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The Probability When a Finite Commutative Ring Is (Weakly) Nil-Neat

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Abstract. We study the probability of (weak) nil-neatness of a finite commutative ring, which is a natural continuation of the probabilities of (weak) nil-cleanliness that are investigated in [4, 5].

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1. Introduction and conventions

Throughout the present paper, we will assume that all rings are finite and commutative with identity, denoted by 1. Our notation and terminology are standard as, for instance, for a ring R, the letter $U(R)$ denotes the group of all units of R, $\mathrm{Nil}(R)$ the nil-radical of R which, in this case, coincides with the Jacobson radical $J(R)$ of R, and $\mathrm{Id}(R)$ is the set of all idempotents of R.

Recall that a ring R, whose elements $x \in R$ satisfy the condition $x = n + e$ for some $n \in \mathrm{Nil}(R)$ and $e \in \mathrm{Id}(R)$, is named *nil-clean* (e.g., [6]). Later on, the so-called *weakly nil-clean* rings were introduced as a possible non-trivial

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generalization of these nil-clean rings. In fact, we say that a ring is weakly nil-clean if each of its elements can be written as the sum or the difference of a nilpotent and an idempotent (see [2]).

An important property of the nil-clean (or, respectively, the weakly nil-clean) notion is that any homomorphic image of is again nil-clean (or, respectively, weakly nil-clean). However, the converse implication of the above statement is not true in the general case. As a matter of fact, it was introduced in [7] the concept of a *nil-neat* ring as such a ring for which every its proper homomorphic image is nil-clean. Moreover, the more generalized concept of a *weakly nil-neat* ring was defined and studied in [3]. Indeed, a ring R is said to be weakly nil-neat, provided every proper homomorphic image of R is weakly nil-clean.

In the way to study the probability when a ring is (weakly) nil-clean (cf. [4, 5]), we find the challenging problem of exploring the probability when a ring is (weakly) nil-neat. So, the purpose of the current article is to investigate in detail this question.

2. Main results

At the beginning of this section, we foremost represent the probability when a ring R is nil-clean, weakly nil-clean, nil-neat or weakly nil-neat, respectively, with the symbols $\mathrm{pn}(R)$, $\mathrm{pwn}(R)$, $\mathrm{pnn}(R)$ or $\mathrm{pwnn}(R)$, respectively.

We next continue with the rephrasing of Theorems 2.13 and 2.17 from [7] in the following form, which shows that the two conditions of being nil-neat and nil-clean are themselves equivalent for an arbitrary ring that is not a domain or for an arbitrary local ring that is not a field. Specifically, the following is fulfilled:

Lemma 2.1. *Let R be a ring which is not a domain or let R be a local ring which is not a field. Then R is nil-neat if, and only if, R is nil-clean.*

We also need the following technical claims.

Lemma 2.2. [4, Proposition 2.2(4)] *For a local ring R the following equality holds:*

$$pn(R) = \frac{2|Nil(R)|}{|R|}.$$

Lemma 2.3. *Any finite ring R with exactly t distinct maximal ideals can be decomposed into the direct product of local rings repeating t-times.*

Proof. See [1, Theorem 8.7]. □

In the following assertion, we calculate the probability when a finite commutative ring is nil-neat.

Theorem 2.4. *For a finite commutative ring R the following is true:*

$$pnn(R) = \begin{cases} 1 & , \text{ if } R \text{ is a field} \\ pn(R) = \frac{2|Nil(R)|}{|R|} & , \text{ otherwise} \end{cases}$$

Proof. Firstly, assume that R is a finite non-local ring. Then R can be written as a direct product of the number of $t > 1$ local rings; say $R \cong \prod_{i=1}^{t} R_i$ by using Lemma 2.3. Hence R is not a domain, and so R is nil-neat if, and only if, R is nil-clean by referring to Lemma 2.1. Thus, in this case,

$$\text{pnn}(R) = \text{pn}(R) = \frac{2|\text{Nil}(R)|}{|R|} = 2\prod_{i=1}^{t} \frac{|m_i|}{|R_i|},$$

by according to Lemma 2.2.

Secondly, suppose (R, m) is a finite local ring which is not a field. Again by utilizing Lemmas 2.3 and 2.2, we will have that

$$\text{pnn}(R) = \text{pn}(R) = \frac{2|\text{Nil}(R)|}{|R|} = \frac{2|m|}{|R|}.$$

Thirdly, let R be a field. It follows directly from the corresponding definition that $\text{pnn}(R) = 1$, as required. □

Here, we examine the possibility when a finite commutative ring is weakly nil-neat. Before doing that, we need the next three technicalities which are our key tools for proving the chief result formulated below.

