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The SL_Φ-integral in Locally Convex Topological Vector Spaces

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Abstract. In this paper, we use the Minkowski functional to introduce an SL-type property or condition and then define a SL-type integral for a function taking values in a locally convex topological vector space (LCTVS). We show that this integral is equivalent to the SH_1 integral, a version of the Henstock-Kurzweil integral in a LCTVS.

Keywords: Strong Lusin condition, Strong Lusin integral, Strongly Henstock integral.

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1. Introduction

Jaroslav Kurzweil and Ralph Henstock independently introduced an integral that bears their name – the Henstock-Kurzweil integral or simply the HK-integral. This integral is a generalization of the Lebesgue integral but possesses the structural definition of the Riemann integral. Various studies of the integral in more abstract spaces have been done in the past decades. However, when extended to Banach spaces, the Henstock-Kurzweil integral no longer satisfies the well-known Henstock Lemma [2]. This shortcoming had led to the definition

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of a stronger version of the HK-integral (called HL-integral) for Banach-valued functions.

To be able to consider or define an integral that would be equivalent to the HK-integral, P.Y. Lee [12] introduced the concept of strong Lusin condition. This concept lies between the Lusin condition N [7, 12] and the absolute continuity property. This property had been used by Lee and Vyborny [11] to introduce the SL-integral. The authors showed that this integral is indeed equivalent to the HK-integral. Recently, Maza and Canoy [5] also introduced an SL-type integral for locally convex topological vector space (LCTVS)-valued functions and showed that such an integral is equivalent to a version of the Henstock integral (called the SH-integral) in a LCTVS. In [6], the authors also defined and studied another integral for LCTVS-valued functions.

In this paper, we make use of the Minkowski's functional Φ to introduce an SL_Φ condition and define an SL-type integral. Specifically, we define the SL_Φ-integral of a function taking values in a locally convex topological vector space. It will be shown, as one of our main results, that this integral is equivalent to the SH_1-integral, a version of the HK integral in the LCTVS setting.

Recall that a **topological vector space** X is real vector space together with a topology defined on it such that scalar multiplication and vector addition are continuous with respect to the topology and that every point of X is closed [10]. Equivalently, (X, τ) is a topological vector space if X is a real vector space and τ is Hausdorff topology on X such that scalar multiplication and vector addition are continuous with respect to τ. Continuity would then imply that for every open set U, there are open sets V_1 and V_2 such that $V_1 + V_2 \subseteq U$. More generally, for every θ-nbd U (an open set containing the zero vector θ of X) and $n \in \mathbb{N}$ there are θ-nbds $V_1, V_2, \ldots, V_n$ such that $V_1 + V_2 + \cdots + V_n \subseteq U$ (see [4] and [10]). Note that X being a Hausdorff space implies that only the zero vector θ is contained in all of the θ-nbds.

A set $A \subseteq X$, where X is a topological vector space, is **absorbing** if for every $x \in X$ there is $t > 0$ such that $x \in tA$; it is **convex** if for every $x, y \in A$ and $0 \leq t \leq 1$, $tx+(1-t)y \in A$; and it is **balanced** if $\alpha A \subseteq A$ for every $|\alpha| \leq 1$. The convex property can be extended as follows: for every $x_1, x_2, \ldots, x_n \in A$ and positive real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $\alpha_1 + \alpha_2 + \cdots + \alpha_n \leq 1$ we have $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n \in A$. A topological vector space X is said to be **locally convex** if there is a local base consisting of convex sets in X. It is known that every locally convex topological vector space has a local base at θ consisting of absorbing, balanced, and convex sets.

For a given set $A \subseteq X$, the **Minkowski functional** of A on X is defined by $\Phi_A(x) = \inf\{\lambda > 0 : x \in \lambda A\}$ for every $x \in X$. If $U \subseteq X$ is a balanced, absorbing and convex set, then (i) $U = \{x \in X : \Phi_U(x) < 1\}$, and (ii) Φ_U is a semi-norm on X, that is, $\Phi_U(u + v) \leq \Phi_U(u) + \Phi_U(v)$ for all $u, v \in X$ (sub-additivity), and $\Phi_U(ku) = |k|\Phi_U(u)$ for any real number k. Also, for any

$V \subseteq X$, $\Phi_{rV}(x) = \frac{1}{r}\Phi_V(x)$ for all positive real number r and $x \in X$. For any given absorbing sets $A \subseteq B \subseteq X$, $\Phi_B(t) \leq \Phi_A(t)$ for all $t \in X$. One may refer to [10, 1] for the definition, the earlier mentioned results, and a detailed discussion of the Minkowski functional.

A function $\delta : [a,b] \to R$ is called a **tight gauge** if it takes on non-negative values and a **gauge** if it takes on positive values [12]. A finite collection of ordered pairs $\{(I_i,t_i)\}_{i=1}^n$ of non-overlapping closed intervals of $[a,b]$ and real numbers is called a **partial partition** (resp. **partition**) of $[a,b]$ if $\bigcup_{i=1}^n I_i \subseteq [a,b]$ (resp. $\bigcup_{i=1}^n I_i = [a,b]$). A collection $\{(I_i,t_i)\}_{i=1}^n$ is called a **δ-fine partial partition** (**δ-fine partition**) of $[a,b]$ if $\{(I_i,t_i)\}_{i=1}^n$ is a partial partition (resp. partition) of $[a,b]$ and $t_i \in I_i \subseteq (t_i - \delta(t_i), t_i + \delta(t_i))$ for each $i \in \{1,2,\ldots,n\}$. We will occasionally denote a given δ-fine partial partition (or partition) $D = \{([u_i,v_i],t_i) : 1, \leq i \leq n\}$ by $D = \{([u,v],t)\}$.

A unitary sequence is a sequence of positive numbers $\{r_i\}_{i=1}^n$ such that $\sum_{i=1}^n r_i = 1$. A function $f : [a,b] \to X$, where X is a LCTVS, is SH_1-integrable (see [8]) if there exists a function $F : [a,b] \to X$ such that for any θ-nbd V, there exists a gauge δ on $[a,b]$ such that for every δ-fine partition $D = \{([x_{i-1},x_i],t_i) : 1 \leq i \leq n\}$ of $[a,b]$, there exists a unitary sequence $\{r_i\}_{i=1}^n$ such that

$$F(x_i) - F(x_{i-1}) - f(t_i)(x_i - x_{i-1}) \in r_i V$$

for each $1 \leq i \leq n$. The difference $F(b) - F(a)$ is the SH_1 integral of f on $[a,b]$ and write $(SH_1)\int_a^b f = F(b) - F(a)$.

A function $F : [a,b] \to X$ is said to satisfy the SL_Φ **condition** if given a subset E of $[a,b]$ of measure zero, a θ-nbd U, and $\varepsilon > 0$, there exists a gauge δ such that for any δ-fine partial partition $D = \{([u,v],t)\}$ with $t_i \in E$, we have

$$(D)\sum \Phi_U(F(u,v)) < \varepsilon$$

where $F(u,v) = F(v) - F(u)$. A function satisfying the SL_Φ condition is called an SL_Φ function.

