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Finite Groups with Given σ-conditionally Permutable Subgroups

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Abstract. Let $\sigma = \{\sigma_i | i \in I\}$ be a partition of the set of all primes $\mathbb{P}$ and G a finite group. A set $\mathcal{H}$ of subgroups of G is said to be a *complete Hall σ-set* of G if every member $\neq 1$ of $\mathcal{H}$ is a Hall σ_i-subgroup of G for some $i \in I$ and $\mathcal{H}$ contains exactly one Hall σ_i-subgroup of G for every i such that $\sigma_i \cap \pi(G) \neq \emptyset$. In this paper, we study the structure of G based on the notion of *σ-conditionally permutable* subgroups.

Keywords: Finite groups, σ-conditionally permutable subgroups, Supersoluble groups, σ-nilpotent subgroups, Complete Hall σ-set.

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1. Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. Let n be an integer. Then $\pi(n)$ denotes the set of all primes dividing n; as usual, $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of G.

In what follows, $\sigma = \{\sigma_i | i \in I\}$ is some partition of all primes $\mathbb{P}$, that is, $\mathbb{P} = \bigcup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$. We write $\sigma(G) = \{\sigma_i | \sigma_i \cap \pi(G) \neq \emptyset\}$.

Following [1, 2, 3, 4], G is said to be *σ-primary* if $|\sigma(G)| \leq 1$; *σ-soluble* if every chief factor of G is σ-primary. A set $\mathcal{H}$ of subgroups of G is said to be a *complete Hall σ-set* of G if every non-identity member of $\mathcal{H}$ is a Hall σ_i-subgroup of G for some σ_i, and $\mathcal{H}$ contains exactly one Hall σ_i-subgroup

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for every $\sigma_i \in \sigma(G)$. G is said to be σ-*full* if G possesses a complete Hall σ-set; σ-*nilpotent* if G has a complete Hall σ-set $\mathcal{H} = \{H_1, \cdots, H_t\}$ such that $G = H_1 \times \cdots \times H_t$. Clearly, a σ-nilpotent group is σ-soluble. G is said to be a σ-*full group of Sylow type* if every subgroup of G is a D_{σ_i}-group for all $\sigma_i \in \sigma(G)$. A complete Hall σ-set $\mathcal{H} = \{H_1, \cdots, H_t\}$ of subgroups of G is said to be a σ-*basis* of G if $H_iH_j = H_jH_i$ for all i, j. A subgroup H of G is said to be σ-*subnormal* in G if there exists a subgroup chain $H = H_0 \leq H_1 \leq \cdots \leq H_n = G$ such that either H_{i-1} is normal in H_i or $H_i/(H_{i-1})_{H_i}$ is σ-primary for all $i = 1, 2, \cdots, n$.

The permutability of subgroups possess a series of interesting properties. For example, a subgroup H of G is said to be: *permute* with a subgroup K if $HK = KH$; *permutable* in G if $HK = KH$ for any subgroup K of G; *conditionally permutable* in G if for any subgroup K of G there exists an element $x \in G$ such that $HK^x = K^xH$ (see [5]); *s-permutable* in G if $HP = PH$ for any Sylow subgroup P of G (see [6]). A set $\mathcal{S}$ of Sylow subgroups of G is said to be a *complete set of Sylow subgroups* of G if $\mathcal{S}$ contains exactly one Sylow p-subgroup of G for every prime $p \in \pi(G)$. Let $\mathfrak{Z}$ be a complete set of Sylow subgroups of G. A subgroup H of G is said to be: $\mathfrak{Z}$-*permutable* if $HA = AH$ for all $A \in \mathfrak{Z}$ (see [7]); *s-conditionally permutable* in G if for any Sylow subgroup P of G there exists an element $x \in G$ such that $HP^x = P^xH$ (see [8]). A subgroup H of G is said to be: σ-*permutable* in G if G possesses a complete Hall σ-set $\mathcal{H}$ such that $HA^x = A^xH$ for all $A \in \mathcal{H}$ and all $x \in G$ (see [3]); $\mathcal{H}$-*permutable* if $HA = AH$ for all $A \in \mathcal{H}$, where $\mathcal{H}$ is a complete Hall σ-set of G (see [2]). Many people have studied the structure of finite groups based on above subgroups and a lot of research has been given; see for example [2, 3, 5, 6, 7, 8, 9]. Recently, Mao et al. introduced the following concept (see [10]):

