Iranian Journal of Mathematical Sciences and Informatics
Vol. 21, No. 1 (2026), pp 107-131
DOI: 10.61882/ijmsi.21.1.107

Fixed Point Theorems for Multivalued Mappings in Banach Algebras and an Application for Fractional Integral Inclusion

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Abstract. In this paper, we establish some fixed point results for the sum and the product of three multivalued mappings, with weakly sequentially closed graph under weak topology features in a Banach algebra. Satisfying a certain sequential condition $(\mathcal{P})$. As an application, our results are used to prove the existence of solutions for a certain non-linear integral inclusion of fractional order.

Keywords: Measures of weak non-compactness, Multivalued mapping, Weakly condensing, Weakly sequentially closed graph, Fixed point theorems, Integral inclusion.

2000 Mathematics subject classification: 47H04, 47H10, 26A33.

1. Introduction

Recently, many authors were concerned in the study of non-linear integral inclusion in a Banach algebra via fixed point techniques. Some of these inclusions can be formulated into non-linear operator inclusion:

$$x \in A(x)B(x) + C(x), \qquad x \in \Omega, \tag{1.1}$$

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Received 28 August 2021; Accepted 16 September 2022

where Ω is a non-empty closed convex subset of a Banach algebra E. Leray-Schauder's (resp. Krasnoselskii) fixed point theorems for the sum and the product of three operators is one of the most important techniques that gives the existence of solutions for (1.1). See for example, [14, 15]. In some type of applications one may encounter the case that the operators A, B and C may be not continuous, and the product of two weakly convergent sequences is not necessary weakly convergent. Ben Amar [5] overcame this problem by presenting a new class of Banach algebras satisfying a sequential condition $(\mathcal{P})$, see Definition 2.8. Also, they initiated the study of fixed point theorems for the sum and the product of weakly sequentially continuous operators, with applications to a non-linear integral equations under weak topology settings. This definition plays an important role in many other works and set a major cornerstone in the field. For instance, in [4] there are some non-linear alternatives of Leray-Schauder type involving three operators in Banach algebra satisfying a sequential condition $(\mathcal{P})$.

Recall the function (β) was introduced by De Blasi [13] as a measure of weak non-compactness that can be regarded as the counterpart for the weak topology of the classical Hausdorff measure of norm non-compactness. A. Ben Amar and D. O'regan, [8] proved that some fixed point theorems for the sum and product of three non-linear weakly sequentially continuous operators, in a certain Banach algebra satisfying sequential condition $(\mathcal{P})$, via the measure of weak non-compactness. Moreover, the single-valued mapping $\left(\frac{I-C}{A}\right)$ and its invertibility play a fundamental role in that argument, where the single-valued mapping A is quasi-regular. An extension of these results to establish some non-linear alternatives of Leray-Schauder type in a Banach algebra satisfying the sequential condition $(\mathcal{P})$ is found in [2].

The multivalued mapping with weak topology has a wide interest in many areas of application. On one hand, in [17], there are some fixed point theorems in Banach algebras for the multivalued mapping AB, where A is Lipschitzian and B is compact and upper semi-continuous. Ben Amar, Boumaiza and O'Regan [10] introduce a new class of multivalued mappings of the form $\left(\frac{I-C}{A}\right)$ where A and C are multivalued mappings acting on Banach algebras. They also use the properties of $\mathcal{D}-$Lipschitzian and $\mathcal{D}-$set-Lipschitzian with respect to the De Blasi measure of weak non-compactness for the mapping A and C. These results generalize, extend and improve that the well known results for weakly sequentially single-valued mappings in [5, 7, 8]. On the other hand, O'Regan and Taoudi [20] give some versions of the Krasnoselskii fixed point theorems in the framework of weak topologies for multivalued mapping $(I-B)^{-1}A$, where A is a multivalued mapping with weakly sequentially closed graph, and B is a weakly sequentially continuous single-valued map. In addition, Ben Amar in

[9] gives some multivalued analogues of Krasnoselskii fixed point theorem for mappings of the form $T+S$ on a non-empty closed convex set of a Banach space, where T is weakly completely continuous and S is weakly condensing (resp. 1-set weakly contractive) mapping with weakly sequentially closed graph. The authors in [1] extend these results to obtain new multivalued analogues of Leray-Schauder alternatives (or Krasnoselskii fixed point theorems) for the sum of two mappings $A+B$, where A is weakly compact with weakly sequentially closed graph and B is $\Phi-$condensing (or hemi-weakly compact) with weakly sequentially closed graph.

In this paper, we establish some new fixed point results to obtain new multivalued analogues of Leray-Schauder alternative (or Krasnoselskii) fixed point theorems in a Banach algebras that satisfies a sequential condition $(\mathcal{P})$, for the sum and the product of three multivalued mappings $AB+C$, where A, B and C are weakly sequentially closed graphs. We also use the properties of $\Phi-$ condensing, Φ-non-expansive and hemi-weakly compact. These results complement the recent literature [1, 2, 4, 5, 7, 8, 9, 10, 17, 20]. The main condition in our results is formulated in terms of axiomatic measures of weak non-compactness. Finally, we apply these fixed point results to study the existence of a solution for the following non-linear integral inclusion of fractional order α.

$$x(t) \in F(t,x(t)) + K(t,x(t)) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} G(s,x(s))ds, \tag{1.2}$$

where F, K and $G : J \times X \longrightarrow P(X)$, $t \in J=[0,T]$ and $\alpha \in (0,1)$.

2. Preliminaries

Throughout this section, we shall introduce some necessary notations and definitions which will be needed to achieve our work. Let E be a Hausdorff linear topological space. Now

$$P(E) = \Big\{ D \subset E : D \text{ is non-empty} \Big\},$$

$$P_{\text{bd}}(E) = \Big\{ D \subset E : D \text{ is non-empty and bounded} \Big\},$$

$$P_{\text{cv}}(E) = \Big\{ D \subset E : D \text{ is non-empty and convex} \Big\},$$

$$P_{\text{cl,bd}}(E) = \Big\{ D \subset E : D \text{ is non-empty closed and bounded} \Big\},$$

$$P_{\text{cl,bd,cv}}(E) = \Big\{ D \subset E : D \text{ is non-empty closed, bounded and convex} \Big\}.$$

Let Z be a non-empty subset of Banach space Y and $F : Z \to P(E)$ be a multivalued mapping. We denote

$$R(F) = \bigcup_{y \in Z} F(y), \ \ and \ GrF = \{(z, x) \in Z \times E : x \in F(z)\}$$

the range and the graph of F respectively, moreover, for every subset A of E, we put

$$F^{-1}(A) = \{z \in Z : F(z) \cap A \neq \phi\}.$$

Suppose that E is a Banach space with θ and Z is weakly closed in Y. Now F is said to have weakly sequentially closed graph if for every sequence $\{x_n\} \subset Z$ with $x_n \rightharpoonup x \in Z$ and for every sequence $\{y_n\}$ with $y_n \in F(x_n)$ for all $n \in \mathbb{N}$, $y_n \rightharpoonup y$ in E implies $y \in F(x)$; here $\rightharpoonup$ denotes weak convergence. F is called weakly compact, if $F(A)$ is a relatively weakly compact subset of E for every bounded subset $A \in P_{bd}(Z)$, in addition, F is called weakly upper semi-continuous if and only if $F^{-1}(A)$ is weakly closed for all weakly closed sets $A \subset E$. If F is single-valued mapping, then F is said to be weakly sequentially continuous if for every sequence $\{x_n\} \subset Z$ with $x_n \rightharpoonup x \in Z$, we have $F(x_n) \rightharpoonup F(x)$. Now F is said to be sequentially weakly upper semi-compact in Z, *s.w.u.sco* for short, if for any weakly convergent sequence $\{x_n\}$ in Z and an arbitrary $y_n \in F(x_n)$, the sequence $\{y_n\}$ has a weakly convergent subsequence in E. If F is a single-valued mapping, F is sequentially weakly upper semi-compact if for any weakly convergent sequence $\{x_n\}$ in Z the sequence $\{Fx_n\}$ has a weakly convergent subsequence in E.

Lemma 2.1 ([10])**.** *If $F : Z \to P(X)$ is a s.w.u.sco. multivalued mapping in Z and Z is relatively weakly compact then,*
1- *The set $F(x)$ is relatively weakly compact for each $x \in Z$,*
2- *The set $F(Z)$ is relatively weakly compact.*

Definition 2.2. Let X be a Banach space and C a lattice with a least element, which is denoted by 0. By a measure of weak non-compactness ($MWNC$) on X we mean a function Φ defined on a set of all bounded subsets of X with values in C, such that for any $\Omega_1, \Omega_2 \in P_{bd}(X)$:
(1) $\Phi(\overline{co}(\Omega_1)) = \Phi(\Omega_1)$, where $\overline{co}$ denotes the closed convex hull of Ω_1,
(2) $\Omega_1 \subseteq \Omega_2$ implies $\Phi(\Omega_1) \leq \Phi(\Omega_2)$,
(3) $\Phi(\Omega_1 \cup \{a\}) = \Phi(\Omega_1)$, for all $a \in X$,
(4) $\Phi(\Omega_1) = 0$ if and only if Ω_1 is relatively weakly compact in X.

If the lattice is a cone of vector space, then the ($MWNC$) Φ is said to positive homogeneous provided $\Phi(\lambda\Omega) = \lambda\Phi(\Omega)$ for all $\lambda > 0$ and $\Omega \in P_{bd}(X)$, and it is called semi-additive iff $\Phi(\Omega_1+\Omega_2) \leq \Phi(\Omega_1)+\Phi(\Omega_2)$ for all $\Omega_1, \Omega_2 \in P_{bd}(X)$. These notations is a generalization of the important well known De Blasi measure of weak non-compactness β [13] which was defined on each bounded set Ω

of X by

$\beta(\Omega) = \inf\{r > 0 : \text{there exists a weakly compact set D such that } \Omega \subseteq D + B_r(0)\}$,

where $B_r(0)$ is the closed ball with radius r and center 0.
It is well known that β enjoys these properties: for any $\Omega_1, \Omega_2 \in P_{bd}(E)$,
(5) $\beta(\Omega_1 \cup \Omega_2) = \max\{\beta(\Omega_1), \beta(\Omega_2)\}$,
(6) $\beta(\lambda\Omega_1) = \lambda\beta(\Omega_1)$ for all $\lambda > 0$,
(7) $\beta(\Omega_1 + \Omega_2) \leq \beta(\Omega_1) + \beta(\Omega_2)$.