A function $f : [a,b] \to X$ is said to be **SL_Φ-integrable** on $[a,b]$ if there exists a SL_Φ function F with the property that for every θ-nbd U and $\varepsilon > 0$, there is a tight gauge δ such that for every δ-fine partial partition $D = \{([u,v],t)\}$ of $[a,b]$, $(D)\sum \Phi_U(F(u,v) - f(t)(v-u)) < \varepsilon$. The vector $F(a,b)$ is the SL_Φ integral of f and is denoted by $(SL_\Phi)\int_a^b f$. We call the function F an **SL_Φ-primitive** of f.

2. MAIN RESULTS

Throughout this section, X is a locally convex topological vector space.

Theorem 2.1. *Let $c \in X$. Then the constant function $F : [a,b] \to X$ given by $F(t) = c$ is an SL_Φ function.*

Proof. Let $E \subseteq [a,b]$ be of measure zero, U a θ-nbd, and $\varepsilon > 0$. Choose any gauge δ on $[a,b]$. Then for any δ-fine partial partition $D = \{([u,v],t)\}$ of $[a,b]$ with $t \in E$,

$$(D)\sum \Phi_U(F(u,v)) = \sum_{i=1}^{n} \Phi_U(c-c) = \sum_{i=1}^{n} \Phi_U(\theta) = 0 < \varepsilon.$$

□

Theorem 2.2. *Let $F, G : [a,b] \to X$ be SL_Φ functions and $c \in R$. Then cF and $F+G$ are SL_Φ functions.*

Proof. By Theorem 2.1, cF is a SL_Φ function if $c = 0$. Suppose $c \neq 0$ and let $E \subset [a,b]$ be a set of measure zero, V a θ-nbd, and $\varepsilon > 0$. Then there is an absorbing, balanced and convex θ-nbd U with $U \subseteq V$. By assumption, there exists a gauge δ_1 such that for any δ_1-fine partial partition $D = \{([u,v],t)\}$ with $t_i \in E$, we have

$$(D)\sum \Phi_U(F(u,v)) < \frac{1}{|c|}\varepsilon.$$

Hence,

$$\begin{aligned}(D)\sum \Phi_V(cF(v) - cF(u)) &\leq (D)\sum \Phi_U(cF(v) - cF(u)) \\ &= (D)\sum \Phi_U\left(c(F(u,v))\right) \\ &= (D)\sum |c|\Phi_U\left(F(u,v)\right) < \varepsilon.\end{aligned}$$

This proves that cF is SL_Φ.

For the second part, again let $E \subset [a,b]$ be a given set of measure zero, V a θ-nbd, and $\varepsilon > 0$. Let $U \subseteq V$ be an absorbing, balanced and convex θ-nbd. Since F and G are SL_Φ functions, there is a common gauge δ on $[a,b]$ such that for any δ-fine partial partition $D = \{([u,v],t)\}$ of $[a,b]$ with $t_i \in E$,

$$(D)\sum \Phi_U(F(u,v)) < \frac{1}{2}\varepsilon \quad \text{and} \quad \sum_{i=1}^{n} \Phi_U(G(u,v)) < \frac{1}{2}\varepsilon$$

It follows that

$$\begin{aligned}(D)\sum \Phi_V((F+G)(u,v)) &\leq (D)\sum \Phi_U((F+G)(u,v)) \\ &\leq (D)\sum \Phi_U(F(u,v)) + (D)\sum \Phi_U(G(u,v)) \leq \varepsilon.\end{aligned}$$

Therefore, $F+G$ satisfies the SL_Φ condition. □

Theorem 2.3. *Let $F : [a,b] \to X$ be an SL_Φ function. Then the restriction $F|_{[c,d]}$ of F to $[c,d] \subseteq [a,b]$ is also a SL_Φ function. function.*

Proof. Let U be a θ-nbd, $\varepsilon > 0$, and $E \subseteq [c,d]$ be of measure zero. Since F is a SL_Φ function on $[a,b]$, there exists a gauge δ on $[a,b]$ such that for any δ-fine partial partition $D = \{([u,v],t)\}$ of $[a,b]$ with $t_i \in E$, we have

$$(D)\sum \Phi_U(F(u,v)) < \varepsilon.$$

Let δ_0 be the restriction of δ to $[c,d]$. Suppose $D' = \{([u',v'],t')\}$ with $t'_i \in E$ is a δ_0-fine partial partition of $[c,d]$. Then D' is a δ-fine partial partition of $[a,b]$. Hence,

$$(D')\sum \Phi_U(F(u',v')) < \varepsilon.$$

Therefore, $F|_{[c,d]} : [c,d] \to X$ is an SL_Φ function. □

Theorem 2.4. *Let $F : [a,b] \to X$ be a function and let $c \in (a,b)$. Suppose the restrictions of F to $[a,c]$ and $[c,b]$ are SL_Φ functions on $[a,c]$ and $[c,b]$, respectively. Then F is an SL_Φ function on $[a,b]$.*

Proof. Let $E \subset [a,b]$ be of measure zero, W a θ-nbd, and $\varepsilon > 0$. Let $U \subseteq W$ be an absorbing, balanced and convex θ-nbd. We may assume that $a < c < b$. Then $E \cap [a,c]$ and $E \cap [c,b]$ are both of measure zero. Since F is a SL_Φ function on $[a,c]$ and $[c,b]$ there are gauges δ_1 on $[a,c]$ and δ_2 on $[c,b]$ that satisfies the SL_Φ-condition with respect to ε and U. Define a function $\delta : [a,b] \to R^+$ by

$$\delta(t) = \begin{cases} \min\{\delta_1(t), c-t\} & t \in [a,c) \\ \min\{\delta_2(t), t-c\} & t \in (c,b] \\ \min\{\delta_1(c), \delta_2(c)\} & t = c. \end{cases}$$

Let $D = \{([u_i,v_i],t_i) : 1 \le i \le n\}$ be a δ-fine partial partition of $[a,b]$.

Case 1. Suppose either $c = u_k$ or $c = v_k$ for some $k \in \{1,2,\ldots,n\}$ or $c \notin [u,v]$ for all $([u,v],t) \in D$. We write $D = D_1 \cup D_2$ where D_1 contains subintervals of $[a,c]$ and D_2 contains subintervals of $[c,d]$. Then D_1 is δ_1-fine partial partition of $[a,c]$ and D_2 is δ_2-fine partial partition of $[c,b]$. Hence,

$$(D_1)\sum \Phi_U(F(u,v)) < \frac{\varepsilon}{2} \quad \text{and} \quad (D_2)\sum \Phi_U(F(u,v)) < \frac{\varepsilon}{2}$$

It follows that

$$(D)\sum \Phi_W(F(u,v) \le (D)\sum_{i=1}^{n} \Phi_U(F(u,v)) < \varepsilon.$$

Case 2. Suppose $c \in (u_k, v_k)$. By the definition of δ, $c = t_k$. A refinement of D' of D can be expressed as $D' = D_1 \cup D_2$ where D_1 a δ_1-fine partial partition of $[a,c]$ and D_2 a δ_2-fine partial partition of $[c,b]$. Hence,

$$(D_1)\sum \Phi_U(F(v)-F(u)) < \frac{\varepsilon}{2} \text{ and } (D_2)\sum \Phi_U(F(v)-F(u)) < \frac{\varepsilon}{2}.$$

