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Approximation Methods for Solving Quasi-equilibrium Problems in Hadamard Spaces

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Abstract. In this paper, we consider quasi-eqilibrium problems which extend equilibrium problems and quasi-variational inequalities as well as variational inequalities in Hadamard spaces. We study Δ-convergence of the sequence generated by the extragradient method to a solution of a quasi-equilibrium problem in Hadamard spaces. Then we show strong convergence of the generated sequence to a solution of the problem by imposing some additional conditions. We also use the Halpern regularization method to prove strong convergence of the generated sequence to a solution of the quasi-equilibrium problem where the equilibrium point is the projection of an arbitrary point u onto the solution set of the problem.

Keywords: Extragradient method, Halpern regularization, Multivalued mapping, Quasi-equilibrium problem, Quasi-nonexpansive.

2000 Mathematics subject classification: 90C33, 74G10.

1. Introduction

Let (X, d) be a metric space. For $x, y \in X$, a mapping $c : [0, l] \to X$, where $l \geq 0$, is called a geodesic with endpoints x, y, if $c(0) = x$, $c(l) = y$, and

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$d(c(t), c(t')) = t - t'$ for all $t, t' \in [0, l]$. If, for every $x, y \in X$, a geodesic with endpoints x, y exists, then we call (X, d) a geodesic metric space. Furthermore, if there exists a unique geodesic for each $x, y \in X$, then (X, d) is said to be uniquely geodesic.

A subset C of a uniquely geodesic space X is said to be convex when for any two points $x, y \in C$, the geodesic joining x and y is contained in C. For each $x, y \in X$, the image of a geodesic c with endpoints x, y is called a geodesic segment joining x and y and is denoted by $[x, y]$.

Let X be a uniquely geodesic metric space. For each $x, y \in X$ and for each $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that $d(x, z) = td(x, y)$ and $d(y, z) = (1-t)d(x, y)$. We will use the notation $(1-t)x \oplus ty$ for denoting the unique point z satisfying the above statement.

Definition 1.1. [11] A geodesic space X is called CAT(0) space if for all $x, y, z \in X$ and $t \in [0, 1]$ it holds that

$$d^2(tx \oplus (1-t)y, z) \leq td^2(x, z) + (1-t)d^2(y, z) - t(1-t)d^2(x, y).$$

A complete CAT(0) space is called an Hadamard space.

Berg and Nikolaev in [4, 5] introduced the concept of quasi-linearization as follows. Let us formally denote a pair $(a, b) \in X \times X$ as $\overrightarrow{ab}$ and call it a vector. Then quasi-linearization is characterized as a map $\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}\{d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d)\}, \qquad a, b, c, d \in X.$$

It is easy to see that $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle$ and $\langle \overrightarrow{ax}, \overrightarrow{cd} \rangle + \langle \overrightarrow{xb}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$ for all $a, b, c, d, x \in X$. We say that X satisfies the Cauchy-Schwarz inequality if $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq d(a, b)d(c, d)$ for all $a, b, c, d \in X$. It is known (Corollary 3 of [5]) that a geodesically connected metric space is a CAT(0) space if and only if it satisfies the Cauchy-Schwarz inequality.

Let (X, d) be an Hadamard space and $\{x_n\}$ be a bounded sequence in X. Take $x \in X$. Let $r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n)$. The asymptotic radius of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) | x \in X\},$$

and the asymptotic center of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X | r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known that in an Hadamard space, $A(\{x_n\})$ consists exactly one point.

Definition 1.2. (see [24], p. 3690) A sequence $\{x_n\}$ in an Hadamard space (X, d) Δ-converges to $x \in X$ if $A(\{x_{n_k}\}) = \{x\}$, for each subsequence $\{x_{n_k}\}$ of $\{x_n\}$.

We denote Δ-convergence in X by $\xrightarrow{\Delta}$ and the metric convergence by $\to$. Now, we present a known result related to the notion of Δ-convergence.

Lemma 1.3 ([24], Proposition 3.6)**.** *Let X be an Hadamard space. Then, every bounded, closed and convex subset of X is Δ-compact; i.e. every bounded sequence in it, has a Δ-convergent subsequence.*

Lemma 1.4 ([11])**.** *Let (X,d) be a CAT(0) space. Then, for all $x,y,z \in X$ and $t \in [0,1]$, it holds that*

$$d(tx \oplus (1-t)y, z) \leq td(x,z) + (1-t)d(y,z).$$

Let $C \subset X$ be nonempty, closed and convex. It is well known for any $x \in X$ there exists a unique $u \in C$ such that

$$d(u,x) = \inf\{d(z,x) : z \in C\}. \tag{1.1}$$

We define the *projection on* C, $P_C : X \to C$, by taking $P_C(x)$ as the unique $u \in C$ which satisfies (1.1). We give next a characterization of the projection.