Definition 2.3. Let $F : \Omega \to P(X)$, Ω be a non-empty subset of Banach space X and Φ a $(MWNC)$ on X, we can say that

- *A-* F is Φ-*condensing* if F is bonded and $\Phi(F(D)) < \Phi(D)$ for all bounded sets $D \subseteq \Omega$ with $\Phi(D) \neq 0$.
- *B-* F is Φ-*non-expansive* if F is bonded and $\Phi(F(D)) \leq \Phi(D)$ for all bounded sets $D \subseteq \Omega$ with $\Phi(D) \neq 0$.
- *C-* F is *hemi-weakly compact* if for each sequence $\{x_n\}$ has a weakly convergent subsequence whenever there exists $y_n \in F(x_n)$ such that the sequence $\{x_n - y_n\}$ is weakly convergent.

In the sequel, we shall need the following theorems.

Theorem 2.4 ([6])**.** *Let Ω be a non-empty, closed, convex subset of Banach space E. Suppose that $F : \Omega \to P_{cv}(\Omega)$ has a weakly sequentially closed graph and $F(\Omega)$ is relatively weakly compact. Then F has a fixed point.*

Theorem 2.5 ([6])**.** *Let Ω be a non-empty, closed, convex subset of Banach space E. and Φ is $(MWNC)$ on E. Suppose that $F : \Omega \to P_{cv}(\Omega)$ has a weakly sequentially closed graph, is Φ-condensing and $F(\Omega)$ is bounded. Then F has a fixed point.*

Theorem 2.6 ([1])**.** *Let Ω be a non-empty closed convex subset of a Banach space E and U be a weakly open subset of Ω with $\theta \in U$. Assume $F : \overline{U^w} \to P_{cv}(\Omega)$ has weakly sequentially closed graph. In addition, suppose that $F(\overline{U^w})$ is relatively weakly compact. Then, either*

- A_1- *F has a fixed point, or*
- A_2- *there is a point $x \in \partial_\Omega U$ (the weak boundary of U in Ω) and $\lambda \in (0, 1)$ with $x \in \lambda F(x)$.*

Theorem 2.7 ([11])**.** ***Eberlien-Šmulian's Theorem****. In the weak topology on a normed space, compactness and sequential compactness coincide. That is, a subset D of a normed space X is relatively weakly compact (respectively, weakly compact) if and only if every sequence in D has a weakly convergent subsequence in X (respectively, in D).*

An algebra is any vector space X equipped with an associative binary operation of multiplication satisfying the condition $(\alpha x)(\beta y) = (\alpha\beta)(xy)$ for any elements $x, y \in X$ and any scalars α, β. A norm algebra is an algebra which is norm, as a vector space, and in which

$$\|xy\| \leq \|x\|.\|y\|$$

for all x, y. A complete norm algebra is called Banach algebra.

Definition 2.8 ([5])**.** We call that the Banach algebra E satisfies a sequential condition $(\mathcal{P})$, if for any sequences $\{x_n\}$ and $\{y_n\}$ in E such that $x_n \rightharpoonup x$ and $y_n \rightharpoonup y$ implies that $x_n y_n \rightharpoonup xy$.

Note that, every finite dimensional Banach algebra satisfies condition $(\mathcal{P})$. If X satisfies condition $(\mathcal{P})$ and K is a hausdorff compact space then $\mathcal{C}(K, X)$ is also a Banach algebra satisfying condition $(\mathcal{P})$. This consequence from Dobrokov's Theorem, see [[18] Theorem (9)].
A Banach space X is said to have the Dunford-Pettis property if for each Banach space Y, every weakly compact linear operator $F : X \to Y$ takes weakly compact sets in X into norm compact sets of Y. It was proved in [3] that every Banach algebra having the Dunford-Pettis property satisfies condition $(\mathcal{P})$.

Remark 2.9. Let E be a Banach algebra with the condition $(\mathcal{P})$, and suppose that A_1 and A_2 be two arbitrary weakly compact subsets of E. Then the product A_1A_2 is weakly compact.

Lemma 2.10 ([7])**.** *Let E be a Banach algebra with a condition $(\mathcal{P})$. Then for any bounded subset D of E and weakly compact subset K of E, we have $\beta(D.K) \leq \|K\|\beta(D)$, where $\|K\| = \sup\{\|x\|, x \in K\}$.*

Definition 2.11 ([10])**.** Let E be a Banach algebra and $A, C : E \to P(E)$ be multivalued mappings. We say that the mapping $\left(\frac{I-C}{A}\right)$ is well defined on $x \in E$ and we write $y \in \left(\frac{I-C}{A}\right)(x)$ if $x \in yA(x) + C(x)$.

3. KRASNOSELSKII TYPE FIXED POINT THEOREMS

Throughout this section we present some existence results for the following non-linear operator inclusion $x \in A(x)B(x) + C(x)$.

Theorem 3.1. *Let Ω be a non-empty closed convex subset of a Banach algebra E satisfying condition $(\mathcal{P})$ and Φ a semi-additive (MWNC) on E. Let $A, B, C : \Omega \to P(E)$ be three multivalued mappings satisfying the following conditions:*

(i)- A, B and C have weakly sequentially closed graphs,
(ii)- A and B are weakly compacts, and C is $\Phi-$condensing,
(iii)- For all $x \in \Omega$, $A(x)B(x) + C(x) \in P_{cv}(\Omega)$,
(iv)- $(AB + C)(\Omega)$ is bounded.

Then there exists $x \in \Omega$ with $x \in A(x)B(x) + C(x)$.

Proof. Let $F := AB + C : \Omega \to P_{cv}(\Omega)$. We show that F has weakly sequentially closed graph. Let $x_n \rightharpoonup x$ and $y_n \in F(x_n)$ such that $y_n \rightharpoonup y$. There exists $u_n \in A(x_n)$, $v_n \in B(x_n)$ and $w_n \in C(x_n)$ such that $y_n = u_n.v_n + w_n$. Since A and B are weakly compacts with weakly sequentially closed graphs and $\{x_n\}$ is bounded, it follows that by the Eberlien-Šmulian's Theorem $u_{n_k} \rightharpoonup u \in A(x)$ and $v_{n_k} \rightharpoonup v \in B(x)$. Since E satisfies a sequential condition $(\mathcal{P})$, and C has a weakly sequentially closed graph. Then

$$w_{n_k} = y_{n_k} - u_{n_k}.v_{n_k} \rightharpoonup y - u.v \in C(x).$$

Hence $y \in A(x).B(x) + C(x)$. Consequently, F has weakly sequentially closed graph.

Now, we claim that F is $\Phi-$condensing. Let D be arbitrary bounded subset of Ω with $\Phi(D) \neq 0$. By using Remark 2.9

$$\Phi(F(D)) \leq \Phi(\overline{A(D)^w}.\overline{B(D)^w}) + \Phi(C(D)) < \Phi(D).$$

Apply Theorem 2.5 to deduce that $F := AB + C$ has a fixed point in Ω. □

Theorem 3.2. *Let Ω be a non-empty closed convex subset of a Banach algebra E satisfying condition $(\mathcal{P})$ and Φ a positive homogeneous semi-additive $(MWNC)$ on E. Let $A, B, C : \Omega \to P(E)$ be three multivalued mappings satisfying the following conditions:*

(i)- A, B and C have weakly sequentially closed graphs,

(ii)- A and B are weakly compacts, and C is $\Phi-$non-expansive hemi-weakly compact,

(iii)- There exists a bounded set Ω_0 of E and a sequence $\{\lambda_n\} \subseteq (0,1)$ such that $\lambda_n \to 1$, for all $x \in \Omega$, $(AB + \lambda_n C)(x) \in P_{cv}(\Omega)$ and $(AB + \lambda_n C)(\Omega) \subset \Omega_0$ for all n.

Then there exists $x \in \Omega$ with $x \in A(x)B(x) + C(x)$.

Proof. Define $F_n := AB + \lambda_n C$, for all $n \in \mathbb{N}$. Then by assumption (iii), $F_n : \Omega \to P_{cv}(\Omega)$ is well defined and $F_n(\Omega)$ is bounded. In view of Theorem 3.1, F_n has weakly sequentially closed graph. Let D be an arbitrary bounded subset of Ω with $\Phi(D) \neq 0$. By using Remark 2.9, and C is $\Phi-$non-expansive then,

$$\Phi(F_n(D)) \leq \Phi(\overline{A(D)^w}.\overline{B(D)^w}) + \lambda_n \Phi(C(D)) < \Phi(D).$$

Therefore, F_n is $\Phi-$ condensing. Theorem 2.5 guarantees that F_n has fixed point $x_n \in \Omega$. That is,

$$x_n \in A(x_n).B(x_n) + \lambda_n C(x_n).$$

Then there exists $u_n \in A(x_n)$, $v_n \in B(x_n)$ and $w_n \in C(x_n)$ such that $x_n = u_n.v_n + \lambda_n w_n$. Since A and B are weakly compacts with weakly sequentially closed graphs, E satisfies a condition $(\mathcal{P})$, and $\{x_n\}$ is bounded. Then we can find subsequences $\{u_{n_k}\}$ with $u_{n_k} \rightharpoonup u \in A(x)$ and $\{v_{n_k}\}$ with $v_{n_k} \rightharpoonup v \in B(x)$ such that $u_{n_k}.v_{n_k} \rightharpoonup u.v$. Obviously, the sequence $\{w_n\}$ is bounded and $\lambda_n \to 1$, then we get

$$x_{n_k} - w_{n_k} = u_{n_k}.v_{n_k} + (\lambda_n - 1)w_{n_k} \rightharpoonup u.v.$$

By assumption C is a hemi-weakly compact. This implies $\{x_{n_k}\}$ has a weakly convergent subsequence, say $\{x_{n_{k_j}}\}$. Also, C with weakly sequentially closed graph. Hence $w_{n_{k_j}} \rightharpoonup x - u.v \in C(x)$. Therefore $x \in A(x).B(x) + C(x)$. $\square$

Remark 3.3. Theorem 3.1 and Theorem 3.2 extends theorem (2.1) and Theorem (2.2) in [9] respectively, for the case of the sum and the product of three multivalued mappings.