Therefore,

$$(D)\sum \Phi_W(F(u,v)) \leq (D)\sum \Phi_U(F(u,v))$$
$$\leq (D_1)\sum \Phi_U(F(u,v)) + (D_2)\sum \Phi_U(F(u,v)) < \varepsilon.$$

Accordingly, F satisfies the SL_Φ condition on $[a,b]$. □

2.1. **The Φ-Strong Lusin integral.**

Theorem 2.5. *Let $f:[a,b]\to X$ be SL_Φ-integrable with SL_Φ-primitive F. For any $[c,d]\subseteq[a,b]$, the restriction $f|_{[c,d]}:[c,d]\to X$ is SL_Φ-integrable with SL_Φ-primitive $F|_{[c,d]}:[c,d]\to X$.*

Proof. Suppose f is SL_Φ-integrable with SL_Φ-primitive F on $[a,b]$. Let U be a θ-nbd and $\varepsilon>0$. Then there is a tight gauge δ_0 on $[a,b]$ associated to f, U, and ϵ in the definition of the SL_Φ integral. The restriction of F on $[c,d]$ satisfies the SL_Φ condition by Theorem 2.3. Let δ be the restriction of δ_0 on $[c,d]$ and let D be δ-fine partial partition of $[c,d]$. Then D is a δ_0-fine partial partition of $[a,b]$. Hence,

$$(D)\sum \Phi_U(F(u,v)-f(t)(v-u)) < \varepsilon,$$

showing that $f|_{[c,d]}$ is SL_Φ-integrable on $[c,d]$ with SL_Φ-primitive $F|_{[c,d]}$: $[c,d]\to X$. □

Theorem 2.6. *Let $f:[a,b]\to X$ be SL_Φ-integrable on $[a,b]$. Suppose F and G are SL_Φ-primitives of f. Then $F(c,d)=G(c,d)$ for every $[c,d]\subseteq[a,b]$. In particular, the SL_Φ-integral of f is unique.*

Proof. Let $[c,d]\subseteq[a,b]$, V a θ-nbd, and $\varepsilon>0$. Let U be an absorbing, balanced and convex θ-nbd with $U\subseteq V$. Since f is SL_Φ-integrable on $[c,d]$, there exists a tight gauge δ_0 such that for a δ_0-fine partial partition D' of $[c,d]$, we have

$$(D')\sum \Phi_U(F_1(u,v)-f(t)(v-u)) < \frac{\varepsilon}{3} \text{ and}$$
$$(D')\sum \Phi_U(f(t)(v-u)-G_1(u,v)) < \frac{\varepsilon}{3}.$$

Let $N_{\delta_0}=\{t\in[a,b]:\delta_0(t)=0\}$. Then $m^*(N_{\delta_0})=0$. Now, $F_1=F|_{[c,d]}$ and $G_1=G|_{[c,d]}$ are SL_Φ-primitives of $f|_{[c,d]}$ by Theorem 2.5. Thus, $H=F_1-G_1$ is a SL_Φ-function on $[c,d]$ by Theorem 2.2. Hence, there exists a gauge δ_1 on $[a,b]$ such that for any δ_1-fine partial partition $D''=\{([u,v],t)\}$ with $t\in N_{\delta_0}$, $(D'')\sum\Phi_U(H(u,v))<\frac{\varepsilon}{3}$. Define $\delta(t)=\delta_0(t)$ if $t\notin N_{\delta_0}$ and $\delta(t)=\delta_1(t)$ if $t\in N_{\delta_0}$. Then δ is a gauge on $[a,b]$. Let D be a δ-fine partition of $[a,b]$. Let $D_0=\{([u,v],t)\in D: t\notin N_{\delta_0}\}$ and $D_1=D\setminus D_0$. Then D_0 is a δ_0-fine partial partition of $[a,b]$ and D_1 is a δ_1-fine partial partition of $[a,b]$. Consequently,

$$\begin{aligned}\Phi_V(F_1(c,d)-G_1(c,d)) &\le \Phi_U(F_1(c,d)-G_1(c,d))\\ &= \Phi_U((D)\sum(F_1(u,v)-G_1(u,v)))\\ &\le \Phi_U g[(D_0)\sum(F_1(u,v)-f(t)(v-u))\\ &\qquad + (D_0)\sum(f(t)(v-u)-G_1(u,v))\\ &\qquad\quad + (D_1)\sum(F_1(u,v)-G_1(u,v))g]\\ &< \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+(D_1)\sum(H(u,v)) \le \varepsilon.\end{aligned}$$

Since V and ε were arbitrarily chosen, $F(c,d)=G(c,d)$. □

Theorem 2.7. *Let $f,g:[a,b]\to X$ be SL_Φ-integrable functions and $c\in R$. Then each of the following holds:*

(i) cf is SL_Φ-integrable on $[a,b]$ and

$$(SL_\Phi)\int_a^b c\cdot f = c\cdot (SL_\Phi)\int_a^b f \ \textit{and,}$$

(ii) $f+g$ is SL_Φ-integrable on $[a,b]$ and

$$(SL_\Phi)\int_a^b (f+g) = (SL_\Phi)\int_a^b f + (SL_\Phi)\int_a^b g.$$

Proof. (i) Let F be a SL_Φ-primitive of f. We may assume that $c\neq 0$. Let V be a given θ-nbd and $\varepsilon>0$. Then there is an absorbing, balanced and convex θ-nbd $U\subseteq V$. Thus, there is a tight gauge $\delta>0$ such that for any δ-fine partial partition $D=\{([u,v],t)\}$, we have

$$(D)\sum \Phi_U(F(u,v)-f(t)(u,v)) < \frac{\varepsilon}{|c|}.$$

Thus,

$$(D)\sum \Phi_V((cF)(u,v)-cf(t)(v-u)) \le \sum_{i=1}^{n}\Phi_U(c(F(u,v)-f(t)(v-u))) < \varepsilon.$$

This and Theorem 2.2 would imply that cf is SL_Φ-integrable with SL_Φ-primitive cF on $[a,b]$. Moreover,

$$(SL_\Phi)\int_a^b c\cdot f = (cF)(a,b) = c(F(a,b)) = c\cdot (SL_\Phi)\int_a^b f.$$

(ii) Let F and G be SL_Φ-primitives for the functions f and g, respectively. Let V be a θ-nbd and $\varepsilon>0$. Let U be an absorbing, balanced and convex θ-nbd with $U\subseteq V$. Let δ_1 and δ_2 be tight gauges on $[a,b]$ associated f, U and $\frac{\varepsilon}{2}$, and g, U and $\frac{\varepsilon}{2}$, respectively, in the definition of the SL_Φ integral. Define $\delta(t)=\min\{\delta_1(t),\delta_2(t)\}$. Then δ is a tight gauge $[a,b]$. Let $D=\{([u,v],t)\}$ be a δ-fine partial partition of $[a, b]$. Note that D is both a δ_1-fine and a δ_2-fine partial partition of $[a, b]$. Hence,

$$(D)\sum \Phi_U(F(u,v) - f(t)(v-u)) < \frac{\varepsilon}{2}$$
$$(D)\sum \Phi_U(G(u,v) - g(t)(v-u)) < \frac{\varepsilon}{2}.$$