Definition 1.1. A subgroup H of G is said to be σ-*conditionally permutable* in G if G possesses a complete Hall σ-set $\mathcal{H}$ and, for any subgroup $A \in \mathcal{H}$, there exists an element $x \in G$ such that $HA^x = A^xH$.

It is clear that every normal subgroup, every permutable subgroup, every conditionally permutable subgroup, every σ-permutable subgroup and every $\mathcal{H}$-permutable subgroup of G are all σ-conditionally permutable in G. In the case when $\sigma = \{\{2\}, \{3\}, \cdots\}$, every s-permutable subgroup, every $\mathfrak{Z}$-permutable subgroup and every s-conditionally permutable subgroup of G are also σ-conditionally permutable in G. However, the converse for s-conditionally permutable subgroup, σ-permutable subgroup and $\mathcal{H}$-permutable subgroup is not ture (see [10, Example 1.2]).

In this paper, we continue the research of σ-conditionally permutable subgroups, and obtain the results in section 3.

2. Preliminaries

Lemma 2.1. [10, Lemma 2.4] *Let H and K be subgroups of a σ-full group G. If K is normal in G and $K \leq H$, then H is σ-conditionally permutable in G if and only if H/K is σ-Conditionally permutable in G/K.*

Lemma 2.2. [11, Lemma 5] *Let H, K and N be pairwise permutable subgroups of G, and suppose that H is a Hall subgroup of G. Then $N \cap HK = (N \cap H)(N \cap K)$.*

We use $\mathfrak{S}_\sigma$ to denote the class of all σ-soluble groups.

Lemma 2.3. [3, Lemma 2.1] *The class $\mathfrak{S}_\sigma$ is closed under taking direct products, homomorphic images and subgroups. Moreover, any extension of a σ-soluble group by a σ-soluble group is a σ-soluble group as well.*

Lemma 2.4. [10, Theorem 1.3] *Let G be a σ-soluble group and $\mathcal{H} = \{H_1, \cdots, H_t\}$ be a complete Hall σ-set of G such that H_i is supersoluble for $i \in \{1, \cdots, t\}$. If every maximal subgroup of any non-cyclic H_i is σ-conditionally permutable in G, then G is supersoluble.*

We use $\mathfrak{N}_\sigma$ to denote the class of all σ-nilpotent groups.

Lemma 2.5. [3, Corollary 2.4 and Lemma 2.5]

(1) *The class $\mathfrak{N}_\sigma$ is closed under taking products of normal subgroups, homomorphic images and subgroups.*

(2) *If G/N and G/R are σ-nilpotent, then $G/N \cap R$ is σ-nilpotent.*

(3) *If E is a normal subgroup of G and $E/E \cap \Phi(G)$ is σ-nilpotent, then E is σ-nilpotent.*

3. Results

Theorem 3.1. *Let G be a σ-full group of Sylow type, $\mathcal{H}$ a complete Hall σ-set of G such that every member of $\mathcal{H}$ is supersoluble. Suppose that G is soluble and for some $\sigma_j \in \sigma(G)$, H_j is nilpotent and $(|G|, p-1) = 1$ for any prime $p \in \sigma_j$. If every maximal subgroup of H_j is σ-conditionally permutable in G, then G is σ_j-nilpotent (that is, G is p-nilpotent for any prime $p \in \sigma_j$).*

Proof. Let $\mathcal{H} = \{H_1, \cdots, H_t\}$ be a complete Hall σ-set of G. We can assume without loss of generality that H_i is a supersoluble σ_i-group for all $i = 1, \cdots, t$, and $j = 1$. Clearly $t > 1$. Assume the contary and let (G, H_1) be a counterexample with $|G| + |H_1|$ minimal. Then

(1) $O_{\sigma_1'}(G) = 1$.