Theorem 3.4. *Let Ω be a non-empty closed convex subset of a Banach algebra E satisfying condition $(\mathcal{P})$. Let $A, C : E \to P(E)$ and $B : \Omega \to P(E)$ be three multivalued mappings satisfying the following conditions:*

(i)- A, B and C have weakly sequentially closed graphs,
(ii)- A and B are weakly compacts, and C is hemi-weakly compact,
(iii)- $\left(\frac{I-C}{A}\right)^{-1}$ exists on $B(\Omega)$,
(iv)- For each $x \in \Omega$, $\left(\frac{I-C}{A}\right)^{-1} B(x) \in P_{cv}(\Omega)$,
(v)- $\left(\frac{I-C}{A}\right)^{-1} B(\Omega)$ is bounded.

Then there exists $x \in \Omega$ with $x \in A(x)B(x) + C(x)$.

Proof. From the hypotheses (iii), the multivalued mapping $\left(\frac{I-C}{A}\right)^{-1}$ exists on $B(\Omega)$. Let $F = \left(\frac{I-C}{A}\right)^{-1} B$. Then by assumption (iv) $F : \Omega \to P_{cv}(\Omega)$ is well defined. To show that F has a weakly sequentially closed graph, let $x_n \rightharpoonup x \in \Omega$ and $y_n \in F(\Omega)$, $y_n \rightharpoonup y$ with $y_n \in F(x_n)$. Then $\left(\frac{I-C}{A}\right)(y_n) \cap B(x_n) \neq \emptyset$, so there exists $z_n \in B(x_n)$ such that $z_n \in \left(\frac{I-C}{A}\right)(y_n)$. From the definition of the mapping $\frac{I-C}{A}$, we have $z_n u_n = y_n - v_n$ where $v_n \in C(y_n)$ and $u_n \in A(y_n)$. Since B is a weakly compact and $\{x_n\}$ is bounded. By Eberlien-Šmulian's Theorem, $\{z_n\}$ has a subsequence $\{z_{n_k}\}$ which weakly converges to some $z \in B(x)$. Also, A is a weakly compact and $\{y_n\}$ is bounded, then $\{u_n\}$ has a subsequence $\{u_{n_k}\}$ which weakly converges to some $u \in A(y)$. In addition, C has a weakly sequentially closed graph, and E satisfies a condition $(\mathcal{P})$. Then we get

$$v_{n_k} = y_{n_k} - z_{n_k} u_{n_k} \rightharpoonup y - zu \in C(y).$$

Hence, $z \in \left(\frac{I-C}{A}\right)(y)$. Consequently, $\left(\frac{I-C}{A}\right)(y) \cap B(x) \neq \emptyset$ and $y \in \left(\frac{I-C}{A}\right)^{-1} B(x)$. Hence F has a weakly sequentially closed graph. Now, let D be an arbitrary bounded subset of Ω. We claim that $F(D)$ is relatively weakly compact. Let $y_n \in F(D)$. Choose $\{x_n\} \subset D$ such that $y_n \in F(x_n)$, that is $y_n \in A(y_n)B(x_n) + C(y_n)$. Thus there exists $z_n \in B(x_n)$, $u_n \in A(y_n)$ and $v_n \in C(y_n)$ such that $y_n = u_n z_n + v_n$. Since A and B are weakly compact, and

E satisfies a condition $(\mathcal{P})$. Tt follows that $y_{n_k} - v_{n_k} = u_{n_k} z_{n_k} \rightharpoonup uz$. Since C is a hemi-weakly compact, then $\{y_{n_k}\}$ has a weakly convergent subsequence. Hence $F(D)$ is relatively weakly compact. Consequently, F is weakly compact. From Theorem 2.4, F has a fixed point. Then there exists $x \in \Omega$ such that $x \in A(x)B(x) + C(x)$. □

Proposition 3.5. *Let E be a Banach algebra satisfying condition $(\mathcal{P})$ and Φ is a $(MWNC)$ on E. Let Ω be a non-empty closed convex subset of E and $A, C : E \to P_{cv}(E)$ be two multivalued mappings satisfying the following conditions:*

(i)- A and C have weakly sequentially closed graphs,
(ii)- A is a weakly compact, and C is $\beta-$ condensing,
(iii)- $A(E)$ and $C(E)$ are bounded,

Then the multivalued operator $\left(\frac{I-C}{A}\right)^{-1}$ exists in E.

Proof. Fix z in E. Consider,

$$\Gamma_z : E \to P_{cv}(E), \ \ x \longmapsto Ax.z + Cx.$$

Since $Ax.z + Cx$ is convex, it is clear that Γ_z is well defined. We prove that the multivalued mapping Γ_z satisfies all the assumptions in the statement of Theorem 2.5. Let $x_n \rightharpoonup x$ and $y_n \in Ax_n.z + Cx_n$ such that $y_n \rightharpoonup y$. There exists $u_n \in Ax_n$ and $v_n \in Cx_n$ such that $y_n = u_n.z + v_n$. Since A is a weakly compact with weakly sequentially closed graph, and $\{x_n\}$ is bounded. By Eberlien-Šmulian's Theorem, $u_{n_k} \rightharpoonup u \in Ax$. The right hand multiplication operator $R_z(x) = x.z$ is a continuous linear operator, so it is weakly continuous. Accordingly,

$$v_{n_k} = y_{n_k} - u_{n_k}.z \rightharpoonup y - u.z.$$

The operator C has a weakly sequentially closed graph, we deduce that $y - u.z \in Cx$. Hence $y \in Ax.z + Cx = \Gamma_z(x)$. Consequently, Γ_z has a weakly sequentially closed graph.

Now, let D be an arbitrary bounded subset of E with $\beta(D) \neq 0$. It is clear that $\Gamma_x(D)$ is bounded. From Lemma 2.10, we get

$$\beta(\Gamma_z(D)) \leq \beta(A(D).z) + \beta(C(D)) \leq \|z\|\beta(A(D)) + \beta(C(D)) < \beta(D).$$

Hence Γ_z is $\beta-$condensing. Moreover, by assumption (iii) $\Gamma_z(E)$ is bounded. Then the multivalued mapping Γ_z satisfies all the assumptions in the statement of Theorem 2.5, so there exists $x \in E$ such that $x \in \Gamma_z(x) = Ax.z + Cx$. Thus $z \in \left(\frac{I-C}{A}\right)(x)$. Therefore, the multivalued operator $\left(\frac{I-C}{A}\right)^{-1}$ exists in E. □

Theorem 3.6. *Let E be a Banach algebra satisfying condition $(\mathcal{P})$ and Φ a $(MWNC)$ on E. Let Ω be a non-empty closed convex subset of E and*

$A, C : E \to P_{cv}(E)$ and $B : \Omega \to P(E)$ be three multivalued mappings satisfying the following conditions:

(i)- A, B and C have weakly sequentially closed graphs,
(ii)- A and C are weakly compacts, and B is s.w.u.sco.,
(iii)- $A(E)$, $B(\Omega)$ and $C(E)$ are bounded,
(iv)- For all $y \in \Omega$, $\left(\frac{I-C}{A}\right)^{-1} B(y) \in P_{cv}(\Omega)$,
(v)- $\left(\frac{I-C}{A}\right)^{-1} B$ is $\Phi-$ condensing.

Then, there exists $x \in \Omega$ with $x \in A(x)B(x) + C(x)$.

Proof. Let $y \in \Omega$ and fix $z \in B(y)$. Consider,

$$\Gamma_z : E \to P_{cv}(E), \ \ x \longmapsto Ax.z + Cx.$$

Since every weakly compact operator is $\beta-$condensing. It follows that C is $\beta-$condensing, and from Proposition 3.5, the multivalued mapping $\left(\frac{I-C}{A}\right)^{-1}$ exists on $B(\Omega)$. Let $F = \left(\frac{I-C}{A}\right)^{-1} B$, then by assumption (iv) $F : \Omega \rightharpoonup P_{cv}(\Omega)$ is well defined. We show that F has a weakly sequentially closed graph. Let $x_n \rightharpoonup x \in \Omega$ and $y_n \in F(\Omega)$, $y_n \rightharpoonup y$ with $y_n \in F(x_n)$. Then $\left(\frac{I-C}{A}\right)(y_n) \cap B(x_n) \neq \emptyset$. Accordingly, there exists $z_n \in B(x_n)$ such that $z_n \in \left(\frac{I-C}{A}\right)(y_n)$. Hence $u_n z_n = y_n - v_n$ where $v_n \in C(y_n)$ and $u_n \in A(y_n)$. Since B is s.w.u.sco. and has a weakly sequentially closed graph, it follows that there exists a subsequence $\{z_{n_k}\}$ such that $z_{n_k} \rightharpoonup z \in B(x)$. Since A and C are weakly compacts, E satisfies a condition $(\mathcal{P})$, and $\{y_n\}$ is bounded. By Eberlien-Šmulian's Theorem, $\{u_n\}$ and $\{v_n\}$ has convergent subsequences $u_{n_k} \rightharpoonup u \in A(y)$ and $v_{n_k} \rightharpoonup v \in C(y)$ respectively. Hence $y_{n_k} = u_{n_k} z_{n_k} + v_{n_k}$. By the uniqueness of weak limit we can get $y = uz + v \in A(y).z + C(y)$, that is $z \in \left(\frac{I-C}{A}\right) y$. Then $\left(\frac{I-C}{A}\right)(y) \cap B(x) \neq \emptyset$. Hence $y \in \left(\frac{I-C}{A}\right)^{-1} B(x)$, and so F has a weakly sequentially closed graph. Using,

$$F(\Omega) \subset A(F(\Omega))B(\Omega) + C(F(\Omega)),$$

and taking into account the assumption (iii). We conclude that $F(\Omega)$ is bounded. Now by Theorem 2.5 F has a fixed point $x \in \Omega$. □

Theorem 3.7. *Let E be a Banach algebra satisfying condition $(\mathcal{P})$ and Φ a $(MWNC)$ on E. Let Ω be a non-empty closed convex subset of E and $A, C : E \to P_{cv}(E)$ and $B : \Omega \to P(E)$ be three multivalued mappings satisfying the following conditions:*

(i)- A, B and C have weakly sequentially closed graphs,
(ii)- A and C are weakly compacts, and B is s.w.u.sco.,
(iii)- $A(E)$, $B(\Omega)$ and $C(E)$ are bounded,
(iv)- For all $y \in \Omega$, $\left(\frac{I-C}{A}\right)^{-1} B(y) \in P_{cv}(\Omega)$,
(v)- $\|A(\Omega)\| \leq 1$, and B is $\beta-$condensing.