Thus,

$$\begin{aligned}(D)\sum \Phi_V((F+G)(u,v)-&(f+g)(t)(u,v)) \\ &\leq (D)\sum \Phi_V((F+G)(u,v) - (f+g)(t)(u,v)) \\ &\leq (D)\sum \Phi_U(F(u,v) - f(t)(u,v)) \\ &\quad + (D)\sum \Phi_U(G(u,v) - g(t)(u,v)) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Thus, $f + g$ is SL_Φ-integrable with SL_Φ-primitive $F + G$ and

$$\begin{aligned}(\Phi\text{-}SL)\int_a^b (f+g) &= (F+G)(a,b) \\ &= F(a,b) + G(a,b) \\ &= (\Phi\text{-}SL)\int_a^b f + (\Phi\text{-}SL)\int_a^b g.\end{aligned}$$

□

Theorem 2.8. *Let $f : [a, b] \to X$ be a function and $c \in (a, b)$. Suppose the restrictions of f on $[a, c]$ and $[c, b]$ are SL_Φ-integrable on $[a, c]$ and $[c, b]$, respectively. Then f is SL_Φ-integrable on $[a, b]$ and*

$$(SL_\Phi)\int_a^b f = (SL_\Phi)\int_a^c f + (SL_\Phi)\int_c^b f.$$

Proof. Let $F_1 : [a, c] \to X$ and $F_2 : [c, b] \to X$ be SL_Φ-primitives of the restrictions of f on $[a, c]$ and $[c, b]$, respectively. Define $F : [a, b] \to X$ by

$$F(t) = \begin{cases} F_1(t) & \text{for } t \in [a,c] \\ F_2(c,t) + F_1(c) & \text{for } t \in [c,b] \end{cases}$$

The restrictions of $F(t)$ to $[a, c]$ and $[c, b]$ are $F_1(t)$ and $G(t) = F_2(c, t) + F_1(c)$, respectively. By Theorem 2.2 and Theorem 2.3, G is a SL_Φ function on $[c, b]$. Thus, F is a SL_Φ function on $[a, b]$ by Theorem 2.4.

It now remains to show that F is a SL_Φ-primitive of f so that f is SL_Φ-integrable on $[a, b]$. To this end, let V be a θ-nbd and $\varepsilon > 0$. Choose any absorbing, balanced, and convex θ-nbd $U \subseteq V$. Let δ_1 and δ_2 be tight gauges associated with the restrictions $f|_{[a,c]}$ and $f|_{[c,b]}$, U, and $\frac{\varepsilon}{2}$ in the definition of SL_Φ integral. Define δ as follows:

$$\delta(t) = \begin{cases} \min\{\delta_1(t), c-t\} & \text{for } t \in [a,c) \\ \min\{\delta_1(c), \delta_2(c)\} & \text{for } t = c \\ \min\{\delta_2(t), t-c\} & \text{for } t \in (c,b] \end{cases}$$

Consider any δ-fine partial partition $D = \{([u,v],t)\}$ of $[a,b]$.

Case 1. Suppose $c = u_k$ or $c = v_k$ for some $k \in \{1,2,\dots,n\}$ or $c \notin [u_i, v_i]$ for all $i \in \{1,2,\dots,n\}$. Then D is a disjoint union of D_1 and D_2 where the elements in D_1 are tagged intervals contained in $[a,c]$ and the elements in D_2 are tagged intervals contained in $[c,b]$. Then D_1 is δ_1-fine partial partition of $[a,c]$ and D_2 is δ_2-fine partial partition of $[c,b]$. Hence,

$$(D_1)\sum \Phi_U(F_1(u,v) - f(t)(v-u)) < \frac{\varepsilon}{2}$$
$$(D_2)\sum \Phi_U(F_2(u,v) - f(t)(v-u)) < \frac{\varepsilon}{2}.$$

It follows that

$$\begin{aligned}(D)\sum \Phi_U(F(u,v) - f(t)(v-u)) \\ &= (D_1)\sum \Phi_U(F_1(u,v) - f(t)(v-u)) \\ &\quad + (D_2)\sum \Phi_U(G(u,v) - f(t)(v-u)) \\ &= (D_1)\sum \Phi_U(F_1(u,v) - f(t)(v-u)) \\ &\quad + (D_2)\sum \Phi_U(F_2(u,v) - f(t)(v-u)) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Case 2. Suppose $c \in (u_k, v_k)$. Then $t_k = c$. Let $D' = D_1 \cup D_2$ be a refinement of D where $D_1 = \{([u_i,v_i],t_i) : 1 \leq i \leq k-1\} \cup \{([u_k,c],c)\}$ and $D_2 = \{([u_i,v_i],t_i) : k+1 \leq i \leq n\} \cup \{([c,v_k],c)\}$. Then D_1 is a δ_1-fine partial partition of $[a,c]$ and D_2 is a δ_2-fine partial partition of $[c,b]$. Hence,

$$\Phi_U(F_1(u_k,c) - f(c)(c-u_k)) + (D_1)\sum \Phi_U(F_1(u,v) - f(t)(v-u)) < \frac{\varepsilon}{2}$$

and

$$\Phi_U(F_2(c,v_k) - f(c)(v_k-c)) + (D_2)\sum \Phi_U(F_2(u,v) - f(t)(v-u)) < \frac{\varepsilon}{2}.$$

Since

$$F(u_k,v_k) - f(t_k)(v_k-u_k) = F_2(u_k,v_k) - f(c)(v_k-c) + F_1(u_k,v_k) - f(c)(c-u_k),$$

it follows that

$$\begin{aligned}&(D)\sum \Phi_U(F(u,v)-f(t)(v-u))\\ &\le (D_1)\sum \Phi_U(F(u,v)-f(t)(v-u)) + (D_2)\sum \Phi_U(F(u,v)-f(t)(v-u))\\ &\quad + \Phi_U(F(u_k,v_k)-f(t_k)(v_k-u_k))\\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

In either case, we have

$$(D)\sum \Phi_V(F(u,v)-f(t)(v-u)) \le (D)\sum \Phi_U(F(u,v)-f(t)(v-u)) < \varepsilon.$$

Hence, F is a SL_Φ-primitive of f on $[a,b]$ and

$$(SL_\Phi)\int_a^b f = F(a,b) = F(a,c)+F(c,b) = (SL_\Phi)\int_a^c f + (SL_\Phi)\int_c^b f.$$

□

Corollary 2.9. *Let $f:[a,b]\to X$ be SL_Φ-integrable on $[a,b]$ with SL_Φ-primitive F. For any $c\in[a,b]$, we have*

$$(SL_\Phi)\int_a^b f = (SL_\Phi)\int_a^c f + (SL_\Phi)\int_c^b f.$$

In what follows, $\Delta_\Phi(U,F,f)$ is given by

$$\begin{aligned}\Delta_\Phi(U,F,f) = \{t\in[a,b] : \forall\delta>0, \exists[u,v]\subseteq[a,b] \text{ with } t\in[u,v] \text{ and}\\ |v-u|<\delta,\ \Phi_U\left(F(v)-F(u)-f(t)(v-u)\right)\ge v-u\}.\end{aligned}$$

