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Vortex Solutions for the 2D Boussinesq Equations Under the Radial Gravity

Mahdi Kamandar[a], Behruz Raesi[b*]

[a]Department of Applied Mathematics, Faculty of Mathematics and Computer, Shahid Bahonar University of Kerman, Kerman, Iran
[b]Department of Mathematics, Shahed University, Tehran, Iran

E-mail: kamandar@uk.ac.ir
E-mail: raesi@shahed.ac.ir

Abstract. The main objective of this article is to find some vortex solutions of finite core size for plane Boussinesq equations under the radial gravity, coupled with a diffusive equation of temperature in a weighted subspace of $L^2(\mathbb{R}^2)$. Solutions are expanded into series of Hermite eigenfunctions. We find the coefficients of the series and show the convergence of them.

Keywords: Boussinesq equations, Vortex solution, Hermite functions.

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1. Introduction

Incompressible flows have two standard representations. The first one is a formulation based on velocity field and pressure. The second one is based on the velocity and vorticity fields [20]. In the case of insignificant boundaries, the velocity-vorticity representation has particular benefits since, interior of a fluid, vorticity cannot be produced or destroyed. In two-dimensional space, the vorticity field reduces to a scalar and is perpendicular to the plane of the flow. In this paper, we present a class of exact vortex solutions for the 2D

*Corresponding Author

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Boussinesq equations under the radial gravity on a rotating plane. As far as we know, these are the first solutions of this kind for this model. The equations describe the evolution of the velocity field $\mathbf{u}$ of an incompressible fluid under the Non-singular radial gravity. Except in the buoyancy term and equation of state, we use Boussinesq approximation, where the density treats as a constant. The standard 2D Boussinesq system under the radial gravity reads as:

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \tau\mathbf{u} + 2\boldsymbol{\Omega}\cdot\mathbf{u} + \nabla p = \nu\triangle\mathbf{u} - \rho_0(1-\alpha(T-T_0))f(r)\mathbf{e}_r, \\ \partial_t T + \mathbf{u}\cdot\nabla T = K_T\triangle T, \\ \nabla\cdot\mathbf{u} = 0, \end{cases} \tag{1.1}$$

where $\mathbf{u} = (u_1, u_2, 0)$ is the velocity field, τ represents the turbulent friction, $\boldsymbol{\Omega} = (0, 0, \Omega)$ is the earth's rotation angular velocity, p is the fluid pressure, ν is the kinematic viscosity, $f(r)$ is a radial function, T stands for temperature, T_0 is the reference temperature, ρ_0 is the density at $T = T_0$ which assumes here $T_0 = 0$, α is thermal expansion coefficient, e_r is the unit vector in the r-direction and, K_T is the diffusion coefficient of temperature.

The Boussinesq equations are studied systematically in different domains of science [26]. They have attracted considerable interest recently due to their broad physical applications and mathematical significance [14, 29]. In this article, we consider two modes for force: 1) $f(r) = g$ and 2) $f(r) = \frac{g}{r^2+\epsilon^2}$, where g is a constant and ϵ is an arbitrary number and small enough. The first one is the most frequently used model for buoyancy-driven fluids, such as many largescale geophysical flows, atmospheric fronts, ocean circulation, clued dynamics (see, e.g.,[8, 17, 19, 28]). In addition, they play an important role in the Rayleigh-Benard convection [5, 15]. The second one is used to model the dynamics of astrophysical flows such as convection in planets and stars (see, e.g.,[4, 18]).

These equations are critically important in mathematics. They are similar to the 3D Navier-Stokes and the Euler equations. The 2D Boussinesq equations preserve some important aspects of the 3D Euler and Navier-Stokes equations such as the vortex stretching mechanism. The inviscid 2D Boussinesq equations are known as the Euler equations for the 3D axisymmetric swirling flows [20]. Mathematically, the vorticity is known as the curl of the flow velocity. Let $\boldsymbol{\omega} = \nabla\cdot\mathbf{u} = (0, 0, \omega)$. When $f(r) = \frac{g}{r^2+\epsilon^2}$, by applying curl on the first equation of (1.1) we arrive at:

$$\begin{cases} \partial_t\omega + \mathbf{u}\cdot\nabla\omega + \tau\omega = \nu\triangle\omega + \frac{\alpha\rho_0 g}{r(r^2+\epsilon^2)}(x_2\partial_{x_1}T - x_1\partial_{x_2}T), \\ \partial_t T + \mathbf{u}\cdot\nabla T = K_T\triangle T. \end{cases} \tag{1.2}$$

By the Biot-Savart law, the velocity field retrieves as follows:

$$\mathbf{u}(\mathbf{x},t) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(\mathbf{x}-\mathbf{z})^{\perp}}{|\mathbf{x}-\mathbf{z}|^2}\boldsymbol{\omega}(\mathbf{z},t)d\mathbf{z}, \tag{1.3}$$

where $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, $\mathbf{x}^{\perp} = (-x_2, x_1)$ and $|\mathbf{x}| = x_1^2 + x_2^2$. For the sake of simplification, we focus on (1.2), but our methods are applicable to the Thermohaline ocean circulation equations too. We employ equations (1.2) and (1.3) to establish a vorticity representation of the two-dimensional viscous flow.

Two-dimensional vortex motion studies go back to the work of Helmholtz [9], and later by Lord Kelvin [12], Sir Lamb [13], Prandtl [27], Milne-Thomson [21], Batchelor [1], and others. Ting and Tung in [32] used the traditional method to consider the movement of a vortex in a two-dimensional incompressible flow. It includes the viscous influence in the internal kernel of the vortex. Bernoff and Lingevitch in [3] achieved that the motion of vortex is the integral of the background irrational current. For a comprehensive survey of the inviscid point vortex model and recent developments, see [24]. Gallay and Wayne in [7] proved that the solutions of vorticity equation tend to Oseen vortex rapidly. Later on, by using the method and results of [7], Nagem et al. in [23] obtained an approximate solution for the vorticity equation. In the next step, they generalized the theory of single point vortex for viscous flow in two dimensions. Their theory finally seized the multi-vortex problem for viscous two-dimensional flows [22]. Jing et al. in [11] represented the viscous progress of a collinear three-vortex structure that previously corresponds to an inviscid point vortex fixed balance. Afterward, Gallay in [6] proved that the solutions of the Navier-Stokes equations converge, as $\nu \to 0$, to a superposition of Lamb-Oseen vortices which the centers evolve at a viscous regularization of the point vortex system. Uminsky et al. in [33] using Hermite eigenfunctions introduced a new multi-moment vortex method (MMVM). By using MMVM, Smith, and Nagem in [31] studied vortex pairs and dipoles. Sharifi and Raesi in [30] presented the first solutions of vortex type for 2D Boussinesq equations under a vertical force.

In this paper, we extend the results of [30] to the Boussinesq equation under the central gravitational force on a rotating plane with turbulent friction terms. We express a moment expansion of the vorticity based on Hermite functions. Then, we establish a convergence criterion of the moment expansion. We show that, if this criterion meets for $t = 0$, then it meets, for all subsequent times $t > 0$. Our convergence criterion relies on the observation that for any value of t, the Hermite functions are the eigenfunctions of a self-adjoint linear operator in a weighted subspace of $L^2(\mathbb{R}^2)$. We prove that if the initial vorticity distribution lies in this space, then the solution of the vorticity equation with that initial condition lies in it. We rewrite the two-dimensional vorticity equation as a system of ODEs with simple, quadratic nonlinear terms whose coefficients

can be evaluated in terms of derivatives of a single explicit function. Furthermore, we establish a sufficient condition on the initial vorticity distribution to guarantee that the expansion of the vorticity generated by the solution of these ODEs converges for all time. Finally, we introduce the hydrodynamical equations governing the atmospheric circulation over the tropics, Boussinesq equation with constant radial gravitational acceleration. In the same way, we obtain the exact solutions of this model. In recent years various dynamical aspects of this system in two and three-dimension were investigated by many authors. For instance, consider [18] for cosmological applications, and [25] for oceanographic applications.
This article is organized as follows. In section 2, we offer an expansion of solutions for the Boussinesq equations in the vorticity form. In section 3, the convergence of the series of the solution is shown. In section 4, We find the ODEs satisfied by the coefficient of the expansion in the Hermite base.