Then, there exists $x \in \Omega$ with $x \in A(x)B(x) + C(x)$.

Proof. According to Theorem 3.6, it suffices to show that the operator $F := \left(\frac{I-C}{A}\right)^{-1} B$ is $\Phi-$condensing. To do this, let D be an arbitrary bounded subset of Ω with $\beta(D) > 0$. Clearly,

$$F(D) \subseteq C(F(D)) + A(F(D)).B(D) \subseteq C(F(D)) + \overline{A(F(D))^w} B(D),$$

and $\overline{A(F(D))^w}$ is weakly compact. For $x \in \overline{A(F(D))^w}$, Eberlein-Šmulian's Theorem says there exists a sequence $\{x_n\} \subset A(F(D))$ with $x_n \rightharpoonup x$. Since for all n, $\|x_n\| \leq 1$ and $\|x\| \leq \liminf \|x_n\|$, then $\|x\| \leq 1$, and hence $\|\overline{A(F(D))^w}\| \leq 1$. By Lemma 2.10 and the hypotheses that C is a weakly compact, we have

$$\beta(F(D)) \leq \beta(C(F(D))) + \beta(\overline{A(F(D))^w}.B(D)) \leq \|\overline{A(F(D))^w}\| \beta(B(D)) < \beta(D).$$

Hence, $\left(\frac{I-C}{A}\right)^{-1} B$ is $\beta-$condensing. The proof is complete. □

Theorem 3.8. *Let E be a Banach algebra satisfying condition $(\mathcal{P})$ and Φ a $(MWNC)$ on E. Let Ω be a non-empty closed convex subset of E and $A, C : E \to P_{cv}(E)$ and $B : \Omega \to P(E)$ be three multivalued mappings satisfying the following conditions:*

(i)- A, B and C have weakly sequentially closed graphs,
(ii)- A and B are weakly compacts, and C is $\beta-$condensing,
(iii)- $A(E)$, $B(\Omega)$ and $C(E)$ are bounded,
(iv)- For all $y \in \Omega$, $\left(\frac{I-C}{A}\right)^{-1} B(y) \in P_{cv}(\Omega)$,

Then, there exists $x \in \Omega$ with $x \in A(x)B(x) + C(x)$.

Proof. From Proposition 3.5, the multivalued mapping $\left(\frac{I-C}{A}\right)^{-1}$ exists on $B(\Omega)$. Let $F = \left(\frac{I-C}{A}\right)^{-1} B$. Then by assumption (iv) $F : \Omega \to P_{cv}(\Omega)$ is well defined. In view of Theorem 3.4, F has a weakly sequentially closed graph. Using

$$F(\Omega) \subset A(F(\Omega))B(\Omega) + C(F(\Omega))$$

and taking into account the assumption (iii), we conclude that $F(\Omega)$ is bounded. Let D be an arbitrary bounded subset of Ω. We claim that F is a weakly compact. From Remark 2.9 we have

$$\beta(F(D)) \leq \beta(\overline{(A(F(D)))^w}.\overline{(B(D))^w}) + \beta(C(F(D))) < \beta(F(D)),$$

a contradiction. Hence $F(D)$ is relatively weakly compact. Consequently, F is weakly compact. From Theorem 2.4, F has a fixed point. □

Recall that, every weakly compact operator is $\beta-$condensing, so an interesting special case of Theorem 3.8 in the applicable form is:

Corollary 3.9. *Let Ω be a non-empty closed convex subset of a Banach algebra E satisfying condition $(\mathcal{P})$. Let $A, C : E \to P_{cv}(E)$ and $B : \Omega \to P(E)$ be three multivalued mappings satisfying the following conditions:*

(i)- A, B and C have weakly sequentially closed graphs,
(ii)- A, B and C are weakly compacts,
(iii)- $A(E)$, $B(\Omega)$ and $C(E)$ are bounded,
(iv)- For each $x \in \Omega$, $\left(\frac{I-C}{A}\right)^{-1} B(x) \in P_{cv}(\Omega)$.

Then, there exists $x \in \Omega$ with $x \in A(x)B(x) + C(x)$.

Proposition 3.10. *Let E be a Banach algebra satisfying condition $(\mathcal{P})$ and Φ a MWNC on E. Let Ω be a non-empty closed convex subset of E and $A, C : E \to P_{cv}(E)$ and $B : \Omega \to P(E)$ be three multivalued mappings satisfying the following conditions:*

(i)- A and C have weakly sequentially closed graphs,
(ii)- C is weakly compact,
(iii)- $A(E)$ and $C(E)$ are bounded,
(iv)- $\|B(\Omega)\| \leq 1$, and A is $\beta-$condensing.

Then the multivalued operator $\left(\frac{I-C}{A}\right)^{-1}$ exists on $B(\Omega)$.

Proof. Let $y \in \Omega$ and fix $z \in B(y)$. Consider,

$$\Gamma_z : E \to P_{cv}(E), \;\; x \longmapsto Ax.z + Cx.$$

Since $Ax.z + Cx$ is convex, it is clear that Γ_z is well defined. We prove that the multivalued mapping Γ_z satisfies all the assumptions in the statement of Theorem 2.5. Let $x_n \rightharpoonup x$ and $y_n \in Ax_n.z + Cx_n$ such that $y_n \rightharpoonup y$. There exists $u_n \in Ax_n$ and $v_n \in Cx_n$ such that $y_n = u_n.z + v_n$. Since C is weakly compact with weakly sequentially closed graph and $\{x_n\}$ is bounded. Then by Eberlien-Šmulian's Theorem $v_{n_k} \rightharpoonup v \in Cx$. Accordingly,

$$u_{n_k}.z = y_{n_k} - v_{n_k} \rightharpoonup y - v.$$

Since A has a weakly sequentially closed graph, it follows that $(y - v) \in Ax.z$. Hence $y \in Ax.z + Cx = \Gamma_z(x)$. Consequently, Γ_z has a weakly sequentially closed graph. Now, let D be an arbitrary bounded subset of E with $\beta(D) \neq 0$. It is clear that $\Gamma_x(D)$ is bounded. From Lemma 2.10, we get

$$\beta(\Gamma_z(D)) \leq \beta(A(D).z) + \beta(C(D)) \leq \|z\|\beta(A(D)) + \beta(C(D)) \leq \beta(A(D)) < \beta(D).$$

Hence Γ_z is $\beta-$condensing. Moreover, by assumption (iii), $\Gamma_z(E)$ is bounded. Then the multivalued mapping Γ_z satisfies all the assumptions in the statement of Theorem 2.5, it follows that there exists $x \in \Omega$ such that $x \in \Gamma_z(x) = Ax.z + Cx$. Thus $z \in \left(\frac{I-C}{A}\right)(x)$, and hence $\left(\frac{I-C}{A}\right)(x) \cap B(y) \neq \emptyset$. Hence, the multivalued operator $\left(\frac{I-C}{A}\right)^{-1}$ exists on $B(\Omega)$. □

Theorem 3.11. *Let E be a Banach algebra satisfying condition $(\mathcal{P})$ and Φ a $(MWNC)$ on E. Let Ω be a non-empty closed convex subset of E and $A, C : E \to P_{cv}(E)$ and $B : \Omega \to P(E)$ be three multivalued mappings satisfying the following conditions:*

(i)- A, B and C have weakly sequentially closed graphs,
(ii)- B and C are weakly compacts, and A is s.w.u.sco.,
(iii)- $A(E)$, $B(\Omega)$ and $C(E)$ are bounded,
(iv)- For all $y \in \Omega$, $\left(\frac{I-C}{A}\right)^{-1} B(y) \in P_{cv}(\Omega)$,
(v)- $\|B(\Omega)\| \leq 1$, and A is $\beta-$condensing.

Then, there exists $x \in \Omega$ with $x \in A(x)B(x) + C(x)$.