Theorem 2.10. *Let $f,F:[a,b]\to X$ be functions where F is a SL_Φ-function. If $\Delta_\Phi(U,F,f)$ is of measure zero for each θ-nbd U, then f is SL_Φ-integrable with SL_Φ-primitive F.*

Proof. Let V be a given θ-nbd and $\varepsilon>0$. Then there is an absorbing, balanced and convex θ-nbd U such that $U\subseteq V$. If $t\notin\Delta_\Phi(\frac{\varepsilon}{b-a}U,F,f)$, then there is a real number $\delta_0(t)>0$ such that $\Phi_{\frac{\varepsilon}{b-a}U}(F(u,v)-f(t)(v-u))<1$ whenever $|v-u|<\delta_0(t)$ and $t\in[u,v]$ or equivalently, $\Phi_U(F(u,v)-f(t)(v-u))<\frac{\varepsilon(v-u)}{b-a}$ whenever $|v-u|<\delta_0(t)$ and $t\in[u,v]$. Define

$$\delta(t)=\begin{cases}0 & \text{if } t\in\Delta_\Phi(\frac{\varepsilon}{b-a}U,F,f)\\ \frac{\delta_0(t)}{2} & \text{if } t\notin\Delta_\Phi(\frac{\varepsilon}{b-a}U,F,f)\end{cases}$$

Then $\delta(t)$ is a tight gauge because $\Delta_\Phi(\frac{\varepsilon}{b-a}U,F,f)$ has measure zero. Consider a given δ-fine partial partition $D=\{([u,v],t)\}$ of $[a,b]$. Suppose there exists $([u,v],t)\in D$ such that $\delta(t)=0$. Then $t\in[u,v]\subseteq(t-\delta(t),t+\delta(t))=\varnothing$, a contradiction. Hence, $\delta(t)\neq 0$ for all $([u,v],t)\in D$. This implies that for each $([u,v],t)\in D$,

$$\Phi_U(F(u,v)-f(t)(v-u)) < \frac{\varepsilon(v-u)}{b-a}.$$

Thus, we have

$$\begin{aligned}(D)\sum \Phi_V(F(u,v)-f(t)(v-u)) &\leq (D)\sum \Phi_U(F(u,v)-f(t)(v-u)) \\ &< (D)\sum\left(\frac{\varepsilon(v-u)}{b-a}\right) \\ &\leq \varepsilon\frac{b-a}{b-a}=\varepsilon.\end{aligned}$$

Therefore, f is SL_Φ-integrable with SL_Φ-primitive F. □

Let $F:[a,b]\to X$ be a function and $t\in[a,b]$. Then F is **differentiable** at t ($F'(t)$ is the derivative F at t) if for every θ-nbd U, there is a $\delta>0$ for which $F(v)-F(u)-F'(t)(v-u)\in(v-u)U$ whenever $t\in[u,v]\subseteq[a,b]$ and $|v-u|<\delta$ (see [9]).

Theorem 2.11. *Let $f,F:[a,b]\to X$ be functions and $\mathcal{D}(F,f)$ be the set of all $t\in[a,b]$ such that $F'(t)$ does not exist or $F'(t)\neq f(t)\}$. Then*

$$\mathcal{D}(F,f)=\bigcup_{\theta\text{-}nbd\ U}\Delta_\Phi(U,F,f).$$

Proof. If $t\in\mathcal{D}(F,f)$, then there is a θ-nbd U such that for each $\delta>0$, there is $[u,v]\subseteq(t-\delta,t+\delta)$ containing t for which $\Phi_U(F(u,v)-f(t)(v-u))>u-v$. Equivalently, $t\in\Delta_\Phi(U,F,f)$. Thus,

$$\mathcal{D}(F,f)\subseteq\bigcup_{\theta\text{-nbd}\ U}\Delta_\Phi(U,F,f).$$

Next, let $t\in\bigcup_{\theta\text{-nbd}\ U}\Delta_\Phi(U,F,f)$. Then there exists a θ-nbd U such that $t\in\Delta_\Phi(U,F,f)$. Let V be an absorbing, balanced, and convex set such that $V\subseteq U$. Then for all $\delta>0$, there exists $[u,v]\subseteq[a,b]$ with $t\in[u,v]$ and $(v-u)<\delta$ such that

$$\Phi_V(F(u,v)-f(t)(v-u))\geq\Phi_U(F(u,v)-f(t)(v-u))\geq(v-u).$$

Since V is absorbing, balanced and convex, it follows that $F(u,v)-f(t)(v-u)\notin(v-u)V$. Thus, $t\in\mathcal{D}(F,f)$, showing that

$$\bigcup_{\theta\text{-nbd}\ U}\Delta_\Phi(U,F,f)\subseteq\mathcal{D}(F,f).$$

This proves the desired equality. □

Example 2.12. The zero function $f:[a,b]\to X$ is SL_Φ-integrable with SL_Φ-primitive $F:[a,b]\to X$ given by the constant function $F=\alpha$ where α is any vector in X. In fact, for any θ-nbd U, $\Delta_\Phi(U,F,f)=\emptyset$. Indeed, if there is a $t\in\Delta(U,F,f)$, then for all $\delta>0$, there exists $[u,v]\subseteq[a,b]$ containing t with $(v-u)<\delta$ and $F(v)-F(u)-f(t)(v-u)\notin(v-u)U$. However, we see that

$F(v) - F(u) - f(t)(v-u) = \alpha - \alpha - \theta(v-u) \in (v-u)U$, a contradiction. By Theorem 2.10, f is SL_Φ-integrable with SL_Φ-primitive $F = \alpha$ and

$$(SL_\Phi)\int_a^b f = F(b) - F(a) = \theta.$$

Theorem 2.13. *Let $F : [a,b] \to X$ be a SL_Φ-function. Suppose that $F'(x) = \theta$ almost everywhere on $[a,b]$. Then F is a constant function.*

Proof. Let $f : [a,b] \to X$ be the zero function. Since $F'(x) = \theta = f(x)$ almost everywhere on $[a,b]$, $\mathcal{D}(F,f)$ has measure zero. Thus, $\Delta_\Phi(U,F,f)$ is of measure zero for each θ-nbd U. By Theorem 2.10 and by Example 2.12,

$$\theta = (SL_\Phi)\int_a^b f = F(b) - F(a).$$

Hence, $F(b) = F(a)$. We know that F is a SL_Φ function on $[a,x]$ by Theorem 2.3 and that $F'(x) = \theta$ almost everywhere on $[a,x]$ for all $x \in (a,b]$. By replacing b with $x \in (a,b]$ in the proof above, we have $F(x) = F(a)$. Therefore, F is a constant function. □