2. Review of the single center vortex method

In this section, we express the solutions of the (1.2), based on Hermite functions. Let

$$\phi_{00}(\mathbf{x},t;\lambda)=\frac{1}{\pi\lambda^2}e^{-|\mathbf{x}|^2/\lambda^2},\qquad T_{00}(\mathbf{x},t;\sigma)=\frac{1}{\pi\sigma^2}e^{-|\mathbf{x}|^2/\sigma^2},\tag{2.1}$$

where $\lambda^2=\lambda_0^2+4\nu t$ and $\sigma^2=\sigma_0^2+4k_T t$ and λ_0 and σ_0 represent the initial core size of our localized vortex structure. Note that for any value of λ_0 and σ_0, ϕ_{00} and T_{00} are exact solutions of the two-dimensional vorticity equation recognized as the Lamb-Oseen vortex. We can choose any value of λ_0 and σ_0 in the definition of our Hermite spectral method; Two numbers λ_0 and σ_0, that represent a typical length scale in the initial vorticity distribution may be chosen arbitrarily. The Hermite functions of degree (k_1,k_2) are defined as follows:

$$\phi_{k_1,k_2}(\mathbf{x},t;\lambda)=D_{x_1}^{k_1}D_{x_2}^{k_2}\phi_{00}(\mathbf{x},t;\lambda),\qquad \psi_{k_1,k_2}(\mathbf{x},t;\sigma)=D_{x_1}^{k_1}D_{x_2}^{k_2}T_{00}(\mathbf{x},t;\sigma),$$

where the operator $D_{x_1}^{k_1}D_{x_2}^{k_2}$ is k_1 the order of the derivative with respect to the component x_1 and k_2 the order of the derivative with respect to the component x_2. An expansion of the solutions of the vorticity field and temperature based on Hermite functions, which are called the moment expansion, are defined as follows:

$$\begin{aligned}\omega(\mathbf{x},t) &= \sum_{k_1,k_2=1}^{\infty} M[k_1,k_2;t]\phi_{k_1,k_2}(\mathbf{x},t;\lambda), \\ T(\mathbf{x},t) &= \sum_{k_1,k_2=1}^{\infty} I[k_1,k_2;t]\psi_{k_1,k_2}(\mathbf{x},t;\sigma).\end{aligned}\tag{2.2}$$

Let $(\omega, T)(\mathbf{x}, t)$ be a solution of the equation (1.2), then Biot-Savrat law implies that the velocity field is as below:

$$\mathbf{V}(\mathbf{x}, t) = \sum_{k_1,k_2=1}^{\infty} M[k_1, k_2; t] V_{k_1,k_2}(\mathbf{x}, t; \lambda),$$

where $V_{k_1,k_2}(\mathbf{x}, t; \lambda) = D_{x_1}^{k_1} D_{x_2}^{k_2} \mathbf{V}_{00}(\mathbf{x}, t; \lambda)$ and $\mathbf{V}_{00}(\mathbf{x}, t; \lambda)$ is the induced velocity from $\phi_{00}(\mathbf{x}, t; \lambda)$ which is as follows:

$$\mathbf{V}_{00}(\mathbf{x}, t; \lambda) = \frac{1}{2\pi} \frac{(-x_2, x_1)}{|\mathbf{x}|^2}(1 - e^{-|\mathbf{x}|^2/\lambda^2}).$$

Hermite polynomials have representations as a partial differentiation of generator functions as below:

$$H_{n_1,n_2}(\mathbf{z}, \lambda) = (D_{t_1}^{n_1} D_{t_2}^{n_2} e^{(\frac{2t\cdot\mathbf{z}-t^2}{\lambda^2})})|_{t=0}, \; F_{n_1,n_2}(\mathbf{z}, \sigma) = (D_{t_1}^{n_1} D_{t_2}^{n_2} e^{(\frac{2t\cdot\mathbf{z}-t^2}{\sigma^2})})|_{t=0}.$$

When $\lambda = 1$ and $\sigma = 1$, we have the standard Hermite multinomial. In this case, they constitute the orthogonal sets:

$$\int_{\mathbb{R}^2} H_{n_1,n_2}(\mathbf{z}, \lambda = 1) H_{m_1,m_2}(\mathbf{z}, \lambda = 1) e^{-\mathbf{z}^2} d\mathbf{z} = \pi 2^{n_1+n_2}(n_1!)(n_2!)\delta_{n_1,m_1}\delta_{n_2,m_2},$$

$$\int_{\mathbb{R}^2} F_{n_1,n_2}(\mathbf{z}, \sigma = 1) F_{m_1,m_2}(\mathbf{z}, \sigma = 1) e^{-\mathbf{z}^2} d\mathbf{z} = \pi 2^{n_1+n_2}(n_1!)(n_2!)\delta_{n_1,m_1}\delta_{n_2,m_2}.$$

Thus, the following projection operators define the coefficients in the expansion (2.2):

$$M[k_1, k_2; t] = (P_{k_1,k_2}\omega)(t) = \rho(k_1, k_2, \lambda) \int_{\mathbb{R}^2} H_{k_1,k_2}(\mathbf{z}, \lambda)\omega(\mathbf{z}, t) d\mathbf{z}, \tag{2.3}$$

$$I[k_1, k_2; t] = (Q_{k_1,k_2}T)(t) = \rho(k_1, k_2, \sigma) \int_{\mathbb{R}^2} F_{k_1,k_2}(\mathbf{z}, \sigma) T(\mathbf{z}, t) d\mathbf{z},$$

where

$$\rho(k_1, k_2, \tau) = \frac{(-1)^{(k_1+k_2)}\tau^{2(k_1+k_2)}}{2^{k_1+k_2}(k_1!)(k_2!)}. \tag{2.4}$$

It can be easily seen, for any value of t, the Hermite functions $\phi_{k_1,k_2}(\mathbf{x}, t; \lambda)$ are the eigenfunctions of the self-adjoint linear operator:

$$L^\lambda \phi = \frac{1}{4}\lambda^2 \triangle \phi + \frac{1}{2}\nabla \cdot (\mathbf{x}\phi).$$

Note that, L^λ can be transformed into the Hamiltonian quantum mechanical harmonic oscillator. The eigenfunctions of L^λ construct an orthogonal set in the $X^\lambda = \{f \in L^2(\mathbb{R}^2) \mid \phi_\lambda^{-1/2} f \in L^2(\mathbb{R}^2)\}$, which is a Hilbert space. let

$$\Phi_\lambda(\mathbf{x}, t) = \phi_{00}(\mathbf{x}, t; \lambda), \qquad \Psi_\sigma(\mathbf{x}, t) = T_{00}(\mathbf{x}, t; \sigma).$$

It is shown in [22] that the convergence of the expansions (2.2), is equivalent to the convergence of the following integrals:

$$\int_{\mathbb{R}^2} \Phi_\lambda^{-1}(\mathbf{x})(\omega(\mathbf{x}, t))^2 d\mathbf{x} < \infty, \qquad \int_{\mathbb{R}^2} \Psi_\sigma^{-1}(\mathbf{x})(T(\mathbf{x}, t))^2 d\mathbf{x} < \infty. \tag{2.5}$$

3. Convergence criterion of the solutions

In this section, we show that if the initial vorticity distribution satisfies (2.5), then the solution of the vorticity satisfies it, for all time t. The Gaussian function ϕ_{00} is important in the convergence proof. To prove Theorem 3.2, we need the following lemma:

Lemma 3.1. *Let (ω, T) satisfies the equations (1.2), $\omega(\mathbf{x}, 0) = \omega_0(\mathbf{x})$ and $T(\mathbf{x}, 0) = T_0(\mathbf{x})$ then the following assertions are true:*
i) For all $1 \leq p \leq \infty$ and $t \geq 0$, $||T(\mathbf{x}, t)||_p \leq ||T_0(\mathbf{x})||_p$.
ii) There exist constant $c = c(\omega_0, T_0, t)$ such that for all $2 \leq q < \infty$ and $t \geq 0$, $||\omega(\mathbf{x}, t)||_q \leq c(\omega_0, T_0)$.
iii) For all $t \geq 0$, $||\nabla \mathbf{u}||_\infty \leq c(\omega_0, T_0, t)$.