Lemma 2.5. [3, Lemma 2.5] *Let R be a ring. Then R is weakly nil-neat if, and only if, R/I is weakly nil-clean for every non-zero semi-prime ideal I.*

Lemma 2.6. [3, Corollary 2.7] *A finite ring R is weakly nil-neat if, and only if, for any non-zero semi-prime ideal I of R it must be that $R/M \cong \mathbb{Z}_3$ for at most one maximal ideal M containing I, while $R/N \cong \mathbb{Z}_2$ for all other maximal ideals N containing I.*

Lemma 2.7. [5, Theorem 1.11] *Let R be a ring and let $t \in \mathbb{N}$ be the number of maximal ideals of R. The following statements are valid:*

(1) *If $2 \nmid |R|$, then*

$$pwn(R) = \frac{3^t}{p_1^{r_1} p_2^{r_2} \cdots p_t^{r_t}},$$

where $R \cong \prod_{i=1}^{t} R_i$ and $p_i^{r_i} = \frac{|R_i|}{|Nil(R_i)|}$.

(2) *If $|R| = 2^s$, then*

$$pwn(R) = \frac{2^{k_1+k_2+\cdots+k_t}}{|R|},$$

where $R \cong \prod_{i=1}^{t} R_i$, $|R_i| = 2^{t_i}$ and $\text{char}(R_i) = 2^{k_i}$.

(3) *If* $|R_i| = 2^{t_i}$ *and* $\mathrm{char}(R_i) = 2^{k_i}$, *for* $1 \leq i \leq l$, *and* $|R_j| = p_j^{k_j}$ *and* $p_j^{r_j} = \frac{|R_j|}{|Nil(R_j)|}$, *for* $p_j \neq 2$ *and* $l+1 \leq j \leq t$, *where* $R \cong \prod_{i=1}^{t} R_i$, *then*

$$pwn(R) = \frac{1}{(2^{t_1-k_1})(2^{t_2-k_2})\cdots(2^{t_l-k_l})} \times \frac{3^{t-l+1}}{p_{l+1}^{r_{l+1}} p_{l+2}^{r_{l+2}} \cdots p_t^{r_t}}.$$

We are now have all the ingredients necessary to establish our main result mentioned above, which motivates the writing of this article.

Theorem 2.8. *For a finite commutative ring* R*, the equality* $pwnn(R) = pnn(R)$ *holds.*

Proof. Firstly, suppose that R is a field. It follows obviously from the corresponding definition that R is weakly nil-neat if, and only if, R is weakly nil-clean, and also these two implications are equivalent to R being isomorphic to $\mathbb{Z}_2$ or to $\mathbb{Z}_3$ owing to Theorem 2.8 in [3]. Therefore, in this case, we conclude with the aid of Theorem 2.8 in [5] that

$$\mathrm{pwnn}(R) = \mathrm{pwn}(R) = \begin{cases} \frac{2}{|R|} & \text{, if } |R| = 2^s \text{ for some positive integer } s \\ \frac{3}{|R|} & \text{, if } 2 \nmid |R| \end{cases}$$

Secondly, assume that R is a ring which is not a field. Let $\mathrm{Spec}(R) = \{p_1, \cdots, p_k\}$ be the set of all prime (and so maximal) ideals of R and also let I_j be an arbitrary intersection of prime ideals of number j for some $1 \leq j \leq k$. Since R is not a field, all of the prime ideals p_i are non-zero and hence the semi-prime ideals of R are of the form I_j as, of course, they are non-zero too. Considering the rings $S_j = \frac{R}{I_j}$, we can say that R is weakly nil-neat if, and only if, S_j is weakly nil-clean for all $1 \leq j \leq k$ and also it follows at once from Lemma 2.6 that these statements are equivalent to $S_j/M \cong \mathbb{Z}_3$ for at most one maximal ideal M containing I_j, while $S_j/N \cong \mathbb{Z}_2$ for all other maximal ideals N containing I_j. One can be easily seen that $\mathrm{Spec}(S_j) = \{\frac{p_{i_1}}{I_j}, \cdots, \frac{p_{i_j}}{I_j}\}$, where $I_j = p_{i_1} \cap p_{i_2} \cap \cdots \cap p_{i_j}$. Consequently, R is weakly nil-neat if, and only if, $R/p_s \cong \mathbb{Z}_3$ for at most one maximal ideal p_s, while $R/p_h \cong \mathbb{Z}_2$ for all other maximal ideals p_h, that is equivalent to R is a weakly nil-clean by virtue of [2, Theorem 1.17]. So, in this case, we have again that $\mathrm{pwnn}(R) = \mathrm{pwn}(R)$, as stated. □

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