Proof. From Proposition 3.10, it follows that $\left(\frac{I-C}{A}\right)^{-1}$ exists on $B(\Omega)$. From assumption (iv) $F := \left(\frac{I-C}{A}\right)^{-1} B : \Omega \to P_{cv}(\Omega)$ is well defined. We show that F has a weakly sequentially closed graph. Let $x_n \rightharpoonup x \in \Omega$ and $y_n \in F(\Omega)$, $y_n \rightharpoonup y$ with $y_n \in F(x_n)$. Then $\left(\frac{I-C}{A}\right)(y_n) \cap B(x_n) \neq \emptyset$, so there exists $z_n \in B(x_n)$ such that $z_n \in \left(\frac{I-C}{A}\right)(y_n)$. Hence $y_n = u_n z_n + v_n$ where $v_n \in C(y_n)$ and $u_n \in A(y_n)$. Since A is s.w.u.sco. and has weakly sequentially closed graph, it follows that there exists a subsequence $\{u_{n_k}\}$ such that $u_{n_k} \rightharpoonup u \in A(y)$. Since B is a weakly compact and $\{x_n\}$ is bounded. By Eberlien-Šmulian's Theorem $\{z_n\}$ has a convergent subsequence $z_{n_k} \rightharpoonup z \in B(x)$. (Similarly, $v_{n_k} \rightharpoonup v \in C(y)$). Keep in your mined that E satisfies a condition $(\mathcal{P})$, hence, $y_{n_k} = u_{n_k} z_{n_k} + v_{n_k}$. By the uniqueness of weak limit we can get $y = uz + v \in A(y).z + C(y)$, that is $z \in \left(\frac{I-C}{A}\right) y$. Consequently, $\left(\frac{I-C}{A}\right)(y) \cap B(x) \neq \emptyset$. Hence $y \in \left(\frac{I-C}{A}\right)^{-1} B(x)$. Accordingly, F has a weakly sequentially closed graph. Next,

$$F(\Omega) \subset A(F(\Omega))B(\Omega) + C(F(\Omega))$$

and taking into account the assumption (ii). We conclude that $F(\Omega)$ is bounded. Now, we claim that $F := \left(\frac{I-C}{A}\right)^{-1} B$ is a weakly compact. To do this, let D be an arbitrary bounded subset of Ω with $\beta(D) > 0$. Using

$$F(D) \subseteq C(F(D)) + A(F(D)).B(D) \subseteq C(F(D)) + A(F(D))\overline{(B(D))^w}$$

Obviously, $\overline{(B(D))^w}$ is a weakly compact. For $x \in \overline{(B(D))^w}$, Eberlein-Šmulian's Theorem implies that there exists a sequence $\{x_n\} \subset B(D)$ with $x_n \rightharpoonup x$. We know that for all n, $\|x_n\| \leq 1$ and $\|x\| \leq \liminf \|x_n\|$, then $\|x\| \leq 1$, and hence $\|\overline{(B(D))^w}\| \leq 1$. By Lemma 2.10 we get,

$$\begin{aligned} \beta(F(D)) &\leq \beta(C(F(D))) + \beta(A(F(D)).\overline{(B(D))^w}) \\ &\leq \|\overline{(B(D))^w}\| \beta(A(F(D))) < \beta(F(D)). \end{aligned}$$

A contradiction. Hence, $\left(\frac{I-C}{A}\right)^{-1} B$ is a weakly compact. Therefore, Theorem 2.4 guarantees that F has a fixed point $x \in \Omega$. □

Proposition 3.12. *Let E be a Banach algebra satisfying condition $(\mathcal{P})$ and Ω be a non-empty closed convex subset of E. Assume Φ is a positive homogeneous semi-additive (MWNC) on E. Let $A, C : E \to P_{cv}(E)$ be two multivalued mappings satisfying the following conditions:*

- (i)- A and C have weakly sequentially closed graph,
- (ii)- A is weakly compact,
- (iii)- $A(E)$ and $C(E)$ are bounded,
- (iv)- C is $\beta-$non-expansive and hemi-weakly compact,

Then, the multivalued operator $\left(\frac{I-C}{A}\right)^{-1}$ exists in E.

Proof. Fix z in E. Consider for each $m \in \mathbb{N}$,

$$\Gamma_{m,z} : E \to P_{cv}(E), \;\; x \longmapsto Ax.z + \lambda_m Cx,$$

where $\{\lambda_m\} \subseteq (0,1)$ such that $\lambda_m \to 1$. Let m be fixed, and since $Ax.z+\lambda_m Cx$ is convex, it is clear that $\Gamma_{m,z}$ is well defined. We prove that the multivalued mapping $\Gamma_{m,z}$ satisfies all the assumptions in the statement of Theorem 2.5. Let $x_n \rightharpoonup x$ and $y_n \in Ax_n.z+\lambda_m Cx_n$ such that $y_n \rightharpoonup y$. There exists $u_n \in Ax_n$ and $v_n \in Cx_n$ such that $y_n = u_n.z + \lambda_m v_n$. Since A is a weakly compact with weakly sequentially closed graph and $\{x_n\}$ is bounded. By Eberlien-Šmulian's Theorem there exists a subsequence $\{u_{n_k}\}$ such that $u_{n_k} \rightharpoonup u \in Ax$. It well known that the right hand multiplication operator $R_z(x) = x.z$ is a continuous linear operator, consequently, it is weakly continuous. Then we can get

$$\lambda_m v_{n_k} = y_{n_k} - u_{n_k}.z \rightharpoonup y - u.z.$$

From hypotheses C has a weakly sequentially closed graph, then $y - u.z \in \lambda_m Cx$. Hence $y \in Ax.z + \lambda_m Cx = \Gamma_{m,z}(x)$. Hence $\Gamma_{m,z}$ has a weakly sequentially closed graph. Next, we claim that $\Gamma_{m,z}$ is $\beta-$condensing. Let D be an arbitrary bounded subset of E with $\beta(D) \neq 0$. From assumption (iii) $\Gamma_{m,z}(E)$ is bounded. Thus $\Gamma_{m,z}(D)$ is bounded. By Lemma 2.10 and C is $\beta-$non-expansive we can get

$$\beta(\Gamma_{m,z}(D)) \leq \beta(A(D).z) + \beta(\lambda_m C(D)) \leq \|z\|\beta(A(D)) + \lambda_m \beta(C(D)) < \beta(D).$$

Hence $\Gamma_{m,z}$ is $\beta-$ condensing. Then the multivalued mapping $\Gamma_{m,z}$ satisfies all the assumptions in the statement of Theorem 2.5. Then there exists $x_m \in E$ such that $x_m \in \Gamma_{m,z}(x_m) = Ax_m.z + \lambda_m Cx_m$. Also, we can find $u_m \in A(x_m)$ and $v_m \in C(x_m)$ such that

$$x_m = u_m.z + \lambda_m v_m.$$

Obviously $\{x_m\}$ is bounded. We can suppose there exists a subsequence $\{u_{m_k}\}$ with $u_{m_k} \rightharpoonup u$. Since $\{v_m\}$ is bounded and $\lambda_m \to 1$ we can get

$$x_{m_k} - v_{m_k} = u_{m_k}.z + (\lambda_{m_k} - 1)v_{m_k} \rightharpoonup u.z.$$

By assumption (iv) C is hemi-weakly compact, this implies $\{x_{m_k}\}$ has a weakly convergent subsequence, say $\{x_{m_{k_j}}\}$. Hence $v_{m_{k_j}} \rightharpoonup x - u.z$. Also $x - u.z \in C(x)$, that is $x \in A(x).z + C(x)$. Consequently, $z \in \left(\frac{I-C}{A}\right)(x)$. Therefore, $\left(\frac{I-C}{A}\right)^{-1}$ exists in E. □

Theorem 3.13. *Let E be a Banach algebra satisfying condition $(\mathcal{P})$ and Ω be a non-empty closed convex subset of E. Assume Φ is a positive homogeneous semi-additive (MWNC) on E. Let $A, C : E \to P_{cv}(E)$ and $B : \Omega \to P(E)$ be three multivalued mappings satisfying the following conditions:*

(i)- A, B and C have weakly sequentially closed graph,
(ii)- A and B are weakly compacts,
(iii)- $A(E)$, $B(\Omega)$ and $C(E)$ are bounded,
(iv)- For all $y \in \Omega$, $\left(\frac{I-C}{A}\right)^{-1} B(y) \in P_{cv}(\Omega)$,
(v)- C is $\beta-$non-expansive and hemi-weakly compact.

Then, there exists $x \in \Omega$ with $x \in A(x)B(x) + C(x)$.

Proof. This is an immediate consequence of Theorem 3.4 and Proposition 3.12. □

4. Non-linear Leray-Schauder alternatives

Depending on the results of Propositions 3.5, 3.10 and 3.12. We prove some new versions of Leray-Schauder type fixed point theorem for the sum and the product of three multivalued mappings.

Theorem 4.1. *Let Ω be a non-empty closed convex subset of a Banach algebra E satisfying condition $(\mathcal{P})$ and U be a weakly open subset of Ω with $\theta \in U$. Assume $A, C : E \to P(E)$ and $B : \overline{U^w} \to P(E)$ be three multivalued mappings satisfying the following conditions:*

(i)- A, B and C have weakly sequentially closed graphs,
(ii)- A and B are weakly compacts, and C is hemi-weakly compact,
(iii)- $\left(\frac{I-C}{A}\right)^{-1}$ exists on $B(\overline{U^w})$,
(iv)- For each $x \in \overline{U^w}$, $\left(\frac{I-C}{A}\right)^{-1} B(x)$ is convex,
(v)- $x \in A(x)B(y) + C(x)$, $y \in \overline{U^w} \Rightarrow x \in \Omega$,
(vi)- $\left(\frac{I-C}{A}\right)^{-1} B(\overline{U^w})$ is bounded.

Then, either

(A_1) the equation $x \in \lambda A(\frac{x}{\lambda})B(x) + \lambda C(\frac{x}{\lambda})$, has a solution for $\lambda = 1$, or
(A_2) there is a point $u \in \partial_\Omega U$ and $\lambda \in (0,1)$ with $u \in \lambda A(\frac{u}{\lambda})Bu + \lambda C(\frac{u}{\lambda})$.

Proof. From the assumption (iii), the multivalued mapping $\left(\frac{I-C}{A}\right)^{-1}$ exists on $B(\overline{U^w})$. First we show that $F := \left(\frac{I-C}{A}\right)^{-1} B : \overline{U^w} \to P_{cv}(\Omega)$ is well defined.