Proof. For (i) see [2], and for (ii) see [10], and for (iii) see [34]. □

Now we can prove the criteria (2.5).

Theorem 3.2. *Define the weighted enstrophy functions*

$$\mu(t) = \int_{\mathbb{R}^2} \Phi_\lambda^{-1}(\omega(\mathbf{x}, t))^2 d\mathbf{x}, \qquad \gamma(t) = \int_{\mathbb{R}^2} \Psi_\sigma^{-1}(T(\mathbf{x}, t))^2 d\mathbf{x}.$$

Let $k_T < 2\nu$, and the primary vorticity and temperature are bounded (in the L^∞ norm), and guarantee that $\mu(0) < \infty$ and $\gamma(0) < \infty$ for some λ_0 and σ_0 respectively, then $\mu(t)$ and $\gamma(t)$ will be finite for all times $t > 0$.

Proof. By using [22], one can prove the following:

$$\frac{d\gamma(t)}{dt} \leq \left(\frac{4c(\omega_0, T_0)}{K_T} + \frac{4K_T}{\sigma^2} \right) \gamma(t),$$

which means that $\gamma(t)$ will be finite for each $t > 0$, if $\gamma(0)$ is finite. First, we obtain a differential inequality for $\mu(t)$. This inequality ensures that if $\mu(0)$ is finite, then $\mu(t)$ will be finite for all t. For this purpose, by differentiating of $\mu(t)$, we obtain:

$$\begin{aligned} \frac{d\mu(t)}{dt} &= \frac{4\nu}{\lambda^2}\mu(t) - \frac{4\nu}{\lambda^4} \int_{\mathbb{R}^2} |\mathbf{x}|^2 \Phi_\lambda^{-1}(\omega(\mathbf{x}, t))^2 d\mathbf{x} + 2 \int_{\mathbb{R}^2} |\mathbf{x}|^2 \Phi_\lambda^{-1} \omega(\mathbf{x}, t) \partial_t \omega(\mathbf{x}, t) d\mathbf{x} \\ &= \frac{4\nu}{\lambda^2}\mu(t) - \frac{4\nu}{\lambda^4} \int_{\mathbb{R}^2} |\mathbf{x}|^2 \Phi_\lambda^{-1}(\omega(\mathbf{x}, t))^2 d\mathbf{x} \qquad (3.1) \\ &+ 2 \int_{\mathbb{R}^2} \Phi_\lambda^{-1} \omega \left(\nu \triangle \omega - \mathbf{u} \cdot \nabla \omega + \tau \omega + \frac{\alpha \rho_0 g (x_2 \partial_{x_1} T - x_1 \partial_{x_2} T)}{r(r^2 + \epsilon^2)} \right) d\mathbf{x}. \end{aligned}$$

Integrating by parts in the last term in (3.1) implies that:

$$2 \int_{\mathbb{R}^2} \Phi_\lambda^{-1} \omega (\nu \triangle \omega) d\mathbf{x} = -2\nu \int_{\mathbb{R}^2} \Phi_\lambda^{-1}(\mathbf{x}) (|\nabla \omega|^2 + \frac{2}{\lambda^2} \omega \mathbf{x} \cdot \nabla \omega) d\mathbf{x}, \qquad (3.2)$$

and the second term in the right-hand side of (3.2) satisfies the following relation:

$$2\nu \int_{\mathbb{R}^2} \Phi_\lambda^{-1}(\mathbf{x})(\frac{2}{\lambda^2}\omega\mathbf{x}\cdot\nabla\omega)d\mathbf{x} \leq \nu \int_{\mathbb{R}^2} \Phi_\lambda^{-1}(x)|\nabla\omega|^2 d\mathbf{x} + \frac{4\nu}{\lambda^4}\int_{\mathbb{R}^2} \Phi_\lambda^{-1}(\mathbf{x})(\mathbf{x}^2\omega^2)d\mathbf{x}. \tag{3.3}$$

According to lemma 2.1 in [7], we have: $||\mathbf{u}||_\infty \leq c||\omega||_p^\alpha ||\omega||_q^{1-\alpha}$ where $1 \leq p < 2 < q \leq \infty$ and $\frac{\alpha}{p} + \frac{1-\alpha}{q} = \frac{1}{2}$. Thus, according to Lemma 3.1, we get $||\mathbf{u}||_\infty \leq c(\omega_0, T_0)$. By using this fact and Cauchy's inequality we have :

$$\begin{aligned} 2\int_{\mathbb{R}^2} \Phi_\lambda^{-1}\omega(\mathbf{u}\cdot\nabla\omega)d\mathbf{x} &\leq 2c(\omega_0,T_0)\int_{\mathbb{R}^2}\Phi_\lambda^{-1}|\omega(\mathbf{x},t)||\nabla\omega|d\mathbf{x} \qquad (3.4)\\ &\leq \frac{c^2(\omega_0,T_0)}{\nu}\int_{\mathbb{R}^2}\Phi_\lambda^{-1}(\omega(\mathbf{x},t))^2 d\mathbf{x} + \nu\int_{\mathbb{R}^2}\Phi_\lambda^{-1}(\mathbf{x})|\nabla\omega|^2 d\mathbf{x}, \end{aligned}$$

also:

$$2\int_{\mathbb{R}^2} \Phi_\lambda^{-1}(\omega\tau\omega d\mathbf{x}) \leq 2\tau\mu(t), \tag{3.5}$$

and

$$\begin{aligned} &2\alpha\rho_0 g\int_{\mathbb{R}^2}\Phi_\lambda^{-1}\omega\left(\frac{x_2\partial_{x_1}T - x_1\partial_{x_2}T}{r(r^2+\epsilon^2)}\right)d\mathbf{x} \qquad (3.6)\\ &\leq 2\alpha\rho_0 g\int_{\mathbb{R}^2}\Phi_\lambda^{-1}\omega\left(\frac{r\partial_{x_1}T + r\partial_{x_2}T}{r(r^2+\epsilon^2)}\right)d\mathbf{x}\\ &\leq \frac{\alpha\rho_0 g}{\epsilon^2}\int_{\mathbb{R}^2}\Phi_\lambda^{-1}\omega^2 d\mathbf{x} + \frac{2\alpha\rho_0 g}{\epsilon^2}\int_{\mathbb{R}^2}\Phi_\lambda^{-1}||\nabla T||^2 d\mathbf{x} = \frac{\alpha\rho_0 g\mu(t)}{\epsilon^2} + \frac{2\alpha\rho_0 g}{\epsilon^2}\delta(t). \end{aligned}$$

Now we define $\delta(t)$ as follow:

$$\delta(t) = \int_{\mathbb{R}^2}\Phi_\lambda^{-1}(\nabla T(\mathbf{x},t))^2 d\mathbf{x}. \tag{3.7}$$

According to [30], if $\delta(0)$ is finite, then $\delta(t)$ will be finite for all $t > 0$.
So according to (3.1)-(3.7) we can write:

$$\frac{d\mu(t)}{dt} \leq \left(\frac{4\nu}{\lambda^2} + \frac{4c(\omega_0,T_0)}{\nu} + \frac{\alpha\rho_0 g}{\epsilon^2} + 2\tau\right)\mu(t) + \frac{2\alpha\rho_0 g}{\epsilon^2}c_1(\omega_0,t_0,t), \tag{3.8}$$

where $||\nabla T||_\lambda^2 \leq c_1(\omega_0, t_0, t)$. Using the Gronwall lemma and (3.8), if $\mu(0)$ is finite then $\mu(t)$ remains finite for all $t > 0$. □