Theorem 2.14. *A function $f : [a,b] \to X$ is SH_1 integrable on $[a,b]$ if and only if there exists a function $F : [a,b] \to X$ with the property that for every θ-nbd U and $\varepsilon > 0$, there is a gauge δ on $[a,b]$ such that for every δ-fine partition $D = \{([u,v],t)\}$ we have*

$$(D)\sum \Phi_U(F(u,v) - f(t)(v-u)) < \varepsilon.$$

In this case,

$$(SH_1)\int_a^b f = F(a,b).$$

Proof. Let $f : [a,b] \to X$ be SH_1-integrable on $[a,b]$ and let F be its SH_1-primitive. Let U be a θ-nbd and $\varepsilon > 0$. Then there is an absorbing, balanced and convex θ-nbd $V \subseteq U$. By the definition of SH_1-integral applied to εV, there is a gauge δ on $[a,b]$ such that for every δ-fine partition $D = \{([u_i,v_i],t_i) : 1 \le i \le n\}$ of $[a,b]$, there is a unitary sequence $\langle r_i \rangle_{i=1}^n$ such that

$$F(u_i,v_i) - f(t_i)(v_i - u_i) \in r_i \varepsilon V$$

for each $i \in \{1,2,\cdots,n\}$. Since V is balanced, convex and absorbing,

$$\Phi_V(F(u_i,v_i) - f(t_i)(v_i-u_i)) \le \Phi_V(F(u_i,v_i) - f(t_i)(v_i-u_i)) < r_i\varepsilon$$

for each $i \in \{1,2,\cdots,n\}$. Hence,

$$(D)\sum \Phi_U(F(u,v) - f(t)(v-u)) < \sum_{i=1}^n r_i\varepsilon = \varepsilon.$$

Conversely, suppose f and F satisfy the condition. Let U be a θ-nbd. Then by assumption there is a gauge δ on $[a,b]$ such that for every δ-fine partition $D = \{([u,v],t)\}$ we have

$$(D)\sum \Phi_U(F(u,v) - f(t)(v-u)) < 1.$$

Choose $\alpha > 0$ so that $\alpha + (D)\sum \Phi_U(F(v) - F(u) - f(t)(v-u)) = 1$ and let $r_i = \frac{\alpha}{n} + \Phi_U(F(v_i) - F(u_i) - f(t_i)(v_i - u_i))$ for each $i \in \{1, 2, \cdots, n\}$. Then $\sum_{i=1}^n r_i = 1$ and

$$F(u_i, v_i) - f(t_i)(v_i - u_i) \in r_i U \text{ for each } i \in \{1, 2, \cdots, n\}.$$

Therefore, f is SH_1 integrable with SH_1 primitive F. □

Theorem 2.15. *Let $f : [a,b] \to X$. If $\Phi_U \circ f = 0$ almost everywhere on $[a,b]$ for each θ-nbd U, then f is SH_1 integrable and $(SH_1)\int_a^b f = \theta$. In particular, if $f(t) = \theta$ almost everywhere on $[a,b]$, then f is SH_1 integrable and $(SH_1)\int_a^b f = \theta$.*

Proof. Let $F : [a,b] \to X$ with $F(t) = \theta$ for all $t \in [a,b]$. Let V be a θ-nbd and let $\varepsilon > 0$. Choose any balanced, convex and absorbing θ-nbd $U \subseteq V$. Let $S = \{t \in [a,b] : \Phi_U(f(t)) \neq 0\}$ and $E_k = \{t \in S : k-1 < \Phi_U(f(t)) \leq k\}$ for each positive integer k. Then $m(S) = 0$ and $m(E_k) = 0$ for each $k > 0$. Thus, there exists an open set G_k such that $E_k \subseteq G_k$ and $m(G_k) < \frac{\varepsilon}{k2^k}$. Also, $\{E_i\}_{i=1}^\infty$ is a pairwise disjoint collection whose union is S. Define $\delta(t) = 1$ if $t \in [a,b] \setminus S$ and $\delta(t) > 0$ be a real number such that $t \in (t - \delta(t), t + \delta(t)) \subseteq G_k$ if $t \in E_k$. Let $D = \{([u,v],t)\}$ be a δ-fine partition of $[a,b]$. Let $D_0 = \{([u,v],t) \in D : t \in [a,b] \setminus S\}$ and let $D_k = \{([u,v],t) \in D : t \in E_k\}$ for each $k > 0$. Since U is convex, $\Phi_U(f(t)) < j$ for each $t \in E_j$. Also, since $\bigcup\{[u_i, v_i] : t_i \in E_j\} \subseteq G_j$ for integer $j > 0$, $\sum_{t_i \in E_j}(v_i - u_i) \leq m(G_j) < \frac{\varepsilon}{j2^j}$. Hence,

$$(D)\sum \Phi_V(F(u,v) - f(t)(v-u)) \leq (D)\sum \Phi_U(F(u,v) - f(t)(v-u))$$
$$= (D_0)\sum \Phi_U(F(u,v) - f(t)(v-u)) + (D \setminus D_0)\sum \Phi_U(F(u,v) - f(t)(v-u)).$$

Since $f(t) = \theta$ for all $t \notin S$ and $F(t) = \theta$ for all $t \in [a,b]$, we have

$$(D)\sum \Phi_V(F(u,v) - f(t)(v-u)) \leq 0 + \sum_{j=1}^\infty \sum_{t \in E_j} (v_i - u_i)\Phi_U(-f(t_i))$$
$$\leq \sum_{j=1}^\infty \sum_{t_i \in E_j} (v_i - u_i) j < \sum_{j=1}^\infty \frac{\varepsilon}{j2^j} j = \varepsilon.$$

By Theorem 2.14, f is SH_1 integrable with SH_1 primitive F and

$$(SH_1)\int_a^b f = F(b) - F(a) = \theta.$$

The second assertion directly follows. □

Theorem 2.16. *Let $f : [a,b] \to X$ be SL_Φ-integrable on $[a,b]$ with SL_Φ-primitive F. Then f is SH_1-integrable on $[a,b]$ with SH_1 primitive F and*

$$(SH_1)\int_a^b f = (SL_\Phi)\int_a^b f.$$

Proof. Let V be a θ-nbd and let $\varepsilon > 0$. Let U be an absorbing, balanced, and convex θ-nbd with $U \subseteq V$. Since f is SL_Φ-integrable with SL_Φ-primitive F, there is a tight gauge δ_1 such that for any δ_1-fine partial partition $D_1 = \{([u,v],t)\}$ of $[a,b]$,

$$(D_1)\sum \Phi_U(F(u,v) - f(t)(v-u)) < \frac{\varepsilon}{3}.$$

Let $Z = \{t \in [a,b] : \delta_1(t) = 0\}$. Then Z has measure zero. Define $f_0 = f \cdot 1_Z$. Then f_0 is zero almost everywhere on $[a,b]$ and hence, f_0 is SH_1-integrable with primitive $F_0 : [a,b] \to X$ given by $F_0 = \theta$ by Theorem 2.15. Thus by Theorem 2.14, there exists a gauge δ_2 such that for any δ_2-fine partition $D_2 = \{([u,v],t)\}$ of $[a,b]$,

$$(D_2)\sum \Phi_U(-f_0(t)(v-u)) < \frac{\varepsilon}{3}.$$

In particular, if $D_2 = \{([u,v],t)\}$ is a δ_2-fine partition of $[a,b]$ with $t \in Z$ for each $([u,v],t) \in D_2$, then

$$(D_2)\sum \Phi_U(-f(t)(v-u)) < \frac{\varepsilon}{3}.$$

Since F is a SL_Φ function, there is a gauge γ such that for every γ-fine partial partition $D_3 = \{([u,v],t)\}$ of $[a,b]$ with $t \in Z$,

$$(D_3)\sum \Phi_U(F(u,v)) < \frac{\varepsilon}{3}.$$

Define

$$\lambda(t) = \begin{cases} \delta_1(t) & \text{if } t \in [a,b] \setminus Z \\ \min\{\delta_2(t), \gamma(t)\} & \text{otherwise.} \end{cases}$$