Proposition 1.5. ([10]) *Let C be a nonempty convex subset of a CAT(0) space X, $x \in X$ and $u \in C$. Then $u = P_C(x)$ if and only if*

$$\langle \overrightarrow{yu}, \overrightarrow{xu} \rangle \leq 0,$$

for all $y \in C$.

Let X be an Hadamard space and $C \subset X$ be a nonempty, closed and convex set, and $K : C \to 2^C$ be a multivalued mapping such that for all $x \in C$, $K(x)$ is a nonempty, closed and convex subset of C. Suppose that $f : X \times X \to \mathbb{R}$ is a bifunction. The quasi-equilibrium problem (QEP(f,K)) is to find $x^* \in K(x^*)$ such that

$$f(x^*, y) \geq 0, \qquad \forall y \in K(x^*). \tag{1.2}$$

The set of all solutions of QEP(f,K) is denoted by $S(f,K)$. Also, the set of all fixed points of K is denoted by Fix(K). The associated Minty quasi-equilibrium problem is to find $x^* \in K(x^*)$ such that $f(y,x^*) \leq 0$ for all $y \in K(x^*)$. When $K(x) = C$ for all $x \in C$, the quasi-equilibrium problem QEP(f,K) becomes a classical equilibrium problem EP(f,C), also the associated Minty quasi-equilibrium problem becomes a classical Minty equilibrium problem (see [12]).

The equilibrium problem encompasses, among its particular cases, convex optimization problems, variational inequalities, fixed point problems, Nash equilibrium problems, and other problems of interest in many applications. Equilibrium problems have been studied extensively in Hilbert, Banach as well as in topological vector spaces by many authors e.g. ([6, 7, 8, 13, 14, 15, 17, 19, 26, 27]).

Recently the extragradient method with and without linesearch for equilibrium problems in Hadamard spaces has been studied in [18] and [21]. Also, the quasi-equilibrium problems have been studied in [1, 12, 25] and [29].

In this paper, we study an extragradient method to approximate a solution of quasi-equilibrium problems in Hadamard spaces. Hence we need the following definitions.

Definition 1.6. The mapping $T : C \to C$ is called quasi nonexpansive whenever $\mathrm{Fix}(T) \neq \emptyset$ and $d(p, Tx) \leq d(p, x)$ for all $(p, x) \in \mathrm{Fix}(T) \times C$.

The following definitions are adapted from [12].

Definition 1.7. Suppose that $K : C \to 2^C$ is a multivalued mapping such that for every $x \in C$, $K(x)$ is nonempty, closed and convex. K is called quasi nonexpansive whenever the mapping $T(\cdot) := P_{K(\cdot)}(\cdot)$ is quasi nonexpansive where P is the projection mapping.

Definition 1.8. We say that $K : C \to 2^C$ is demiclosed, whenever we have $x_k \xrightarrow{\Delta} \bar{x}$ and $\lim_{k\to\infty} d(x_k, K(x_k)) = 0$, then $\bar{x} \in \mathrm{Fix}(K)$.

Lemma 1.9. (Lemma 3.18 of [22]) *Let $T : C \to C$ be a quasi-nonexpansive mapping, then* $\mathrm{Fix}(T)$ *is closed and convex.*

We introduce now some conditions on the bifunction f and the multivalued mapping K which are needed in the convergence analysis.

$B1$: $f(x, \cdot) : X \to$ is convex and lower semicontinuous for all $x \in X$.

$B2$: $f(\cdot, y)$ is Δ-upper semicontinuous for all $y \in X$.

$B3$: f is Lipschitz-type continuous, i.e. there exist two positive constants c_1 and c_2 such that

$$f(x, y) + f(y, z) \geq f(x, z) - c_1 d^2(x, y) - c_2 d^2(y, z), \quad \forall x, y, z \in X.$$

$B4$: f is pseudo-monotone, i.e. whenever $f(x, y) \geq 0$ with $x, y \in X$, it holds that $f(y, x) \leq 0$.

$B5$: $K : C \to 2^C$ is a quasi-nonexpansive and demiclosed mapping with nonempty, closed and convex values.