4. ODEs of the coefficients of the expansion

In this section, we study the coefficients of expansion, which satisfy a system of ODEs. These coefficients can explicitly be expressed as derivatives of ϕ_{00} and T_{00}. Suppose that, $(\omega, T)(\mathbf{x}, t)$ is a solution of (1.2). Define the followings:

$$\boldsymbol{\omega}^m(\mathbf{x},t) := \sum_{k_1,k_2}^{m} M[k_1,k_2;t]\phi_{k_1,k_2}(\mathbf{x},t;\lambda),$$

$$\begin{aligned}
\mathbf{u}^m(\mathbf{x},t) &:= \sum_{k_1,k_2}^{m} M[k_1,k_2;t]V_{k_1,k_2}(\mathbf{x},t;\lambda),\\
T^m(\mathbf{x},t) &:= \sum_{k_1,k_2}^{m} I[k_1,k_2;t]\psi_{k_1,k_2}(\mathbf{x},t;\sigma),
\end{aligned}$$

where $\boldsymbol{\omega}^m$, $\mathbf{u}^m$, and T^m are Hermite approximations of order m (Galerkin approximation by Hermite functions). Now, using a standard Galerkin approximation, one can derive the equation for the evolution of the coefficients in the moment expansion. This derivation is difficult for nonlinear Partial differential equations since passing the nonlinearity through the operators (2.3) cannot be calculated explicitly. It is also more true in vorticity situations, where the nonlinearity involves a product of vorticity and velocity. By applying Galerkin standard approximation to equation (1.2) and differentiating from that equation, we obtain:

$$\begin{aligned}
\partial_t\omega^m &= \sum_{k_1,k_2}^{m} \frac{dM[k_1,k_2;t]}{dt}\phi_{k_1,k_2}(x,t;\lambda) + \sum_{k_1,k_2}^{m} M[k_1,k_2;t]\partial_t\phi_{k_1,k_2} \qquad (4.1)\\
&= \sum_{k_1,k_2}^{m} M[k_1,k_2;t](\nu\triangle\phi_{k_1,k_2}(x,t;\lambda))\\
&- P^m\left[(\sum_{l_1,l_2}^{m} M[l_1,l_2;t]V_{l_1,l_2}(x,t;\lambda))\cdot\nabla(\sum_{k_1,k_2}^{m} M[k_1,k_2;t]\phi_{k_1,k_2}(x,t;\lambda))\right]\\
&+ P^m\left[\frac{\alpha\rho_0 g x_2}{r(r^2+\epsilon^2)}\partial_{x_1}(\sum_{k_1,k_2}^{m} I[k_1,k_2;t]\psi_{k_1,k_2}(\mathbf{x},t;\sigma))\right]\\
&- P^m\left[\frac{\alpha\rho_0 g x_1}{r(r^2+\epsilon^2)}\partial_{x_2}(\sum_{k_1,k_2}^{m} I[k_1,k_2;t]\psi_{k_1,k_2}(\mathbf{x},t;\sigma))\right]\\
&+ 2\tau\sum_{k_1,k_2}^{m} M[k_1,k_2;t]\phi_{k_1,k_2}(\mathbf{x},t;\lambda),\\
\partial_t T^m &= \sum_{k_1,k_2}^{m} \frac{dI[k_1,k_2;t]}{dt}\psi_{k_1,k_2}(x,t;\sigma) + \sum_{k_1,k_2}^{m} I[k_1,k_2;t]\partial_t\psi_{k_1,k_2} \qquad (4.2)\\
&= \sum_{k_1,k_2}^{m} I[k_1,k_2;t](K_T\triangle\psi_{k_1,k_2}(x,t;\sigma))\\
&- P^m\left[(\sum_{l_1,l_2}^{m} M[l_1,l_2;t]V_{l_1,l_2}(x,t;\lambda))\cdot\nabla(\sum_{k_1,k_2}^{m} I[k_1,k_2;t]\psi_{k_1,k_2}(x,t;\sigma))\right],
\end{aligned}$$

where $P^m[\cdot]$ is a projector on the subspace given by Hermite functions of degree less or equal to m which is a weighted subspace of L^2, with the norm is defined by the weighted enstrophy function in Theorem 3.2. Note that:

$$\partial_t \phi_{k_1,k_2} = \nu \triangle \phi_{k_1,k_2},\ \partial_t \psi_{k_1,k_2} = K_T \triangle \psi_{k_1,k_2}.$$

When $k_1 + k_2 \leq m$, by applying the projection operators P_{k_1,k_2} and Q_{k_1,k_2}, defined in (2.3) on the equations (4.1) and (4.2), give rise:

$$\begin{aligned}
\frac{dM[k_1,k_2;t]}{dt} = & \qquad (4.3)\\
& P_{k_1,k_2}\left[(\sum_{l_1,l_2}^{m} M[l_1,l_2;t] V_{l_1,l_2}(\mathbf{x},t;\lambda)) \cdot \nabla (\sum_{m_1,m_2}^{m} M[m_1,m_2;t]\phi_{m_1,m_2}(\mathbf{x},t;\lambda)) \right] \\
& + P_{k_1,k_2}\left[\frac{\alpha\rho_0 g x_2}{r(r^2+\epsilon^2)} \partial_{x_1} (\sum_{m_1,m_2}^{m} I[m_1,m_2;t]\psi_{m_1,m_2}(\mathbf{x},t;\sigma)) \right] \\
& - P_{k_1,k_2}\left[\frac{\alpha\rho_0 g x_1}{r(r^2+\epsilon^2)} \partial_{x_2} (\sum_{m_1,m_2}^{m} I[m_1,m_2;t]\psi_{m_1,m_2}(\mathbf{x},t;\sigma)) \right] \\
& + 2P_{k_1,k_2}\left[\tau \sum_{k_1,k_2}^{m} M[k_1,k_2;t]\phi_{k_1,k_2}(\mathbf{x},t;\lambda) \right],
\end{aligned}$$

$$\begin{aligned}
\frac{dI[k_1,k_2;t]}{dt} = & \qquad (4.4)\\
& - Q_{k_1,k_2}\left[(\sum_{l_1,l_2}^{m} M[l_1,l_2;t] V_{l_1,l_2}(\mathbf{x},t;\lambda)) \cdot \nabla (\sum_{m_1,m_2}^{m} I[m_1,m_2;t]\psi_{m_1,m_2}(\mathbf{x},t;\sigma)) \right].
\end{aligned}$$

For the sake of simplification, define:

$$\phi_{m_1,m_2}(\mathbf{x},t;\lambda) \quad := \quad (D_{a_1}^{m_1} D_{a_2}^{m_2} \phi_{00}(\mathbf{x}+\mathbf{a},\lambda))|_{\mathbf{a}=0}. \qquad (4.5)$$

