \text{ID}(L) = 0.$$

In this case, if  $\mathcal{D} = \text{Der}(L)$ , then  $H = 0$  and  $\text{Der}^H(L) = 0$ , if  $\mathcal{D} = \text{IDer}(L)$ , then  $H = Z(L) = L$  thus  $\text{Der}^H(L) \neq \text{Der}_c(L)$  and  $\text{Der}_{Z(L)}^H(L) \neq \text{ID}^*(L)$ . Therefore in the sequel, we consider non-abelian Lie algebras.

**Corollary 3.1.** *Let  $L$  be a non-abelian nilpotent Lie algebras with  $\dim L \leq 6$ . Then*

- (i) *If  $\mathcal{D} = \text{IDer}(L)$ , then  $\text{Der}^H(L) = \text{Der}_c(L)$  if and only if  $L$  is one of the following Lie algebra.*

$$H(1), H(2) \quad \text{or} \quad L_{6,22}(\varepsilon), \varepsilon \neq 0.$$

- (ii) *If  $\mathcal{D} = \text{Der}_c(L)$ , then  $\text{Der}^H(L) = \text{Der}_c(L)$  if and only if  $L$  is one of the following Lie algebra.*

$$H(1), H(2) \quad \text{or} \quad L_{6,22}(\varepsilon), \varepsilon \neq 0.$$

- (iii) *If  $\mathcal{D} = \text{ID}^*(L)$ , then  $\text{Der}^H(L) = \text{Der}_c(L)$  if and only if  $L$  is one of the following Lie algebra.*

$$H(1), H(2) \quad \text{or} \quad L_{6,22}(\varepsilon), \varepsilon \neq 0.$$

- (iv) *If  $\mathcal{D} = \text{ID}(L)$ , then  $\text{Der}^H(L) = \text{Der}_c(L)$  if and only if  $L$  is one of the following Lie algebra.*

$$H(1), H(2) \quad \text{or} \quad L_{6,22}(\varepsilon), \varepsilon \neq 0.$$

**Corollary 3.2.** *Let  $L$  be a non-abelian nilpotent Lie algebras with  $\dim L \leq 6$ . Then*

- (i) *If  $\mathcal{D} = \text{IDer}(L)$ , then  $\text{Der}_{Z(L)}^H(L) = \text{ID}^*(L)$  if and only if  $L$  is one of the following Lie algebras.*

$$H(1), H(2), L_{5,8}, L_{6,22}(\varepsilon) \quad \text{or} \quad L_{6,26}, \varepsilon \neq 0.$$

- (ii) *If  $\mathcal{D} = \text{Der}_c(L)$ , then  $\text{Der}_{Z(L)}^H(L) = \text{ID}^*(L)$  if and only if  $L$  is one of the following Lie algebras.*

$$H(1), H(2), L_{5,8}, L_{6,22}(\varepsilon) \quad \text{or} \quad L_{6,26}, \varepsilon \neq 0.$$

- (iii) If  $\mathcal{D} = \text{ID}^*(L)$ , then  $\text{Der}_{Z(L)}^H(L) = \text{ID}^*(L)$  if and only if  $L$  is one of the following Lie algebras.

$$H(1), H(2), L_{5,8}, L_{6,22}(\varepsilon) \quad \text{or} \quad L_{6,26}, \varepsilon \neq 0.$$

- (iv) If  $\mathcal{D} = \text{ID}(L)$ , then  $\text{Der}_{Z(L)}^H(L) = \text{ID}^*(L)$  if and only if  $L$  is one of the following Lie algebras.

$$H(1), L_{4,2}, L_{5,2}, H(2), L_{5,8}, L_{6,2}, L_{6,4}, L_{6,8}, L_{6,22}(\varepsilon) \quad \text{or} \quad L_{6,26}, \varepsilon \neq 0.$$

**Corollary 3.3.** Let  $L$  be a non-abelian nilpotent Lie algebras with  $\dim L \leq 6$ . Then

- (i) If  $\mathcal{D} = \text{IDer}(L)$ , then  $\text{Der}_{Z(L)}^H(L) = \text{IDer}(L)$  if and only if  $L$  is one of the following Lie algebras.

$$H(1), H(2).$$

- (ii) If  $\mathcal{D} = \text{Der}_c(L)$ , then  $\text{Der}_{Z(L)}^H(L) = \text{IDer}(L)$  if and only if  $L$  is one of the following Lie algebras.

$$H(1), H(2).$$

- (iii) If  $\mathcal{D} = \text{ID}^*(L)$ , then  $\text{Der}_{Z(L)}^H(L) = \text{IDer}(L)$  if and only if  $L$  is one of the following Lie algebras.