Let $x \in \left(\frac{I-C}{A}\right)^{-1} B(y)$ for $y \in \overline{U^w}$. Then $x \in A(x)B(y)+C(x)$. By assumption (v) we have $x \in \Omega$. Hence $\left(\frac{I-C}{A}\right)^{-1} B(y) \subseteq \Omega$ for $y \in \overline{U^w}$. This together with (iv) guarantees that $F : \overline{U^w} \to P_{cv}(\Omega)$. Next, we show that F has a weakly sequentially closed graph. Let $x_n \rightharpoonup x \in \overline{U^w}$ and $y_n \in F(\overline{U^w})$, $y_n \rightharpoonup y$ with $y_n \in F(x_n)$. Then $\left(\frac{I-C}{A}\right)(y_n) \cap B(x_n) \neq \emptyset$, so we can find $z_n \in B(x_n)$ such that $z_n \in \left(\frac{I-C}{A}\right)(y_n)$. Chose $v_n \in C(y_n)$ and $u_n \in A(y_n)$ such that $u_n z_n = y_n - v_n$. Since B is a weakly compact and $\{x_n\}$ is bounded. The Eberlien-Šmulians Theorem implies that $\{z_n\}$ has a subsequence $\{z_{n_k}\}$ which weakly converges to some $z \in B(x)$. Since A is a weakly compact and $\{y_n\}$ is bounded, $\{u_n\}$ has a subsequence $\{u_{n_k}\}$ which weakly converges to some $u \in A(y)$. Also, C has a weakly sequentially closed graph and E satisfies the condition $(\mathcal{P})$. It follows that

$$v_{n_k} = y_{n_k} - z_{n_k} u_{n_k} \rightharpoonup y - zu \in C(y),$$

that is, $z \in \left(\frac{I-C}{A}\right)(y)$. Then $\left(\frac{I-C}{A}\right)(y) \cap B(x) \neq \emptyset$ and consequently, $y \in \left(\frac{I-C}{A}\right)^{-1} B(x)$. Hence F has a weakly sequentially closed graph. Finally, we claim that $F(\overline{U^w})$ is a relatively weakly compact. Let $y_n \in F(\overline{U^w})$. Choose $\{x_n\} \subset \overline{U^w}$ such that $y_n \in F(x_n)$, that is $y_n \in A(y_n)B(x_n) + C(y_n)$. Thus there exists $z_n \in B(x_n)$, $u_n \in A(y_n)$ and $v_n \in C(y_n)$ such that $y_n = u_n z_n + v_n$. Since A and B are weakly compact and E satisfies a condition $(\mathcal{P})$. It follows that $y_{n_k} - v_{n_k} = u_{n_k} z_{n_k} \rightharpoonup uz$. Since C is hemi-weakly compact, then $\{y_{n_k}\}$ has a weakly convergent subsequence, say $\{y_{n_{k_j}}\}$. Hence $F(\overline{U^w})$ is relatively weakly compact. From Theorem 2.6, either F has a fixed point or there exists $u \in \partial_\Omega U$ and $\lambda \in (0,1)$ with $u \in \lambda F(u)$. The proof is complete. □

Theorem 4.2. *Let Ω be a non-empty closed convex subset of a Banach algebra E satisfying condition $(\mathcal{P})$ and Φ a $(MWNC)$ on E. Let U be a weakly open subset of Ω with $\theta \in U$. Assume $A, C : E \to P_{cv}(E)$ and $B : \overline{U^w} \to P(E)$ are three multivalued mappings satisfying the following conditions:*

- (i)- A, B and C have weakly sequentially closed graphs,
- (ii)- A and C are weakly compacts, and B is s.w.u.sco.,
- (iii)- For each $x \in \overline{U^w}$, $\left(\frac{I-C}{A}\right)^{-1} B(x)$ is convex,
- (iv)- $x \in A(x)B(y) + C(x)$, $y \in \overline{U^w} \Rightarrow x \in \Omega$,
- (v)- $A(E)$, $C(E)$ and $B(\overline{U^w})$ are bounded,
- (vi)- $\left(\frac{I-C}{A}\right)^{-1} B(\overline{U^w})$ is relatively weakly compact.

Then, either

- (A_1) the equation $x \in \lambda A(\frac{x}{\lambda})B(x) + \lambda C(\frac{x}{\lambda})$, has a solution for $\lambda = 1$, or
- (A_2) there is a point $u \in \partial_\Omega U$ and $\lambda \in (0,1)$ with $u \in \lambda A(\frac{u}{\lambda})Bu + \lambda C(\frac{u}{\lambda})$.

Proof. From Proposition 3.5, it follows that the multivalued mapping $\left(\frac{I-C}{A}\right)^{-1}$ exists on $B(\overline{U^w})$. By assumption (iii) and (iv), $F := \left(\frac{I-C}{A}\right)^{-1} B : \overline{U^w} \to P_{cv}(\Omega)$

is well defined. In view of Theorem 3.6, it easy to see that F has weakly sequentially closed graph. Using,

$$F(\overline{U^w}) \subseteq A(F(\overline{U^w}))B(\overline{U^w}) + C(F(\overline{U^w})).$$

Hence assumption (v) guarantees that $F(\overline{U^w})$ is bounded. Applying Theorem 2.6. The proof is complete. □

Theorem 4.3. *Let* Ω *be a non-empty closed convex subset of a Banach algebra* E *satisfying condition* $(\mathcal{P})$ *and* Φ *a* $(MWNC)$ *on* E*. Let* U *be a weakly open subset of* Ω *with* $\theta \in U$*. Assume* $A, C : E \to P_{cv}(E)$ *and* $B : \overline{U^w} \to P(E)$ *are three multivalued mappings satisfying the following conditions:*

(i)- A, B and C have weakly sequentially closed graphs,
(ii)- A and B are weakly compacts,
(iii)- C is $\Phi-$ condensing,
(iv)- For each $x \in \overline{U^w}$, $\left(\frac{I-C}{A}\right)^{-1} B(x)$ is convex,
(v)- $x \in A(x)B(y) + C(x)$, $y \in \overline{U^w} \Rightarrow x \in \Omega$,
(vi)- $A(E)$, $C(E)$ and $B(\overline{U^w})$ are bounded.

Then, either

(A_1) the equation $x \in \lambda A(\frac{x}{\lambda})B(x) + \lambda C(\frac{x}{\lambda})$, has a solution for $\lambda = 1$, or
(A_2) there is a point $u \in \partial_\Omega U$ and $\lambda \in (0,1)$ with $u \in \lambda A(\frac{u}{\lambda})Bu + \lambda C(\frac{u}{\lambda})$.

Proof. From Proposition 3.5, and as the same argument of the proof of the Theorem 4.1, $F := \left(\frac{I-C}{A}\right)^{-1} B : \overline{U^w} \to P_{cv}(\Omega)$ is well defined and has a weakly sequentially closed graph. Now we show that $F(\overline{U^w})$ is a relatively weakly compact. By assumptions (vi) we see that $F(\overline{U^w})$ is bounded. Let $\{y_n\} \subset F(\overline{U^w})$ and choose $\{x_n\} \subset \overline{U^w}$ such that $y_n \in F(x_n)$. Accordingly, $y_n \in A(y_n)B(x_n)+C(y_n)$, and hence $(I-C)y_n \in Ay_nBx_n$. Taking into account assumption (ii) and the condition $(\mathcal{P})$. We can get a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $(I-C)y_{n_k} \rightharpoonup z$ in E. Let $D = \{y_{n_k}\}$ and put $D \subseteq (I-C)D + CD \subseteq \overline{(I-C)D^w} + CD$, hence

$$\Phi(D) \leq \Phi(\overline{((I-C)D)^w}) + \Phi(C(D)).$$

Obviously, $\overline{((I-C)D)^w}$ is a weakly compact. Since C is Φ-condensing. then we get $D = \{y_{n_k}\}$ is a relatively weakly compact. Accordingly, $\{y_{n_k}\}$ has a subsequence $\{y_{n_{k_j}}\}$ which converges weakly to some y_0 in Ω. Hence $F(\overline{U^w})$ is relatively weakly compact. From Theorem 2.6, either F has a fixed point or there exists $u \in \partial_\Omega U$ and $\lambda \in (0,1)$ with $u \in \lambda F(u)$. □

Theorem 4.4. *Let* Ω *be a non-empty closed convex subset of a Banach algebra* E *satisfying condition* $(\mathcal{P})$ *and* Φ *a* $(MWNC)$ *on* E*. Let* U *be a weakly open subset of* Ω *with* $\theta \in U$*. Assume* $A, C : E \to P_{cv}(E)$ *and* $B : \overline{U^w} \to P(E)$ *are three multivalued mappings with weakly sequentially closed graph and satisfying the following conditions:*

(i)- A, B and C have weakly sequentially closed graphs,
(ii)- B and C are weakly compacts,
(iii)- $\|B(\overline{U^w})\| \leq 1$, and A is $\Phi-$ condensing,
(iv)- For each $x \in \overline{U^w}$, $\left(\frac{I-C}{A}\right)^{-1} B(x)$ is convex,
(v)- $x \in A(x)B(y) + C(x)$, $y \in \overline{U^w} \Rightarrow x \in \Omega$,
(vi)- $A(E)$, $C(E)$ and $B(\overline{U^w})$ are bounded.

Then, either

(A_1) the equation $x \in \lambda A(\frac{x}{\lambda})B(x) + \lambda C(\frac{x}{\lambda})$, has a solution for $\lambda = 1$, or
(A_2) there is a point $u \in \partial_\Omega U$ and $\lambda \in (0,1)$ with $u \in \lambda A(\frac{u}{\lambda})Bu + \lambda C(\frac{u}{\lambda})$.

Proof. From Proposition 3.10 the multivalued operator $\left(\frac{I-C}{A}\right)^{-1}$ exists on $B(\overline{U^w})$. Using the same argument of the proof of the Theorem 3.11 we can get $\left(\frac{I-C}{A}\right)^{-1} B(\overline{U^w})$ is a relatively weakly compact. Then applying Theorem 2.6. The proof is complete. ☐

Theorem 4.5. *Let Ω be a non-empty closed convex subset of a Banach algebra E satisfying condition $(\mathcal{P})$ and Φ a MWNC on E. Let U be a weakly open subset of Ω with $\theta \in U$. Assume $A, C : E \to P_{cv}(E)$ and $B : \overline{U^w} \to P(E)$ are three multivalued mappings with weakly sequentially closed graph and satisfying the following conditions:*

(i)- A, B and C have weakly sequentially closed graphs,
(ii)- A and B are weakly compacts,
(iii)- C is $\Phi-$non-expansive and hemi-weakly compact,
(iv)- For each $x \in \overline{U^w}$, $\left(\frac{I-C}{A}\right)^{-1} B(x)$ is convex,
(v)- $x \in A(x)B(y) + C(x)$, $y \in \overline{U^w} \Rightarrow x \in \Omega$,
(vi)- $A(E)$ $C(E)$ and $B(\overline{U^w})$ are bounded.