The system of ordinary differential equations (4.3) and (4.4) becomes:

$$\begin{aligned}
\frac{dM[k_1,k_2;t]}{dt} = & -\rho(k_1,k_2\lambda) \sum_{l_1,l_2=1}^{m} \sum_{m_1,m_2=1}^{m} M[l_1,l_2;t] M[m_1,m_2;t] \qquad (4.6)\\
& \cdot \int_{\mathbb{R}^2} H_{k_1,k_2}(\mathbf{x}) (D_{x_1}^{l_1} D_{x_2}^{l_2} V_{00}(\mathbf{x},\lambda)) \cdot \nabla_{\mathbf{x}} (D_{x_1}^{m_1} D_{x_2}^{m_2} \phi_{00}(\mathbf{x},\lambda)) d\mathbf{x} \\
& + \rho(k_1,k_2,\lambda)\alpha\rho_0 g \sum_{m_1,m_2=1}^{m} I[m_1,m_2;t] \\
& \cdot \int_{\mathbb{R}^2} H_{k_1,k_2}(\mathbf{x}) \frac{x_2}{r(r^2+\epsilon^2)} (D_{x_1}^{m_1+1} D_{x_2}^{m_2} T_{00}(\mathbf{x},\sigma)) d\mathbf{x} \\
& - \rho(k_1,k_2,\lambda)\alpha\rho_0 g \sum_{m_1,m_2=1}^{m} I[m_1,m_2;t] \\
& \cdot \int_{\mathbb{R}^2} H_{k_1,k_2}(\mathbf{x}) \frac{x_1}{r(r^2+\epsilon^2)} (D_{x_1}^{m_1} D_{x_2}^{m_2+1} T_{00}(\mathbf{x},\sigma)) d\mathbf{x}
\end{aligned}$$

$$+ 2\tau\rho(k_1,k_2,\lambda)\sum_{k_1k_2=1}^{m} M[k_1,k_2;t]\int_{\mathbb{R}^2} H_{k_1,k_2}(\mathbf{x})(D_{x_1}^{k_1}D_{x_2}^{k_2}\phi_{00}(\mathbf{x},\lambda))d\mathbf{x},$$

$$\frac{dI[k_1,k_2;t]}{dt} = -\rho(k_1,k_2,\sigma)\sum_{l_1,l_2=1}^{m}\sum_{m_1,m_2=1}^{m} M[l_1,l_2;t]I[m_1,m_2;t] \quad (4.7)$$

$$\cdot\int_{\mathbb{R}^2} F_{k_1,k_2}(\mathbf{x})(D_{x_1}^{m_1}D_{x_2}^{m_2}V_{00}(\mathbf{x},\lambda))\cdot\nabla_{\mathbf{x}}(D_{x_1}^{l_1}D_{x_2}^{l_2}T_{00}(\mathbf{x},\sigma))d\mathbf{x},$$

where $\rho(k_1, k_2, \tau)$ is defined in (2.4). The first integral in (4.6) is calculated in [33], and the three remaining integrals are calculated in the appendix. Also the integral of (4.7) is calculated in [30]. Finally, by using the appendix calculations and equations (4.6)-(4.7), we get the differential equations for $M[k_1, k_2, t]$ and $I[k_1, k_2, t]$ to:

$$\begin{aligned}\frac{dM[k_1,k_2;t]}{dt} &= \rho(k_1,k_2,\lambda)\sum_{l_1,l_2=1}^{m}\sum_{m_1,m_2=1}^{m} M[l_1,l_2;t]M[m_1,m_2;t] \quad (4.8)\\ &\cdot \Gamma[k_1,k_2,l_1,l_2,m_1,m_2;\lambda] + \rho(k_1,k_2,\lambda)\alpha\rho_0 g\\ &\cdot \sum_{m_1,m_2}^{m} I[m_1,m_2;t]B[k_1,k_2,m_1,m_2;\lambda,\sigma]\\ &+ 2\tau\rho(k_1,k_2,\lambda)\sum_{k_1,k_2=1}^{m} M[k_1,k_2;t]A[k_1,k_2,m_1,m_2;\lambda],\\ \frac{dI[k_1,k_2;t]}{dt} &= \rho(k_1,k_2,\sigma)\sum_{l_1,l_2=1}^{m}\sum_{m_1,m_2=1}^{m} M[l_1,l_2;t]I[m_1,m_2;t] \quad (4.9)\\ &\cdot \theta[k_1,k_2,l_1,l_2,m_1,m_2;\lambda,\sigma],\end{aligned}$$

where θ, $\tilde{\Gamma}$, A, B introduced in the appendix. Thus, every solution of ODEs (4.8) and (4.9) gives rise to a solution of (1.2).

4.1. **Atmospheric circulation over the tropics.** The atmosphere and ocean around the earth are rotating geophysical fluids. They are two essential parts of the climate system [17]. Now consider the case where $f(r) = g$, then the system (1.1) is reduced to:

$$\begin{cases} \partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \tau\mathbf{u} + 2\mathbf{\Omega}\cdot\mathbf{u} + \nabla p = \nu\triangle\mathbf{u} - \rho_0(1-\alpha(T-T_0))g\mathbf{e}_r,\\ \partial_t T + \mathbf{u}\cdot\nabla T = K_T\triangle T,\\ \nabla\cdot\mathbf{u} = 0.\end{cases} \quad (4.10)$$

By applying curl on the first equation (4.10), we obtain:

$$\begin{cases} \partial_t\omega + \mathbf{u}\cdot\nabla\omega + \tau\omega = \nu\triangle\omega + \frac{\alpha\rho_0 g}{r}(x_2\partial_{x_1}T - x_1\partial_{x_2}T),\\ \partial_t T + \mathbf{u}\cdot\nabla T = K_T\triangle T.\end{cases}$$

Similarly, Theorem 3.2 is true for this system. To obtain the exact solutions, it is enough to replace in (4.8), $\tilde{B}$ which is calculated in the appendix, instead of B.

5. CONCLUTION

In this paper, we have presented simplified equations for a new multi-moment vortex method for computing solutions to the 2D vorticity equation. The novel feature of MMVM as compared to classical vortex methods is that the higher Hermite moments allow the vortex particles to convect and deform without any of the usual computational difficulties associated to calculating the Biot-Savart kernel for anisotropic vortex elements.

Appendix A

In this appendix we calculate explicitly the integral term in equation (4.6).

$$A[k_1,k_2,m_1,m_2;\lambda]=\int_{\mathbb{R}^2}H_{k_1,k_2}(\mathbf{x})(D_{x_1}^{m_1}D_{x_2}^{m_2}\phi_{00}(\mathbf{x},\lambda))d\mathbf{x} \tag{A.1}$$

Note that:

$$H_{n,m}=(-1)^{n+m}\phi_{00}^{-1}D_{x_1}^{n}D_{x_2}^{m}\phi_{00}. \tag{A.2}$$

Now by using equations (A.2) and (4.5) we have:

$$\begin{aligned}A[k_1,k_2,m_1,m_2;\lambda] &= \qquad\qquad (A.3)\\ &(-1)^{k_1+k_2}D_{a_1}^{m_1+k_1}D_{a_2}^{m_2+k_2}\int_{\mathbb{R}^2}\phi_{00}^{-1}(\mathbf{x})\phi_{00}(\mathbf{x}+\mathbf{a},\lambda)\phi_{00}(\mathbf{x}+\mathbf{a},\lambda))d\mathbf{x}|_{\mathbf{a}=0}.\end{aligned}$$

Note that, in the last integral, all three factors are Gaussians. Thus the integral can be calculated explicitly. Now by using (2.1) we obtain:

$$\begin{aligned}&\int_{\mathbb{R}^2}\phi_{00}^{-1}(\mathbf{x})\phi_{00}(\mathbf{x}+\mathbf{a},\lambda)\phi_{00}(\mathbf{x}+\mathbf{a},\lambda))d\mathbf{x}=\\ &\int_{\mathbb{R}^2}\pi\lambda^2 e^{\frac{x_1^2+x_2^2}{\lambda^2}}\cdot\frac{1}{\pi\lambda^2}e^{\frac{-x_1^2-2a_1x_1-a_1^2-x_2^2-2a_2x_2-a_2^2}{\lambda^2}}\\ &\cdot\frac{1}{\pi\lambda^2}e^{\frac{-x_1^2-2a_1x_1-a_1^2-x_2^2-2a_2x_2-a_2^2}{\lambda^2}}d\mathbf{x}=\\ &\frac{1}{\pi\lambda^2}\cdot e^{\frac{-2a_1^2-2a_2^2}{\lambda^2}}\cdot\int_{\mathbb{R}^2}e^{\frac{-x_1^2-4a_1x_1-x_2^2-4a_2x_2}{\lambda^2}}d\mathbf{x}\\ &=\frac{1}{\pi\lambda^2}\cdot e^{\frac{+2a_1^2+2a_2^2}{\lambda^2}}\int_{\mathbb{R}}e^{\frac{-(x_1+2a_1)^2}{\lambda^2}}dx_1\cdot\int_{\mathbb{R}}e^{\frac{-(x_2+2a_2)^2}{\lambda^2}}dx_2\\ &=\frac{1}{\pi\lambda^2}\cdot e^{\frac{+2a_1^2+2a_2^2}{\lambda^2}}\cdot\pi\lambda^4.\end{aligned}$$