In connection with $B2$, it is valuable to mention that a concave and upper semicontinuous function is always Δ-upper semicontinuous. Also, note that $B3$ together with $B4$ imply that $f(x, x) = 0$ for all $x \in X$. Finally, in order to well definedness and boundedness of the generated sequences by our algorithm in this paper, we assume that

$$S^* = \Big\{ x \in K(x) : f(x, y) \geq 0, \quad \forall y \in C \Big\} \neq \emptyset.$$

Note that under B1–B5, S^* is closed and convex. It is easy to see that $S^* \subset S(f, K)$.

The paper is organized as follows. In Section 2, we present an extragradient method for quasi-equilibrium problems in Hadamard spaces for proving

Δ-convergence of the generated sequence to a solution of the problem. In Section 3, we prove the strong convergece of the generated sequence to a solution of the problem by imposing some additional conditions. In Section 4, we propose a variant of the extragradient method for which the generated sequence is strongly convergent to a solution of the problem without any additional conditions on it.

2. Δ-convergence of the extragradient method

In this section, we study Δ-convergence of the sequence generated by the extragradient method to a solution of a quasi-equilibrium problem under appropriate assumptions on the problem. Let $C \subset X$ be a nonempty, closed and convex set of an Hadamard space X, and $K : C \to 2^C$ be a multivalued quasi-nonexpansive mapping, and $f : X \times X \to$. We assume that the bifunction f satisfies $B1, B2, B3$, $B4$, the multivalued mapping K satisfies $B5$. Next, we propose the following extragradient method for solving QEP(f, K).

Initialization: $z_0 \in C$, $n := 0$, $0 < \alpha \leq \lambda_k \leq \beta < \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\}$ and $0 < \gamma < \beta_k < \delta < 1$ for all k.
Iterative Step : Given z_n, define

$$w_n = P_{K(z_n)}(z_n), \tag{2.1}$$

$$x_n = \beta_n z_n \oplus (1 - \beta_n) w_n, \tag{2.2}$$

$$y_n = \operatorname{argmin}_{y \in C} \left\{ f(x_n, y) + \frac{1}{2\lambda_n} d^2(x_n, y) \right\}, \tag{2.3}$$

$$z_{n+1} = \operatorname{argmin}_{y \in C} \left\{ f(y_n, y) + \frac{1}{2\lambda_n} d^2(x_n, y) \right\}. \tag{2.4}$$

Since K is a multivalued mapping with nonempty, closed and convex values, it is easy to see that the sequences $\{w_n\}$ and $\{x_n\}$ are well defined. Now we claim that the sequences $\{y_n\}$ and $\{z_n\}$ are well defined. Let $\varphi : X \to (-\infty, +\infty]$ be a proper, convex and lower semicontinuous function. The resolvent of φ of order $\lambda > 0$ is defined at each point $x \in X$ as follows.

$$J_\lambda^\varphi x := \operatorname{argmin}_{y \in X} \left\{ \varphi(y) + \frac{1}{2\lambda} d^2(y, x) \right\}.$$

By Lemma 3.1.2 of [20] (see also Lemma 2.2.19 of [2]) for each $x \in X$, $J_\lambda^\varphi x$ exists. This shows the sequences $\{y_n\}$ and $\{z_n\}$ are well defined.

In order to prove Δ-convergence of the sequences generated by the extragradient method to a solution of the problem, we need to adapt the following lemma from [21].

Lemma 2.1. *Assume that* $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ *and* $\{w_n\}$ *are generated by the algorithm and* $x^* \in S^*$, *then*

$$\begin{aligned} d^2(z_{n+1}, x^*) &\leq d^2(z_n, x^*) - \beta_n(1-\beta_n)d^2(z_n, w_n) \\ &\quad - (1-2c_1\lambda_n)d^2(x_n, y_n) - (1-2c_2\lambda_n)d^2(y_n, z_{n+1}). \end{aligned}$$

Proof. Take $x^* \in S^*$ and suppose that $y = tz_{n+1} \oplus (1-t)x^*$ where $t \in [0,1)$. Then, by (2.4) and B1, we have

$$\begin{aligned} & f(y_n, z_{n+1}) + \frac{1}{2\lambda_n}d^2(x_n, z_{n+1}) \leq f(y_n, y) + \frac{1}{2\lambda_n}d^2(x_n, y) \\ & \leq tf(y_n, z_{n+1}) + (1-t)f(y_n, x^*) \\ & + \frac{1}{2\lambda_n}\{td^2(x_n, z_{n+1}) + (1-t)d^2(x_n, x^*) - t(1-t)d^2(z_{n+1}, x^*)\}. \end{aligned}$$