Then λ is a gauge on $[a,b]$. Let $D = \{([u,v],t)\}$ be a λ-fine partition of $[a,b]$. Write D as a disjoint union of D_1' and D_2', where $D_1' = \{([u,v],t) \in D : t \in [a,b] \setminus Z\}$ and $D_2' = D \setminus D_1'$. Then D_1' is δ_1-fine partial partition of $[a,b]$ and D_2' is both a δ_2-fine and a γ-fine partial partition of $[a,b]$. Thus,

$$\begin{aligned}(D)\sum \Phi_U(F(u,v) - f(t)(v-u)) &\le (D_1')\sum \Phi_U(F(u,v) - f(t)(v-u)) \\ &+(D_2')\sum \Phi_U(-f(t)(v-u)) + (D_2')\sum \Phi_U(F(u,v)) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

Consequently,

$$(D)\sum \Phi_V(F(v)-F(u)-f(t)(v-u)) \le (D)\sum \Phi_U(F(v)-F(u)-f(t)(v-u)) < \varepsilon.$$

Therefore, f is SH_1-integrable with primitive F and

$$(SH_1)\int_a^b f = F(b) - F(a) = (SL_\Phi)\int_a^b f.$$

This proves the assertion. □

Theorem 2.17. *Let $f:[a,b]\to X$ be SH_1-integrable with SH_1 primitive F. Then f is SL_Φ-integrable with SL_Φ-primitive F and*

$$(SL_\Phi)\int_a^b f = (SH_1)\int_a^b f.$$

Proof. Let $S \subseteq [a,b]$ be of measure zero, V a θ-nbd, and $\varepsilon > 0$. Let U be balanced, convex, and absorbing θ-nbd with $U \subseteq V$. Let $f_0 = f \cdot 1_S$. Since $f_0 = \theta$ almost everywhere, f_0 is SH_1-integrable with primitive $F_0 = \theta$ by Theorem 2.15. Hence by Theorem 2.14, there exists a gauge δ_1 such that for any δ_1-fine partition $D_1 = \{([u,v],t)\}$ on $[a,b]$,

$$(D_1)\sum_{i=1}^{n} \Phi_U(-f_0(t_i)(v_i-u_i)) = (D_1^*)\sum_{i=1}^{n} \Phi_U(-f(t_i)(v_i-u_i)) < \frac{\varepsilon}{2},$$

where $D_1^* = \{([u,v],t)\in D_1 : t\in S\}$. Since f is SH_1-integrable, there exists a gauge δ_2 such that for every δ_2-fine partition $D_2 = \{([u,v],t)\}$ of $[a,b]$,

$$(D_2)\sum \Phi_U(F(u,v) - f(t)(v-u)) < \frac{\varepsilon}{2}.$$

Let $\delta(t) = \min\{\delta_1(t), \delta_2(t)\}$. Then δ is a gauge. If $D = \{([u,v],t) : 1 \le i \le n\}$ is a partial partition of $[a,b]$ for which the tags are in S, then D is both a δ_1-fine and a δ_2-fine partial partition of $[a,b]$. Thus,

$$(D)\sum \Phi_U(-f(t)(v-u)) < \frac{\varepsilon}{2}$$

and

$$(D)\sum \Phi_U(F(u,v) - f(t)(v-u)) < \frac{\varepsilon}{2}.$$

This implies that

$$(D)\sum \Phi_V(F(u,v)) \le (D)\sum \Phi_U(F(u,v))$$
$$\le (D)\sum \Phi_U(f(t)(v-u)) + (D)\sum \Phi_U(F(u,v) - f(t)(v-u)) < \varepsilon.$$

This shows that F is a SL_Φ function. Finally, let V be a θ-nbd and $\varepsilon > 0$. Let U be a balanced, convex and absorbing θ-nbd with $U \subseteq V$. Since f is SH_1 integrable on $[a,b]$, there is a gauge δ on $[a,b]$ such that for every δ-fine partition $D = \{([u,v],t)\}$ of $[a,b]$, we have

$$(D)\sum \Phi_U(F(u,v) - f(t)(v-u) < \varepsilon.$$

Clearly, δ is a tight gauge and for each δ-fine partial partition $D' = \{([u,v],t)\}$ of $[a,b]$,

$$(D')\sum \Phi_V(F(u,v) - f(t)(v-u) \le (D')\sum \Phi_U(F(u,v) - f(t)(v-u) < \varepsilon.$$

Accordingly, f is SL_Φ-integrable with SL_Φ-primitive F on $[a,b]$ and

$$(SL_\Phi)\int_a^b f = F(b) - F(a) = (SH_1)\int_a^b f.$$

This completes the proof of the theorem. □

Theorem 2.18. *Let $F, f : [a,b] \to X$ be functions. Then the following statements are equivalent:*

(*i*) *f is SH_1-integrable with SH_1 primitive F.*
(*ii*) *f is SL_Φ-integrable with SL_Φ-primitive F.*
(*iii*) *F is a SL_Φ and $\Delta_\Phi(U,F,f)$ is of measure zero for each θ-nbd U.*

In this case,

$$\big(SL_\Phi\big)\int_a^b f = \big(SH_1\big)\int_a^b f.$$

Proof. The equivalence of (i) and (ii) follows immediately from Theorems 2.16 and 2.17. By Theorem 2.10, (iii) implies (ii). It remains only to show that (ii) implies (iii) and we only need to prove that $\Delta_\Phi(U,F,f)$ is of measure zero for all θ-nbd U. Assume (ii) and let U be a θ-nbd. Then f is SH_1-integrable with SH_1 primitive F.