Assume that this is false. Let $K = O_{\sigma_1'}(G)$. Clearly, $\sigma_1 \in \sigma(G/K)$ and H_1K/K is a nilpotent Hall σ_1-subgroup of G/K. Since $(|G|, p-1) = 1$ for any $p \in \sigma_1$, we have that $(|G/K|, p-1) = 1$. Let U/K be any maximal subgroup of H_1K/K, then U is a maximal subgroup of H_1K. Since $U = U \cap H_1K$ and H_1K/K is supersoluble, so

$$\begin{aligned} |H_1K/K : U/K| &= |H_1K : U| \\ &= |H_1K : (U \cap H_1)K| \\ &= |H_1||K||(U \cap H_1) \cap K| : |U \cap H_1||K||H_1 \cap K| \\ &= |H_1 : U \cap H_1| \end{aligned}$$

is a prime, and so $U \cap H_1$ is a maximal subgroup of H_1. Then by the hypothesis and Lemma 2.1, U/K is σ-conditionally permutable in G/K. This shows that $(G/K, H_1K/K)$ satisfies the hypothesis, and so G/K is σ_1-nilpotent by the choice of (G, H_1). It follows that G is σ_1-nilpotent, a contradiction. Hence we have claim (1).

(2) *Let R be a minimal normal subgroup of G, then R is an elementary abelian p-group for some prime $p \in \sigma_1$ and G/R is σ_1-nilpotent.*

Since G is soluble and by claim (1), we have that R is an elementary abelian p-group for some prime $p \in \sigma_1$, and so $R \leq H_1$. If $R = H_1$, then G/R is a σ_1'-group and hence G/R is σ_1-nilpotent. Now we consider that $R \neq H_1$. Then $\sigma_1 \in \sigma(G/R)$ and H_1/R is a nilpotent Hall σ_1-subgroup of G/R. Since $(|G|, p-1) = 1$ for any $p \in \sigma_1$, we have that $(|G/R|, p-1) = 1$. Let W/R be a maximal subgroup of H_1/R. Then W is a maximal subgroup of H_1. Hence by the hypothesis and Lemma 2.1, W/R is σ-conditionally permutable in G/R. This shows that $(G/R, H_1/R)$ satisfies the hypothesis. Therefore G/R is σ_1-nilpotent by the choice of (G, H_1).

(3) *R is the unique minimal normal subgroup of G, $\Phi(G) = 1$ and $R = O_p(G) = F(G) = C_G(R)$.*

This directly follows from G is soluble, claim (2) and [12, Chapter A, Theorem 15.2].

(4) *R is non-cyclic.*

Assume that R is cyclic. Then R is a cyclic group of order p by claim (2). It follows that $G/R = N_G(R)/C_G(R) \lesssim Aut(R)$ is a cyclic group of order $p-1$. But as $(|G|, p-1) = 1$, $G = R$ is a cyclic group of order p, a contradiction. Hence we have claim (4).

(5) *H_1 is a p-group and so $\sigma_1 \cap \pi(G) = \{p\}$.*

It directly follows from $C_G(R) = R$ for H_1 is nilpotent.

(6) *Final contradiction.*

Since $\Phi(G) = 1$, $R \not\leq \Phi(H_1)$ by [13, Chapter III, Lemma 3.3]. Hence there exists a maximal subgroup M of H_1 such that $H_1 = RM$. Let $E = R \cap M$. Then by claim (2), we have that $E \trianglelefteq H_1$. Since H_1 is supersoluble, $|R : E| = |RM : M| = |H_1 : M|$ is a prime. Hence E is a maximal subgroup of R, and so $E \neq 1$ by claim (4). By the hypothesis, G possesses a complete Hall σ-set $\mathcal{H} = \{H_1, \cdots, H_t\}$ and for any subgroup $H_i \in \mathcal{H}$, there exists an element $x_i \in G$ such that $MH_i^{x_i} = H_i^{x_i}M$. Assume that H_i is not σ_1-group. Then $R \cap MH_i^{x_i} = (R \cap M)(R \cap H_i^{x_i}) = (R \cap M) = E$ by Lemma 2.2. Hence $H_i^{x_i} \leq N_G(E)$. Moreover, as $E \trianglelefteq H_1$, we obtain that $E \trianglelefteq G$, and therefore $E = 1$. This follows that R is cyclic, contrary to claim (4). This final contradiction completes the proof. □