$$H(1), H(2).$$

- (iv) If  $\mathcal{D} = \text{ID}(L)$ , then  $\text{Der}_{Z(L)}^H(L) = \text{IDer}(L)$  if and only if  $L$  is one of the following Lie algebras.

$$H(1), L_{4,2}, L_{5,2}, H(2), L_{6,2}, L_{6,4}.$$

**Corollary 3.4.** Let  $L$  be a non-abelian nilpotent Lie algebras with  $\dim L \leq 6$ . Then

- (i) If  $\mathcal{D} = \text{IDer}(L)$ , then  $\text{Der}^H(L) = \text{Der}_{Z(L)}(L)$  if and only if  $L$  is one of the following Lie algebras.

$$H(1), L_{4,2}, L_{4,3}, L_{5,i}(2 < i < 9), L_{6,i}(2 \leq i \leq 20), L_{6,21}(\varepsilon), L_{6,22}(\varepsilon) \quad \text{or} \quad L_{6,24}(\varepsilon), \varepsilon \neq 0.$$

- (ii) If  $\mathcal{D} = \text{Der}_c(L)$ , then  $\text{Der}^H(L) = \text{Der}_{Z(L)}(L)$  if and only if  $L$  is one of the following Lie algebras.

$$H(1), L_{4,2}, L_{4,3}, L_{5,i}(2 \leq i \leq 9), L_{6,i}(2 \leq i \leq 20), L_{6,21}(\varepsilon), L_{6,22}(\varepsilon) \quad \text{or} \quad L_{6,24}(\varepsilon), \varepsilon \neq 0.$$

- (iii) If  $\mathcal{D} = \text{ID}^*(L)$ , then  $\text{Der}^H(L) = \text{Der}_{Z(L)}(L)$  if and only if  $L$  is one of the following Lie algebras.

$$H(1), L_{4,2}, L_{4,3}, L_{5,i}(2 \leq i \leq 9), L_{6,i}(10 \leq i \leq 20), L_{6,21}(\varepsilon), L_{6,22}(\varepsilon) \quad \text{or} \quad L_{6,24}(\varepsilon), \varepsilon \neq 0.$$

- (iv) If  $\mathcal{D} = \text{ID}(L)$ , then  $\text{Der}^H(L) = \text{Der}_{Z(L)}(L)$  if and only if  $L$  is one of the following Lie algebras.

$$H(1), L_{4,3}, L_{5,i}(4 \leq i \leq 9), L_{6,i}(10 \leq i \leq 20), L_{6,3}, L_{6,21}(\varepsilon) \quad \text{or} \quad L_{6,24}(\varepsilon), \varepsilon \neq 0.$$

**Corollary 3.5.** Let  $L$  be a non-abelian nilpotent Lie algebras with  $\dim L \leq 6$ . Then

(i) If  $\mathcal{D} = \text{IDer}(L)$ , then  $\text{Der}_{Z(L)}^H(L) = \text{Der}_z(L)$  if and only if  $L$  is one of the following Lie algebras.

$$H(1), L_{4,3}, L_{5,i}(4 \leq i \leq 9), L_{6,i}(10 \leq i \leq 26), \varepsilon \neq 0.$$

(ii) If  $\mathcal{D} = \text{Der}_c(L)$ , then  $\text{Der}_{Z(L)}^H(L) = \text{Der}_z(L)$  if and only if  $L$  is one of the following Lie algebras.

$$H(1), L_{4,3}, L_{5,i}(4 \leq i \leq 9), L_{6,i}(10 \leq i \leq 26), \varepsilon \neq 0.$$

(iii) If  $\mathcal{D} = \text{ID}^*(L)$ , then  $\text{Der}_{Z(L)}^H(L) = \text{Der}_z(L)$  if and only if  $L$  is one of the following Lie algebras.

$$H(1), L_{4,3}, L_{5,i}(4 \leq i \leq 9), L_{6,i}(10 \leq i \leq 26), \varepsilon \neq 0.$$

(iv) If  $\mathcal{D} = \text{ID}(L)$ , then  $\text{Der}_{Z(L)}^H(L) = \text{Der}_z(L)$  if and only if  $L$  is one of the following Lie algebras.

$$H(1), L_{4,3}, L_{5,i}(4 \leq i \leq 9), L_{6,i}(10 \leq i \leq 20), \varepsilon \neq 0.$$

#### ACKNOWLEDGMENTS

We thank the two referees for careful readings of the manuscript and for a number of constructive corrections and suggestions.

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