Then, either

(A_1) the equation $x \in \lambda A(\frac{x}{\lambda})B(x) + \lambda C(\frac{x}{\lambda})$, has a solution for $\lambda = 1$, or
(A_2) there is a point $u \in \partial_\Omega U$ and $\lambda \in (0,1)$ with $u \in \lambda A(\frac{u}{\lambda})Bu + \lambda C(\frac{u}{\lambda})$.

Proof. This is an immediate consequence of Proposition 3.12, Theorem 3.4 and Theorem 2.6. ☐

5. FUNCTIONAL INTEGRAL INCLUSION OF FRACTIONAL ORDER

In this section, we prove an existence theorem for the quadratic integral inclusion 1.2. Let $J = [0,T]$ be a closed and bounded interval subset of $\mathbb{R}$, $L^1(J,X)$ the space of all integrable X-valued functions with norm

$$\|x\|_{L^1} = \int_0^T |x(t)|dt.$$

$E := C(J, X)$ denote the Banach algebra of all continuous X-valued functions defined on J endowed with the norm

$$\|x\| = \sup_{t\in J} |x(t)|.$$

Throughout this section X will be a finite dimensional Banach algebra which satisfying a sequential condition $(\mathcal{P})$. Assume that (X, Σ, μ) is a measurable space The multivalued map $F : X \to P(Y)$ (where Y is a separable metric space) with closed values is called measurable if $F^{-1}(V) \in \Sigma$ for each open subset $V \subseteq Y$, and is called weakly measurable if $F^{-1}(U) \in \Sigma$ for each closed subset $U \subseteq Y$. Furthermore, F is weakly measurable if and only if the distance functions $f_y : X \to \mathbb{R}$, $f_y = dist(y, F(x)) = \inf\{|y - z| : z \in F(x)\}$ is measurable for all $y \in Y$.

Definition 5.1. A multivalued map $F : J\times X \to P(X)$ is called L^1_X-carathéodory if

- (i)- For each $x \in X$, $F_x = F(., x)$ is weakly measurable,
- (ii)- For each $t \in J$, $F_t = F(t, .)$ is weakly upper semi-continuous,
- (iii)- There exists a function $\alpha \in L^1(J, X)$ such that $\|F(t, x)\| \le \alpha(t)$, a.e $t \in J$ and for all $x \in X$, where α is the growth function of F on $J \times X$.

For a function x defined on J we define the set $S_F(x) = \{u \in L^1(J, X) : u(t) \in F(t, x(t))$ for a.e $t \in J\}$ which is known as the set of selection functions. Also, denote that $\|F(t, x(t))\| = \sup\{|u(t)| : u(t) \in F(t, x(t))\}$.

Lemma 5.2 ([16])**.** *Let X be a Banach space, if* $\dim(X) < \infty$ *and* $F : J\times X \to P_{cl,bd}(X)$ *is* L^1_X*-carathéodory , then* $S_F(x) \neq \phi$ *for all* $x \in X$.

Lemma 5.3 ([12])**.** *Let $F : X \to P(Y)$, where X, Y be any topological vector space and D be a subset of X then the graph of F is convex if and only if the set D is convex and $\lambda F(x) + (1 - \lambda)F(y) \subseteq F(\lambda x + (1 - \lambda)y)$ for all $x, y \in D$ and $\lambda \in (0, 1)$.*

Now to discus the functional integral inclusion 1.2 we list the following hypotheses,

- (H_1) A multivalued mapping $G : J \times X \to P_{cl,bd,cv}(X)$ is L^1_X-carathéodory with a growth function $\eta \in L^1(J, X)$ such that $\|G(t, x(t))\| \le \eta(t)$ a.e $t \in J$ for all $x \in E$.
- (H_2) For each $t \in J$, $K_t = K(t, .) : X \longrightarrow P_{cl,bd,cv}(X)$ is L^1_X-carathéodory, and there exist a bounded function $\beta \in L^1(J, X)$ with bound $\|\beta\|$ such that for all $x, y \in E$,

$$\|K(t, x(t)) - K(t, y(t))\| \le \beta(t)|x(t) - y(t)| \quad a.e. \quad t \in J.$$

- (H_3) For each $t \in J$, $F_t = F(t, .) : X \longrightarrow P_{cl,bd,cv}(X)$ is L^1_X-carathéodory, and there exist a bounded function $\xi \in L^1(J, X)$ with bound $\|\xi\|$ such that for all $x, y \in E$,

$$\|F(t,x(t)) - F(t,y(t))\| \leq \xi(t)|x(t)-y(t)| \quad a.e. \quad t \in J.$$

(H_4) For each $t \in J$, $F_t = F(t,.)$ and $K_t = K(t,.)$ have convex graph.

(H_5) For each $x \in X$, $K_x = K(.,x)$ and $F_x = F(.,x)$ are continuous.

Consider $Q = \{u \in E : \|u\| \leq r\}$. Clearly Q is a non-empty, closed, bounded and convex set. Let us consider the operators A, B and C defined on Q by

$$Ax(t) = K(t,x(t)),$$
$$Cx(t) = F(t,x(t)),$$
$$Bx(t) = I^{\alpha}G(t,x(t)) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} G(s,x(s))ds.$$

Accordingly, the inclusion 1.2 is equivalent to the operator inclusion

$$x(t) \in Ax(t)Bx(t) + Cx(t).$$

By Lemma 5.2 it is clear that $S_K(x)$, $S_F(x)$, and $S_G(x)$ are non-empty sets then A, B and C are well defined.

Theorem 5.4. *Assume that the hypotheses* $(H_1) - (H_5)$ *holds. Suppose that there is a real number* $r > 0$ *such that*

$$r = \frac{F_1\Gamma(\alpha+1) + K_1 T^{\alpha}\|\eta\|_{L1}}{(1-\|\xi\|)\Gamma(\alpha+1) - \|\beta\|T^{\alpha}\|\eta\|_{L1}},$$

where $F_1 = \sup_{t\in J}|F(t,0)|$, $K_1 = \sup_{t\in J}|K(t,0)|$, *and* $\frac{\|\beta\|T^{\alpha}\|\eta\|_{L_1}}{1-\|\xi\|} < \Gamma(\alpha+1)$. *Then the inclusion 1.2 has a solution on* J.

Proof. Since X is a Banach algebra with a sequential condition $(\mathcal{P})$, then $E = C(J,X)$ is also Banach algebra satisfying the condition $(\mathcal{P})$ [4]. Now, we claim that the mappings A, B and C satisfy all the assumptions of Corollary 3.9. To this end, we split the proof into sequence of steps.

Step1: A has weakly sequentially closed graph and $A(Q)$ is relatively weakly compact. First, we show that A has weakly sequentially closed graph. Let $\{x_n\} \subset Q$ and $x_n \rightharpoonup x \in Q$, $y_n \in A(x_n)$ for all $n \in \mathbb{N}$, $y_n \rightharpoonup y$. Then we get $y_n(t) \rightharpoonup y(t)$ (similarly, $x_n(t) \rightharpoonup x(t)$) [18]. Now for $y_n \in A(x_n)$ there exists $v_n \in S_K(x_n)$ such that

$$y_n(t) = v_n(t).$$

Fix $t \in J$. Without loss of generality we may assume that $y_n(t) \neq 0$. In view of Hahn-Banach theorem there exists $f \in X^*$ such that $f(y_n(t)) = |y_n(t)|$ and $|f|_* = 1$. Since $v_n(t) \in K(t,(x_n(t)))$ a.e. $t \in J$, and K with bounded values. By the reflexivity of X there exists a subsequence $\{v_{n_k}(t)\}$ converges weakly to some $v(t)$. i.e. $f(v_{n_k}(t)) \to f(v(t))$. Then $y(t) = v(t)$. Moreover, by hypotheses (H_2) $K(t,.)$ is weakly upper semi-continuous, so $K(t,.)$ has a weakly closed graph [19]. Then $v_{n_k}(t) \in K(t, x_{n_k}(t))$ implies that $v(t) \in K(t,x(t))$. Hence $v \in S_K(x)$ and therefore $y \in A(x)$ consequently A has a weakly sequentially

closed graph.
Next, we show that $A(Q)$ is relatively weakly compact. Let $t \in J$ be fixed, and $\{u_n\}$ be a sequence in $A(Q)$. Thus there exists $x_n \in Q$ such that $u_n \in A(x_n)$, and hence there exists $v_n \in S_F(x_n)$ such that $u_n(t) = v_n(t)$. Thus

$$\begin{aligned} |u_n(t)| &= |v_n(t)| \leq \|K(t, x_n(t))\| \\ &\leq \|K(t, x_n(t)) - K(t, 0)\| + \|K(t, 0)\| \\ &\leq \|\beta\| r + K_1. \end{aligned}$$

Therefore $\{u_n(t)\}$ is weakly equi-bounded. For all $t \in J$, the reflexivity of X implies that the set $\{u_n(t) : n \in \mathbb{N}\}$ is relatively weakly sequentially compact [22]. Now we show that $A(Q)$ is weakly equi-continuous. Let $t_1, t_2 \in J$ and assume that $u_n(t_1) \neq u_n(t_2)$. Then there exists $f \in X^*$ such that
$f(u_n(t_1) - u_n(t_2)) = |u_n(t_1) - u_n(t_2)|$ and $|f|_* = 1$. Thus

$$\begin{aligned} |u_n(t_1) - u_n(t_2)| &= |v_n(t_1) - v_n(t_2)| \leq |K(t_1, x_n(t_1)) - K(t_2, x_n(t_2))| \\ &\leq |K(t_1, x_n(t_1)) - K(t_2, x_n(t_1))| \\ &+ |K(t_2, x_n(t_1)) - K(t_2, x_n(t_2))| \\ &\leq |K(t_1, x_n(t_1)) - K(t_2, x_n(t_1))| \\ &+ \beta(t)|x_n(t_1) - x_n(t_2)|. \end{aligned}$$