Then

$$A[k_1,k_2,m_1,m_2;\lambda]=(-1)^{k_1+k_2}D_{a_1}^{m_1+k_1}D_{a_2}^{m_2+k_2}[\lambda^2 e^{\frac{2a_1^2+2a_2^2}{\lambda^2}}],$$

and

$$e^{\frac{2a_1^2+2a_2^2}{\lambda^2}} = \sum_{n=0}^{\infty} \frac{2^n}{\lambda^{2n}} \cdot \frac{1}{n!}(a_1^2+a_2^2)^n = \sum_{n=0}^{\infty} \frac{2^n}{n!\lambda^{2n}} \cdot \sum_{r=0}^{n} \binom{n}{r} (a_1)^{2r}(a_2)^{2(n-r)}.$$

So we have

$$D_{a_1}^{m_1+k_1} D_{a_2}^{m_2+k_2}(e^{\frac{2a_1^2+2a_2^2}{\lambda^2}})|_{\mathbf{a}=0} = \sum_{n=0}^{\infty} \frac{2^n}{n!\lambda^{2n}} \cdot \sum_{r=0}^{n} \binom{n}{r} \frac{(2r)!}{(2r-m_1-k_1-2)!} a_1^{2r-m_1-k_1-2} \cdot \frac{(2n-2r)!}{(2n-2r-m_2-k_2)!} a_2^{2n-2r-m_2-k_2}|_{\mathbf{a}=0}.$$

Now assume that $r = \frac{m_1+k_1}{2}$ and $n = \frac{m_2+k_2-m_1-k_1}{2}$. In this case, A obtained as follows:

$$A[k_1,k_2,m_1,m_2;\lambda] = \begin{cases} (-1)^{k_1+k_2} \cdot \frac{1}{(\frac{\alpha_2-\alpha_1}{2})!} & \text{if } \alpha_1 \text{ and} \\ \cdot \frac{2^{\frac{\alpha_2-\alpha_1}{2}}}{\lambda^{(\alpha_2-\alpha_1+2)}} \cdot \binom{\frac{\alpha_2-\alpha_1}{2}}{\frac{\alpha_1}{2}} & \alpha_2 \text{ be positive} \\ (\alpha_1)! \cdot (\alpha_2-2\alpha_1)! & \text{and } \alpha_2 \geq \alpha_1 \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha_1 = m_1+k_1$ and $\alpha_2 = m_2+k_2$.

Appendix B

In this appendix we calculate explicitly the integral term in equation (4.6).

$$B[k_1,k_2,m_1,m_2;\lambda;\sigma] = \int_{\mathbb{R}^2} H_{k_1,k_2}(\mathbf{x}) \frac{(x_2 D_{x_1}^{m_1+1} D_{x_2}^{m_2} T_{00}(\mathbf{x},\sigma)) - (x_1 D_{x_1}^{m_1} D_{x_2}^{m_2+1} T_{00}(\mathbf{x},\sigma)) d\mathbf{x}}{r(r^2+\epsilon^2)} \tag{B.1}$$

$$= \int_{\mathbb{R}^2} \Big[(-1)^{k_1+k_2} \phi_{00}^{-1}(\mathbf{x}) D_{x_1}^{k_1} D_{x_2}^{k_2} \phi_{00}(\mathbf{x}) \cdot \frac{\big(x_2 D_{x_1}^{m_1+1} D_{x_2}^{m_2} T_{00}(\mathbf{x},\sigma) - x_1 D_{x_1}^{m_1} D_{x_2}^{m_2+1} T_{00}(\mathbf{x},\sigma)\big)}{r(r^2+\epsilon^2)} \Big] d\mathbf{x},$$

where the above equality comes from (A.2). Now to calculate the last integral, by using (2.1) we obtain:

$$B[k_1,k_2,m_1,m_2;\lambda;\sigma] = \tag{B.2}$$

$$(-1)^{k_1+k_2} \int_{\mathbb{R}^2} \pi\lambda^2 e^{\frac{x_1^2+x_2^2}{\lambda^2}} D_{b_1}^{k_1} D_{b_2}^{k_2} \frac{1}{\pi\lambda^2} e^{\frac{-x_1^2-2b_1x_1-b_1^2-x_2^2-2b_2x_2-b_2^2}{\lambda^2}}$$

$$\cdot \left[x_2 D_{a_1}^{m_1+1} D_{a_2}^{m_2} \frac{1}{\pi\sigma^2} e^{\frac{-x_1^2-2a_1x_1-a_1^2-x_2^2-2a_2x_2-a_2^2}{\sigma^2}} \right.$$
$$\left. -x_1 D_{a_1}^{m_1} D_{a_2}^{m_2+1} \frac{1}{\pi\sigma^2} e^{\frac{-x_1^2-2a_1x_1-a_1^2-x_2^2-2a_2x_2-a_2^2}{\sigma^2}} \right] \frac{d\mathbf{x}}{r(r^2+\epsilon^2)} \Bigg|_{\mathbf{a}=0,\mathbf{b}=0}$$
$$= \frac{(-1)^{k_1+k_2}}{\pi\sigma^2} \int_{\mathbb{R}^2} D_{b_1}^{k_1} D_{b_2}^{k_2}(\beta_1) \left(x_2 D_{a_1}^{m_1+1} D_{a_2}^{m_2} \beta_2 - x_1 D_{a_1}^{m_1} D_{a_2}^{m_2+1} \beta_2 \right) \frac{e^{\frac{-r^2}{\sigma^2}} d\mathbf{x}}{r(r^2+\epsilon^2)},$$

where:

$$\beta_1 = e^{\frac{-b_1^2-b_2^2-2b_1x_1-2b_2x_2}{\lambda^2}}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\lambda^{2n}} \left(\sum_{r=0}^{n} \binom{n}{r} (b_1^2+b_2^2)^{(n-r)} (2b_1x_1+2b_2x_2)^r \right)$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\lambda^{2n}} \sum_{r=0}^{n} \binom{n}{r} \left(\sum_{\ell=0}^{n-r} \binom{n-r}{\ell} b_1^{2(n-r-\ell)} b_2^{2\ell} \right) \left(2^r \sum_{i=0}^{r} \binom{r}{i} (b_1x_1)^{r-i} (b_2x_2)^i \right)$$
$$= \sum_{n=0}^{\infty} \sum_{r=0}^{n} \sum_{\ell=0}^{n-r} \sum_{i=0}^{r} \frac{(-1)^n 2^r}{n!\lambda^{2n}} \binom{n}{r} \binom{n-r}{\ell} \binom{r}{i} b_1^{2(n-r-\ell)+r-i} b_2^{2\ell+i} (x_1^{r-i} x_2^i).$$

So

$$D_{b_2}^{k_2} D_{b_1}^{k_1}(\beta_1) =$$
$$\sum_{n=0}^{\infty} \sum_{r=0}^{n} \sum_{\ell=0}^{n-r} \sum_{i=0}^{r} \frac{(-1)^n 2^r}{n!\lambda^{2n}} \binom{n}{r} \binom{n-r}{\ell} \binom{r}{i} \frac{(2n-2r-2\ell)!}{(2n-2r-2\ell-k_1)!}$$
$$b_1^{2n-2r-2\ell-k_1} \frac{(2\ell+i)!}{(2\ell+i-k_2)!} b_2^{2\ell+i-k_2} \cdot (x_1^{r-i} \cdot x_2^i).$$

Then

$$D_{b_2}^{k_2} D_{b_1}^{k_1}(\beta_1)|_{\mathbf{b=0}} = \tag{B.3}$$
$$\sum_{n=0}^{\infty} \sum_{r=0}^{n} \frac{(-1)^n 2^r}{n!\lambda^{2n}} \binom{n}{r} \binom{r}{k_2-2\ell} \binom{n-r}{\ell} (k_1)!k_2!(x_1^{r-k_2+2\ell} \cdot x_2^{k_2-2\ell}),$$

where $\ell = \frac{2n-2r-k_1}{2}$ and $i = k_2 - 2\ell$.