Recall that $G^{\mathfrak{N}_\sigma}$ denotes the *σ-nilpotent residual* of G, that is, the intersection of all normal subgroups N of G with σ-nilpotent quotient G/N.

Theorem 3.2. *Let G be a σ-full group of Sylow type, $\mathcal{H}$ a complete Hall σ-set of G such that every member of $\mathcal{H}$ is supersoluble. If every maximal subgroup of any noncyclic H_i is σ-conditionally permutable in G, then the derived subgroup $G^{'}$ of G is σ-nilpotent.*

Proof. Let $\mathcal{H} = \{H_1, \cdots, H_t\}$ be a complete Hall σ-set of G. We can assume without loss of generality that H_i is a supersoluble σ_i-group for all $i = 1, \cdots, t$. Suppose that this assertion is false and let G be a counterexample of minimal order. Then G is not σ-nilpotent, so $|\sigma(G)| > 1$. Let $D = G^{\mathfrak{N}_\sigma} \neq 1$ be the σ-nilpotent residual of G, and so G is σ-soluble by Lemma 2.3.

(1) *G is soluble and so G is not simple group.*

Let q be the smallest prime dividing the order of G. Without loss of generality, we may assume that $q \in \pi(H_1)$. If H_1 is cyclic, then sylow q-subgroup of G is cyclic. Hence G is q-nilpotent by [13, Chapter IV, Theorem 2.8] and so G is soluble. Now assume that H_1 is noncyclic, then by Lemma 2.4 G is soluble.

(2) *Let R be a minimal normal subgroup of G. Then R is a elementary abelian for some $p \in \sigma_i$ and $(G/R)^{'}$ is σ-nilpotent.*

It is clear that $\overline{\mathcal{H}} = \{H_1R/R, H_2R/R, \cdots, H_tR/R\}$ is a complete Hall σ-set of G/R and $H_iR/R \cong H_i/H_i \cap R$ is supersoluble. By claim (1), R is an elementary abelian p-group for some prime p. Without loss of generality, we can assume that $R \leq H_1$. If H_1/R is non-cyclic. Then H_1 is non-cyclic. For every maximal subgroup V/R of H_1/R, we have that V is a maximal subgroup of H_1. Then by the hypothesis and Lemma 2.1, V/R is σ-conditionally permutable in G/R. Now assume that H_iR/R is non-cyclic for $i \neq 1$, and let V_i/R be a maximal subgroup of H_iR/R. Then $V_i = (H_i \cap V_i)R$ is a maximal subgroup of H_iR. If $V_i \cap H_i$ is not a maximal subgroup of H_i, then there exists a subgroup M of H_i such that $V_i \cap H_i < M < H_i$. Since $(|H_i|, |R|) = 1$, $V_i < MR < H_iR$. This contradiction shows that $H_i \cap V_i$ is a maximal subgroup of H_i. By the hypothesis and Lemma 2.1, V/K is σ-conditionally permutable in G/R. This shows that G/R satisfies the hypothesis. Hence $(G/R)^{'} = G^{'}R/R = G^{'}/G^{'} \cap R$ is σ-nilpotent by the choice of G.