Since $K_x = K(., x)$ is continuous and so as $t_1 \to t_2$, we get $|u_n(t_1) - u_n(t_2)| \to 0$. Hence $A(Q)$ is weakly equi-continuous. By Arzela Ascoli theorem, $u_{n_j} \rightharpoonup u \in A(Q)$, and hence by Eberlien-Šmulian theorem we conclude that $A(Q)$ is relatively weakly compact.
Step2: As an argument similar to that in Step 1, $C(Q)$ is relatively weakly compact and C has a weakly sequentially closed graph.
Step3: B has weakly sequentially closed graph and $B(Q)$ is relatively weakly compact. First, we show that B has weakly sequentially closed graph. Let $\{x_n\} \subset Q$ and $x_n \rightharpoonup x \in Q$, $y_n \in B(x_n)$ for all $n \in \mathbb{N}$, $y_n \rightharpoonup y$. Then we get $y_n(t) \rightharpoonup y(t)$ (similarly, $x_n(t) \rightharpoonup x(t)$) [18]. Now for $y_n \in B(x_n)$ there exists $v_n \in S_G(x_n)$ such that

$$y_n(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v_n(s) ds.$$

Fix $t \in J$. Without loss of generality we may assume that $y_n(t) \neq 0$. In view of Hahn-Banach theorem there exists $f \in X^*$ such that $f(y_n(t)) = |y_n(t)|$ and $|f|_* = 1$. Since $v_n(t) \in G(t, (x_n(t)))$ a.e. $t \in J$, and G with bounded values. Then by the reflexivity of X there exists a subsequence $\{v_{n_k}(t)\}$ converges weakly to some $v(t)$. i.e. $f(v_{n_k}(t)) \to f(v(t))$. An application of Lebesgue dominated convergence theorem [21] yields

$$|y_{n_k}(t)| = f\left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v_{n_k}(s) ds\right) \to f\left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v(s) ds\right).$$

Then

$$y(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v(s) ds.$$

Moreover, by hypotheses (H_2), $K(t,.)$ is weakly upper semi-continuous, so $K(t,.)$ has a weakly closed graph [19]. Then $v_{n_k}(t) \in G(t, x_{n_k}(t))$ implies that $v(t) \in G(t, x(t))$. Hence $v \in S_G(x)$ and therefore $y \in B(x)$ consequently B has a weakly sequentially closed graph.

Next we show that $B(Q)$ is relatively weakly compact. Let $t \in J$ be fixed, and $\{u_n\}$ be a sequence in $B(Q)$. Thus there exists $x_n \in Q$ such that $u_n \in B(x_n)$, and hence there exists $v_n \in S_G(x_n)$ such that

$$u_n(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v_n(s) ds.$$

Thus

$$\begin{aligned} |u_n(t)| &= f\left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v_n(s) ds\right) \\ &\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |v_n(s)| ds \leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \|G(s, x_n(s))\| ds \\ &\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |\eta(s)| ds \leq \frac{T^\alpha \|\eta\|_{L_1}}{\Gamma(\alpha+1)}. \end{aligned}$$

Therefore $\{u_n(t)\}$ is weakly equi-bounded. For all $t \in J$, the reflexivity of X implies that the set $\{u_n(t) : n \in \mathbb{N}\}$ is relatively weakly sequentially compact ([22] P.782). Now we show that $B(Q)$ is weakly equi-continuous. Let $t_1, t_2 \in J$ and assume that $u_n(t_1) \neq u_n(t_2)$. Then there exists $f \in X^*$ such that $f(u_n(t_1) - u_n(t_2)) = |u_n(t_1) - u_n(t_2)|$ and $|f|_* = 1$. Thus
$|u_n(t_1) - u_n(t_2)|$

$$\begin{aligned} &= f\left(\int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} v_n(s) ds - \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} v_n(s) ds\right) \\ &\leq \left|\int_{t_2}^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} v_n(s) ds - \int_0^{t_2} \frac{(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}}{\Gamma(\alpha)} v_n(s) ds\right| \\ &\leq \int_{t_2}^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} \|G(s, x_n(s))\| ds \\ &+ \int_0^{t_2} \frac{(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}}{\Gamma(\alpha)} \|G(s, x_n(s))\| ds \\ &\leq \frac{\|\eta\|_{L_1}}{\Gamma(\alpha)} \left(\int_{t_2}^{t_1} (t_1-s)^{\alpha-1} ds + \int_0^{t_2} ((t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}) ds\right) \\ &\leq \frac{\|\eta\|_{L_1}}{\Gamma(\alpha+1)} (2(t_1-t_2)^\alpha + t_2^\alpha - t_1^\alpha). \end{aligned}$$

As $t_1 \to t_2$, we get $|u_n(t_1) - u_n(t_2)| \to 0$. Hence $B(Q)$ is weakly equi-continuous. By (Arzela Ascoli Theorem), $u_{n_j} \rightharpoonup u \in B(Q)$ and hence by

Eberlien-Šmulian's Theorem, we conclude that $B(Q)$ is relatively weakly compact.

Step4: If $y \in AyBx + Cy$, $x \in Q$ then $y \in Q$. Let $y \in AyBx + Cy$. Then there exists $k \in S_K(y)$, $f \in S_F(y)$ and $g \in S_G(x)$ such that

$$y(t) = k(t)I^\alpha g(t) + f(t).$$

Now

$$\begin{aligned}
|y(t)| &\leq |k(t)| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |g(s)| ds + |f(t)| \\
&\leq \|K(t, y(t))\| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \|G(s, x(s))\| ds + \|F(t, y(t))\| \\
&\leq (\|K(t, y(t)) - K(t, 0)\| + \|K(t, 0)\|) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |\eta(t)| ds \\
&+ \|F(t, y(t)) - F(t, 0)\| + \|F(t, 0)\| \\
&\leq (\|\beta\||y(t)| + K_1)(\frac{\|\eta\|_{L_1} T^\alpha}{\Gamma(\alpha+1)}) + \|\xi\||y(t)| + F_1,
\end{aligned}$$

and we can do this for all $t \in J$. Then

$$\|y\| \leq (\|\beta\|\|y\| + K_1)(\frac{\|\eta\|_{L_1} T^\alpha}{\Gamma(\alpha+1)}) + \|\xi\|\|y\| + F_1.$$

Hence,

$$\|y\| \leq \frac{F_1 \Gamma(\alpha+1) + K_1 T^\alpha \|\eta\|_{L1}}{(1 - \|\xi\|)\Gamma(\alpha+1) - \|\beta\| T^\alpha \|\eta\|_{L1}}.$$

Therefore $y \in Q$. Thus $\left(\frac{I-C}{A}\right)^{-1} B(Q) \subset Q$.

Step5: $\left(\frac{I-C}{A}\right)^{-1} B(x)$ is convex for each $x \in Q$. Let $u_1, u_2 \in \left(\frac{I-C}{A}\right)^{-1} B(x)$. Then $u_1 \in A(u_1)B(x) + C(u_1)$ and $u_2 \in A(u_2)B(x) + C(u_2)$. Thus there exists $k_1 \in S_K(u_1)$, $k_2 \in S_K(u_2)$, $f_1 \in S_F(u_1)$, $f_2 \in S_F(u_2)$ and $g \in S_G(x)$ such that for all $t \in J$

$$\begin{aligned}
u_1(t) &= k_1(t)I^\alpha g(t) + f_1(t), \\
u_2(t) &= k_2(t)I^\alpha g(t) + f_2(t).
\end{aligned}$$

For all $\lambda \in [0, 1]$,

$$\begin{aligned}
\lambda u_1(t) + (1-\lambda)u_2(t) &= \lambda k_1(t)I^\alpha g(t) \\
&+ \lambda f_1(t) + (1-\lambda)k_2(t)I^\alpha g(t) + (1-\lambda)f_2(t) \\
&= [\lambda k_1(t) + (1-\lambda)k_2(t)]I^\alpha g(t) \\
&+ [\lambda f_1(t) + (1-\lambda)f_2(t)] \\
&= k(t)I^\alpha g(t) + f(t).
\end{aligned}$$

Note that

$$k(t) = \lambda k_1(t) + (1-\lambda)k_2(t) \in \lambda K(t, u_1(t)) + (1-\lambda)K(t, u_2(t)),$$

$$f(t) = \lambda f_1(t) + (1-\lambda)f_2(t) \in \lambda F(t, u_1(t)) + (1-\lambda)F(t, u_2(t)).$$

From (H_4) $K(t,.)$ and $F(t,.)$ have convex graph and so

$$\lambda K(t,u_1(t))+(1-\lambda)K(t,u_2(t))\subseteq K(t,\lambda u_1(t)+(1-\lambda)u_2(t)),$$

$$\lambda F(t,u_1(t))+(1-\lambda)F(t,u_2(t))\subseteq F(t,\lambda u_1(t)+(1-\lambda)u_2(t)).$$

Obviously, $k=\lambda k_1+(1-\lambda)k_2\in L_1(J,X)$ and $f=\lambda f_1+(1-\lambda)f_2\in L_1(J,X)$. Then $k\in S_K(\lambda u_1+(1-\lambda)u_2)$ and $f\in S_F(\lambda u_1+(1-\lambda)u_2)$. Hence

$$\lambda u_1+(1-\lambda)u_2\in A(\lambda u_1+(1-\lambda)u_2)B(x)+C(\lambda u_1+(1-\lambda)u_2),$$

and therefore

$$\lambda u_1+(1-\lambda)u_2\in\left(\frac{I-C}{A}\right)^{-1}B(x).$$

□

ACKNOWLEDGMENTS

I wish to thank Prof. Dr. Maher Mnief, and Prof. Dr. Radu Precup for all the efforts and advice they offered to complete this work. I also wish to thank the Mathematical laboratory, the college of science at the university of Safax, Tunisia for providing the rich scientific environment and the best workplace leading to this work.

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