And the same way:

$$\beta_2 = e^{\frac{-a_1^2-a_2^2-2a_1x_1-2a_2x_2}{\sigma^2}}$$
$$= \sum_{n_2=0}^{\infty} \frac{(-1)^{n_2}}{n_2!\sigma^{2n_2}} \left(\sum_{r_2=0}^{n_2} \binom{n_2}{r_2} (a_1^2+a_2^2)^{(n_2-r_2)} (2a_1x_1+2a_2x_2)^{r_2} \right)$$

$$= \sum_{n_2=0}^{\infty} \frac{(-1)^{n_2}}{n_2!\sigma^{2n_2}} \sum_{r_2=0}^{n_2} \binom{n_2}{r_2} \left(\sum_{\ell_2=0}^{n_2-r_2} \binom{n_2-r_2}{\ell_2} a_1^{2(n_2-r_2-\ell_2)} a_2^{2\ell_2} \right) \cdot \left(2^{r_2} \sum_{i_2=0}^{r_2} \binom{r_2}{i_2} (a_1x_1)^{r_2-i_2} (a_2x_2)^{i_2} \right).$$

So

$$D_{a_2}^{m_2} D_{a_1}^{m_1+1}(\beta_2) = \sum_{n_2=0}^{\infty} \sum_{r_2=0}^{n_2} \frac{(-1)^{n_2} 2^{r}}{n_2!\sigma^{2n_2}} \binom{n_2}{r_2} \binom{r_2}{i_2} \binom{n_2-r_2}{\ell_2} \frac{(2n_2-2r_2-2\ell_2)!}{(2n_2-2r_2-2\ell_2-(m_1+1))!} a_1^{2n_2-2r_2-2\ell_2-(m_1+1)} \frac{(2\ell_2+i_2)!}{(2\ell_2+i_2-m_2)!} a_2^{2\ell_2+i_2-m_2} \cdot (x_1^{r_2-i_2} \cdot x_2^{i_2}),$$

where $\ell_2 = \frac{2n_2-2r_2-(m_1+1)}{2}$ and $i_2 = m_2 - 2\ell_2$. Then we have

$$D_{a_2}^{m_2} D_{a_1}^{m_1+1}(\beta_2)|_{\mathbf{a=0}} = \sum_{n_2=0}^{\infty} \sum_{r_2=0}^{n_2} \frac{(-1)^{n_2} 2^{r_2}}{n_2!\sigma^{2n_2}} \binom{n_2}{r_2} \binom{r_2}{m_2-2\ell_2} \binom{n_2-r_2}{\ell_2} (m_1+1)! m_2! \cdot (x_1^{r_2-m_2+2\ell_2} \cdot x_2^{m_2-2\ell_2}). \tag{B.4}$$

So

$$D_{a_2}^{m_2+1} D_{a_1}^{m_1}(\beta_2)|_{\mathbf{a=0}} = \sum_{n_2=0}^{\infty} \sum_{r_2=0}^{n_2} \frac{(-1)^{n_2} 2^{r_2}}{n_2!\sigma^{2n_2}} \binom{n_2}{r_2} \binom{r_2}{m_2+1-2\ell_2} \binom{n_2-r_2}{\ell_2} \cdot (m_1)!(m_2+1)!(x_1^{r_2-(m_2+1)+2\ell_2} \cdot x_2^{m_2+1-2\ell_2})$$
$$= \sum_{n_2=0}^{\infty} \sum_{r_2=0}^{n_2} \frac{(-1)^{n_2} 2^{r_2}}{n_2!\sigma^{2n_2}} \binom{n_2}{r_2} \frac{r_2-m_2+2\ell_2}{m_2-2\ell_2+1} \binom{r_2}{m_2-2\ell_2} \binom{n_2-r_2}{\ell_2} \cdot (m_1)!(m_2+1)(m_2)!(x_1^{r_2-(m_2+1)+2\ell_2} \cdot x_2^{m_2+1-2\ell_2}),$$

where $\ell_2 = \frac{2n_2-2r_2-m_1}{2}$ and $i_2 = m_2 + 1 - 2\ell_2$. Then we have:

$$D_{a_2}^{m_2+1} D_{a_1}^{m_1}(\beta_2)|_{\mathbf{a=0}} = \frac{(r_2-m_2+2\ell_2)(m_2+1)x_2}{(m_2-2\ell_2+1)(m_1+1)x_1} D_{a_2}^{m_2} D_{a_1}^{m_1+1}(\beta_2)|_{\mathbf{a=0}}. \tag{B.5}$$

So

$$\left(x_2 D_{a_1}^{m_1+1} D_{a_2}^{m_2} \beta_2 - x_1 D_{a_1}^{m_1} D_{a_2}^{m_2+1} \beta_2 \right)|_{\mathbf{a=0}} = x_2 D_{a_1}^{m_1+1} D_{a_2}^{m_2}(\beta_2)|_{\mathbf{a=0}} - \frac{(r_2-m_2+2\ell_2)(m_2+1)x_2}{(m_2-2\ell_2+1)(m_1+1)} D_{a_2}^{m_2} D_{a_1}^{m_1+1}(\beta_2)|_{\mathbf{a=0}} \tag{B.6}$$

$$= \left(1 - \frac{(r_2 - m_2 + 2\ell_2)(m_2+1)}{(m_2 - 2\ell_2 + 1)(m_1+1)}\right) x_2 D_{a_2}^{m_2} D_{a_1}^{m_1+1}(\beta_2)|_{\mathbf{a}=\mathbf{0}}.$$

Then the last integral of (B.2) is as below:

$$\begin{aligned} &\frac{(-1)^{k_1+k_2}}{\pi\sigma^2} \sum_{n=0}^{\infty}\sum_{r=0}^{n}\sum_{n_2=0}^{\infty}\sum_{r_2=0}^{n_2} \binom{n}{r}\binom{r}{k_2-2\ell}\binom{n-r}{\ell} \\ &\cdot \binom{n_2}{r_2}\binom{r_2}{m_2-2\ell_2}\binom{n_2-r_2}{\ell_2} \frac{(-1)^{n+n_2}2^{r+r_2}}{n!n_2!\lambda^{2n}\sigma^{2n_2}} k_1!k_2!(m_1+1)!m_2! \\ &\cdot \left(1 - \frac{(r_2 - m_2 + 2\ell_2)(m_2+1)}{(m_2 - 2\ell_2 + 1)(m_1+1)}\right) \int_{\mathbb{R}^2} x_1^p x_2^q \frac{e^{\frac{-r^2}{\sigma^2}}}{r(r^2+\epsilon^2)} dx, \end{aligned} \tag{B.7}$$

where $q := m_2 - 2\ell_2 + k_2 - 2\ell + 1$, $p := r_2 - m_2 + 2\ell_2 + r - k_2 + 2\ell$. Now by integrating in polar coordinate, we have:

$$\int_{\mathbb{R}^2} x_1^p x_2^q \frac{e^{\frac{-r^2}{\sigma^2}}}{r(r^2+\epsilon^2)} dx = 2\xi_1\xi_2\pi \int_0^{\infty} \frac{r^{p+q} e^{\frac{-r^2}{\sigma^2}}}{r^2+\epsilon^2} dr, \tag{B.8}$$

such that

$$\xi_1 := \prod_{i=0}^{\frac{q}{2}-1} \frac{q-(2i+1)}{q+p-2i}, \qquad \xi_2 := \prod_{i=0}^{\frac{p}{2}-1} \frac{p-(2i+1)}{p-2i}.$$