Let $\varepsilon > 0$. By Theorem 2.14, there is a gauge δ on $[a,b]$ such that for every δ-fine partition $D = \{([u_i,v_i],t_i) : 1 \le i \le n\}$ we have

$$\sum_{i=1}^{n} \Phi_U\big(F(v_i) - F(u_i) - f(t_i)(v_i - u_i)\big) < \frac{\varepsilon}{2}.$$

Let $\mathcal{F} = \{[u,v] \subseteq [a,b] : \exists t \in [a,b], \Phi_U\big(F(v) - F(u) - f(t)(v-u)\big) \ge (v - u)$ and $|v-u| < \delta(t)\}$. Let $t \in \Delta_\Phi(U,F,f)$ and $\eta > 0$. Then by the defintion of $\Delta_\Phi(U,F,f)$, there is $[u,v] \subseteq [a,b]$ with $t \in [u,v]$, $m(I) = v - u < \eta$ and $\Phi_U\big(F(v) - F(u) - f(t)(v-u)\big) \ge (v-u)$. Hence, $\mathcal{F}$ is a Vitali cover for $\Delta_\Phi(U,F,f)$. By Vitali's theorem, there is a finite collection of non-degenerate close intervals $\{I_i\}_{i=1}^n \subseteq \mathcal{F}$ such that $m^*\big(\Delta_\Phi(U,F,f) \setminus \bigcup_{i=1}^n I_i\big) < \frac{\varepsilon}{2}$. Let $t_i \in I_i$ so that $I_i \in \mathcal{F}$ and $I_i \subseteq \Big(t_i - \delta(t_i), t_i + \delta(t_i)\Big)$ for each $i \in \{1,2,\dots,n\}$. Let $D = \{([x_i,y_i],s_i) : 1 \le i \le n\}$ be a δ-fine partition so that for each $i \in \{1,2,\dots,n\}$, $(I_i,t_i) \in D$. Thus,

$$\begin{aligned} m^*\big(\Delta_\Phi(U,F,f)\big) &< m^*\Big(\bigcup_{i=1}^n I_i\Big) + m^*\Big(\Delta_\Phi(U,F,f) \setminus \bigcup_{i=1}^n I_i\Big) \\ &\le \sum_{i=1}^n m(I_i) + \frac{\varepsilon}{2} \\ &\le \sum_{i=1}^n \Phi_U\big(F(v_i) - F(u_i) - f(t_i)(v_i - u_i)\big) + \frac{\varepsilon}{2} \\ &\le \varepsilon. \end{aligned}$$

Since ε was arbitrary, it follows that $\Delta_\Phi(U, F, f)$ is of measure zero. $\square$

Corollary 2.19. *Let X be a first countable LCTVS and let $f : [a, b] \to X$ be a function. Then f is SL_Φ-integrable on $[a, b]$ if and only if there is a SL_Φ function $F : [a, b] \to X$ such that $F' = f$ almost everywhere on $[a, b]$.*

Proof. Suppose f is SL_Φ-integrable with SL_Φ-primitive $F : [a, b] \to X$. By Theorem 2.18, $\Delta_\Phi(U, F, f)$ is of measure zero for each θ-nbd U. Since X is first countable, there is a countable local basis $\mathcal{B}$ at θ for the topology associated with X. By Theorem 2.11,

$$\mathcal{D}(F, f) \subseteq \bigcup_{U \in \mathcal{B}} \Delta_\Phi(U, F, f),$$

implying that $\mathcal{D}(F, f)$ is of measure zero. Thus, $F'(x) = f(x)$ almost everywhere on $[a, b]$.

Conversely, if F is a SL_Φ function such that $F'(x) = f(x)$ almost everywhere on $[a, b]$, then $\mathcal{D}(F, f)$ is of measure zero. By Theorem 2.11,

$$\mathcal{D}(F, f) = \bigcup_{\theta\text{-nbd } U} \Delta_\Phi(U, F, f).$$

Therefore, $\Delta_\Phi(U, F, f)$ is of measure zero for each θ-nbd U. $\square$

Example 2.20. The collection $\mathbb{R}^{[a,b]}$ of all functions from $[a, b]$ to $\mathbb{R}$ is a real vector space with respect to the usual addition and scalar multiplication of functions. For each $\alpha \in [a, b]$, define the evaluation map $\rho_\alpha : \mathbb{R}^{[a,b]} \to \mathbb{R}$ by $\rho_\alpha(f) = |f(\alpha)|$. Note that for every $f, g \in \mathbb{R}^{[a,b]}$ and $c \in \mathbb{R}$,

$$\rho_\alpha(f + g) = |f(\alpha) + g(\alpha)| \le |f(\alpha)| + |g(\alpha)| = \rho_\alpha(f) + \rho_\alpha(g)$$
$$\rho_\alpha(cf) = |cf(\alpha)| = |c||f(\alpha)| = |c|\rho_\alpha(f).$$

This shows that each evaluation map ρ_α is a semi-norm on $\mathbb{R}^{[a,b]}$ making the real vector space $\mathbb{R}^{[a,b]}$ into a locally convex space with absorbing, balance, and convex θ-nbds $V(\rho_\alpha) = \{f \in \mathbb{R}^{[a,b]} : \rho_\alpha(f) < 1\}$.

Now, from the family of functions $\{e_\alpha : \alpha \in [a, b]\} \subseteq \mathbb{R}^{[a,b]}$, where

$$e_\alpha(x) = \begin{cases} 1 & \text{if} \quad x = \alpha \\ 0 & \text{otherwise,} \end{cases}$$

define the function $\Theta : [a, b] \to \mathbb{R}^{[a,b]}$ by $\Theta(t) = e_t$ for each $t \in [a, b]$. Then $\rho_\alpha\big(\Theta(t)\big) = 0$ for all $\alpha \ne t \in [a, b]$. Here we find that Θ satisfies the assumption of Theorem 2.15. Hence, the zero function $F(t) = \theta$ on $[a, b]$ is a SH_1-primitive of Θ. Notice that F has a derivative equal to itself everywhere on $[a, b]$ and F is not equal to Θ at every point on $[a, b]$. Thus, the second condition in (iii) of Theorem 2.18 is not equivalent to $F'(t) = \Theta(t)$ almost everywhere on $[a, b]$ as stated in Corollary 2.19.

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References

1. V. I. Bogachev, O. G. Smolyanov, *Topological Vector Spaces and Their Applications*, Springer International Publishing AG, Switzerland, 2017.
2. S. Cao, The Henstock Integral for Banach-Valued Functions, *Southeast Asian Bull. Math.*, **16**(1), (1992), 35–40.
3. S. Canoy, Jr., M. Navarro, A Denjoy-type integral for Banach-valued functions, *Rend. Circ. Mat. Palermo*, **44**(2), (1995), 330-336.
4. J. Dugundji, *Topology*, Allyn and Bacon, Inc., Boston, 1966.
5. R. Maza, S. Canoy, Jr., On SL-integral of LCTVS-valued Functions, *Real Analysis Exchange*, **46**(2), (2020), 505–522.
6. R. Maza, S. Canoy, Jr., Denjoy-type Integrals in Locally Convex Topological Vector Space, *European Journal of Pure and Applied Mathematics*, **14**(4), (2021), 1169–1183.
7. Lusin, N., Sur les proprietes de l'integrale de M. Denjoy, *Comptes Rendus De l'Academie des Sciences*, **155**, (1912), 1475-1478.
8. R. Paluga, S. Canoy, Jr., The Henstock Integral in Topological Vector Spaces, *Matimyas Matematika*, **24**(3), (2001), 34–47.
9. R. Paluga, S. Canoy, Jr., On the Strongly Henstock integral in topological vector spaces, *Journal of Research in Science and Engineering*, **1**(3), (2004), 46.
10. W. Rudin, *Functional Analysis*, 2nd edition, McGraw-hill, Inc., Singapore, 1991.
11. L. P. Yee, *Lanzhou Lectures on Henstock Integration,* Series in Real Analysis, **2**, World Scientific, 1989.
12. L. P. Yee, R. Výborný, *Integral: An Easy Approach after Kurzweil and Henstock,* Cambridge University Press, New York, 2000.