(3) *R is the unique minimal normal subgroup of G, $R \nleq \Phi(G)$ and $C_G(R) \leq R$.*

By claim (2) $R \leq G^{'}$ and $R \nleq \Phi(G)$ by Lemma 2.5(3). Moreover, if G has a minimal normal subgroup $N \neq R$, then $N \leq G^{'}$ and $G^{'} \cong G^{'}/1 = G^{'}/R \cap N$

is σ-nilpotent by Lemma 2.5(2), contrary to the choice of G. Therefore R is the unique minimal normal subgroup of G and $C_G(R) \leq R$ by [12, Chapter A, Theorem 15.2].

(4) $|R| > p$.

If $|R| = p$ for some prime p. But then $C_G(R) = R$ and $G/R = G/C_G(R) \lesssim Aut(R)$ is a cyclic. Hence $G^{'}$ is nilpotent. This contradiction shows that $|R| > p$.

(5) *Final contradiction.*

Without loss of genrality, we may assume that $R \leq H_1$. Then $|R| > p$ by claim (4), and $R \nleq \Phi(H)$ by claim (3) and [13, Chapter III, Lemma 3.3]. Therefore there exists a maximal subgroup V of H_1 such that $H_1 = RV$. It follows from R is elementary abelian that $R_1 = R \cap V \trianglelefteq H_1$. It is clear that R_1 is a maximal subgroup of R. Since R is noncyclic by claim (4), and $R \leq H_1$, H_1 is noncyclic. Then by the hypothesis, there exists an element $x_i \in G$ such that $VH_i^{x_i} = H_i^{x_i}V$ for all $i \in \{2, \cdots, t\}$. Then $R \cap VH_i^{x_i} = (R \cap V)(R \cap H_i^{x_i}) = (R \cap V) = R_1$ by Lemma 2.2. This implies that $H_i^{x_i} \leq N_G(R_1)$. Hence $R_1 \trianglelefteq G$, and so $R_1 = 1$. This implies that $|R| = p$, a contradiction to claim (3). This completes the proof. □

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References

1. W. Guo, A. N. Skiba, On Π-quasinormal Subgroups of Finite Groups, *Monatsh. Math.*, **185**, (2018), 443-453.
2. W. Guo, C. Cao, A. N. Skiba, D. A. Sinitsa, Finite Groups with $\mathcal{H}$-permutable Subgroups, *Commun. Math. Stat.*, **5**, (2017), 83-92.
3. A. N. Skiba, On σ-subnormal and σ-permutable Subgroups of Finite Groups, *J. Algebra*, **436**, (2015), 1-16.
4. A. N. Skiba, On Some Results in the Theory of Finite Partially Soluble Groups, *Commun. Math. Stat.*, **4**, (2016), 281-309.
5. W. Guo, K. P. Shum, A. N. Skiba, Conditionally Permutable Subgroups and Supersolubility of Finite Groups, *Southeast Asian Bull. Math.*, **29**, (2005), 493-510.
6. O. H. Kegel, Sylow Gruppen and Subnormalteiler Endlicher Gruppen, *Math. Z.*, **87**, (1962), 205-221.
7. M. Asaad, A. A. Heliel, On Permutable Subgroups of Finite Groups, *Arch. Math.*, **80**(2), (2003), 113-118.
8. M. Zha, W. Guo, B. Li, On p-supersolubility of Finite Groups (in Chinese), *J. Math.* (Wuhan), **27**(5), (2007), 563-568.
9. J. Huang, W. Guo, s-conditionally Permutable Subgroups of Finite Groups (in Chinese), *Chinese Ann. Math. Ser. A*, **28**(1), (2007), 17-26.
10. Y. Mao, C. Cao, W. Guo, On σ-conditionally Permutable Subgroups of Finite Groups, *Commun. Algebra*, **47**(1), (2019), 1-12.
11. V. N. Knyagina, V. S. Monakhov, On $\pi^{'}$-properties of Finite Groups Having a Hall π-subgroup, *Sib. Math. J.*, **52**(2), (2011), 297-309.

12. K. Doerk, T. Hawkes, *Finite Soluble Groups*, Walter de Gruyter, Berlin New York, 1992.
13. B. Huppert, *Endliche Gruppen I*, Springer-verlag, Berlin Heidelberg New York, 1967.