We know that the last integral and the series of (B.7) are convergence. Then by using (B.1)-(B.8) we have:

$$B[k_1,k_2,m_1,m_2;\lambda;\sigma] = \begin{cases} \xi_1\xi_2 k_1!k_2!(m_1+1)!m_2! & \text{if } p \text{ and} \\ \cdot \sum_{r=0}^{n}\sum_{\ell=0}^{n-r}\sum_{r_2=0}^{n_2}\sum_{\ell_2=0}^{n_2-r_2} \frac{2^{r+r_2+1}(-1)^{k_1+k_2+n+n_2}}{n!n_2!\lambda^{2n}\sigma^{2n_2+2}} & \\ \cdot \binom{n}{r}\binom{r}{k_2-2\ell}\binom{n-r}{\ell}\binom{n_2}{r_2}\binom{r_2}{m_2-2\ell_2}\binom{n_2-r_2}{\ell_2} & q \text{ be even} \\ \cdot \left(1 - \frac{(r_2-m_2+2\ell_2)(m_2+1)}{(m_2-2\ell_2+1)(m_1+1)}\right) \int_0^{\infty} \frac{r^{p+q} e^{\frac{-r^2}{\sigma^2}}}{r^2+\epsilon^2} dr & \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\tilde{B}[k_1,k_2,m_1,m_2;\lambda;\sigma]=\begin{cases}\xi_1\xi_2 k_1!k_2!(m_1+1)!m_2!\Gamma(\frac{1}{2}(p+q+1))|\sigma|^{p+q+1} & \text{if } p \text{ and}\\ \cdot\sum_{r=0}^{n}\sum_{\ell=0}^{n-r}\sum_{r_2=0}^{n_2}\sum_{\ell_2=0}^{n_2-r_2}\frac{(-1)^{k_1+k_2+n+n_2}2^{r+r_2}}{n!n_2!\lambda^{2n}\sigma^{2n_2+2}} & \\ \cdot\binom{n}{r}\binom{r}{k_2-2\ell}\binom{n-r}{\ell}\binom{n_2}{r_2}\binom{r_2}{m_2-2\ell_2}\binom{n_2-r_2}{\ell_2} & q \text{ be even}\\ \cdot\left(1-\frac{(r_2-m_2+2\ell_2)(m_2+1)}{(m_2-2\ell_2+1)(m_1+1)}\right) & \\ 0 & \text{otherwise,}\end{cases}$$

where $\ell=\frac{2n-2r-k_1}{2}$ and $\ell_2=\frac{2n_2-2r_2-(m_1+1)}{2}$ and Γ is the gamma function.

$$\theta[k_1,k_2,l_1,l_2,m_1,m_2;\lambda,\sigma]:=\theta^1[k_1,k_2,l_1,l_2,m_1,m_2;\lambda,\sigma]+\theta^2[k_1,k_2,l_1,l_2,m_1,m_2;\lambda,\sigma].$$

$$\theta^1[k_1,k_2,l_1,l_2,m_1,m_2;\lambda,\sigma]=\sum_{i=0}^{\min(m_1,k_1-1)}\sum_{j=0}^{\min(m_2,k_2)}\frac{\sigma^2}{\pi\lambda^2}\varepsilon\binom{i}{m_1}\binom{j}{m_2}(-1)^{m_1+m_2}\left(\frac{2^{i+1}k_1!}{\sigma^{2(i+1)}(k_1-i-1)!}\right)\cdot\left(\frac{2^jk_2!}{\sigma^{2(j)}(k_2-j)!}\right)\mathcal{R}_1(l_1+k_1-i-1+m_1-i,l_2+k_2-j+m_2-j).$$

$$\theta^2[k_1,k_2,l_1,l_2,m_1,m_2;\lambda,\sigma]=\sum_{i=0}^{\min(m_1,k_1-1)}\sum_{j=0}^{\min(m_2,k_2)}\frac{\sigma^2}{\pi\lambda^2}\varepsilon\binom{i}{m_1}\binom{j}{m_2}(-1)^{m_1+m_2}\left(\frac{2^{i}k_1!}{\sigma^{2i}(k_1-i)!}\right)\cdot\left(\frac{2^{j+1}k_2!}{\sigma^{2(j+1)}(k_2-j-1)!}\right)\mathcal{R}_2(l_1+k_1-i+m_1-i,l_2+k_2-j-1+m_2-j).$$

And $\mathcal{R}_1$ and $\mathcal{R}_2$ give rise to the following :

$$\mathcal{R}_1(\alpha_1,\alpha_2)=\begin{cases}\eta(\alpha_1,\alpha_2,\varepsilon)\binom{\frac{\alpha_1+\alpha_2-1}{2}}{\frac{\alpha_1}{2}} & \text{if } \alpha_1 \text{ even and } \alpha_2 \text{ is odd}\\ 0 & \text{otherwise.}\end{cases}$$

$$\mathcal{R}_2(\alpha_1,\alpha_2)=\begin{cases}\eta(\alpha_1,\alpha_2,\varepsilon)\binom{\frac{\alpha_1+\alpha_2-1}{2}}{\frac{\alpha_1-1}{2}} & \text{if } \alpha_1 \text{ odd and } \alpha_2 \text{ is even}\\ 0 & \text{otherwise,}\end{cases}$$

where $\eta(\alpha_1,\alpha_2,\delta)=-\frac{1}{2\pi}\left(\frac{1}{2\delta^2}\right)^{\frac{\alpha_1+\alpha_2+1}{2}}\frac{(-1)^{(\alpha_1+\alpha_2-1)/2}}{((\alpha_1+\alpha_2+1)/2)!}(\alpha_1)!(\alpha_2)!$.

$$\tilde{\Gamma}[k_1,k_2,l_1,l_2,m_1,m_2;\lambda]:=\Gamma^1[k_1,k_2,l_1,l_2,m_1,m_2;\lambda]+\Gamma^2[k_1,k_2,l_1,l_2,m_1,m_2;\lambda].$$

$$\Gamma^1[k_1,k_2,l_1,l_2,m_1,m_2;\lambda] = \sum_{i=0}^{\min(m_1,k_1-1)} \sum_{j=0}^{\min(m_2,k_2)} \binom{m_1}{i}\binom{m_2}{j}(-1)^{m_1+m_2}\left(\frac{2^{i+1}k_1!}{\lambda^{2(i+1)}(k_1-i-1)!}\right)\left(\frac{2^{j}k_2!}{\lambda^{2(j)}(k_2-j)!}\right)\cdot \mathcal{H}_1(l_1+k_1-i-1+m_1-i, l_2+k_2-j+m_2-j).$$

$$\Gamma^2[k_1,k_2,l_1,l_2,m_1,m_2;\lambda] = \sum_{i=0}^{\min(m_1,k_1)} \sum_{j=0}^{\min(m_2,k_2-1)} \binom{m_1}{i}\binom{m_2}{j}(-1)^{m_1+m_2}\left(\frac{2^{i}k_1!}{\lambda^{2(i)}(k_1-i)!}\right)\left(\frac{2^{j+1}k_2!}{\lambda^{2(j+1)}(k_2-j-1)!}\right)\cdot \mathcal{H}_2(l_1+k_1-i+m_1-i, l_2+k_2-j-1+m_2-j).$$

And $\mathcal{H}_1$ and $\mathcal{H}_2$ give rise to the following :

$$\mathcal{H}_1(\alpha_1,\alpha_2) = \begin{cases} \eta(\alpha_1,\alpha_2,\lambda)\binom{\frac{\alpha_1+\alpha_2-1}{2}}{\frac{\alpha_1}{2}} & \text{if } \alpha_1 \text{ even and } \alpha_2 \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathcal{H}_2(\alpha_1,\alpha_2) = \begin{cases} \eta(\alpha_1,\alpha_2,\lambda)\binom{\frac{\alpha_1+\alpha_2-1}{2}}{\frac{\alpha_1-1}{2}} & \text{if } \alpha_1 \text{ odd and } \alpha_2 \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

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