Since $f(y_n, x^*) \leq 0$, by $B4$, we get

$$f(y_n, z_{n+1}) \leq \frac{1}{2\lambda_n}\{d^2(x_n, x^*) - d^2(x_n, z_{n+1}) - td^2(z_{n+1}, x^*)\}.$$

By letting $t \to 1^-$ we get

$$f(y_n, z_{n+1}) \leq \frac{1}{2\lambda_n}\{d^2(x_n, x^*) - d^2(x_n, z_{n+1}) - d^2(z_{n+1}, x^*)\}. \tag{2.5}$$

Now, let $y = ty_n \oplus (1-t)z_{n+1}$ such that $t \in [0,1)$, then by (2.3), we have

$$\begin{aligned} & f(x_n, y_n) + \frac{1}{2\lambda_n}d^2(x_n, y_n) \leq f(x_n, y) + \frac{1}{2\lambda_n}d^2(x_n, y) \\ & \leq tf(x_n, y_n) + (1-t)f(x_n, z_{n+1}) \\ & + \frac{1}{2\lambda_n}\{td^2(x_n, y_n) + (1-t)d^2(x_n, z_{n+1}) - t(1-t)d^2(y_n, z_{n+1})\}. \end{aligned}$$

Then we obtain

$$f(x_n, y_n) - f(x_n, z_{n+1}) \leq \frac{1}{2\lambda_n}\{d^2(x_n, z_{n+1}) - d^2(x_n, y_n) - td^2(y_n, z_{n+1})\}.$$

Now, if $t \to 1^-$ we get

$$f(x_n, y_n) - f(x_n, z_{n+1}) \leq \frac{1}{2\lambda_n}\{d^2(x_n, z_{n+1}) - d^2(x_n, y_n) - d^2(y_n, z_{n+1})\}. \tag{2.6}$$

Since f is Lipschitz-type continuous with constants c_1 and c_2, we have

$$-c_1d^2(x_n, y_n) - c_2d^2(y_n, z_{n+1}) + f(x_n, z_{n+1}) - f(x_n, y_n) \leq f(y_n, z_{n+1}). \tag{2.7}$$

Note that by (2.6) and (2.7), we obtain

$$(\frac{1}{2\lambda_n} - c_1)d^2(x_n, y_n) + (\frac{1}{2\lambda_n} - c_2)d^2(y_n, z_{n+1}) - \frac{1}{2\lambda_n}d^2(x_n, z_{n+1}) \leq f(y_n, z_{n+1}). \tag{2.8}$$

Now (2.5) and (2.8) imply that

$$(1-2c_1\lambda_n)d^2(x_n, y_n) + (1-2c_2\lambda_n)d^2(y_n, z_{n+1}) \leq d^2(x_n, x^*) - d^2(z_{n+1}, x^*). \tag{2.9}$$

In the sequel, since $x_n = \beta_n z_n \oplus (1-\beta_n)w_n$ by (2.2), we have

$$d^2(x_n, x^*) \leq \beta_n d^2(z_n, x^*) + (1-\beta_n)d^2(w_n, x^*) - \beta_n(1-\beta_n)d^2(z_n, w_n). \tag{2.10}$$

Now since $x^* \in K(x^*)$, $w_n = P_{K(z_n)}(z_n)$ and K is a quasi nonexpansive mapping, hence we have

$$d(w_n, x^*) \leq d(z_n, x^*).$$

Therefore (2.10) implies that

$$d^2(x_n, x^*) \leq d^2(z_n, x^*) - \beta_n(1-\beta_n)d^2(z_n, w_n). \tag{2.11}$$

Now, (2.9) and (2.11) show that

$$\begin{aligned} d^2(z_{n+1}, x^*) \leq\, & d^2(z_n, x^*) - \beta_n(1-\beta_n)d^2(z_n, w_n) \\ & - (1-2c_1\lambda_n)d^2(x_n, y_n) - (1-2c_2\lambda_n)d^2(y_n, z_{n+1}). \end{aligned}$$

□

Remark 2.2. In Lemma 2.1, it is obvious that $\lim_{n\to\infty} d(z_n, x^*)$ exists and hence $\{z_n\}$ is bounded. Note that $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$ and $\liminf_{n\to\infty}(1-2c_i\lambda_n) > 0$ for $i = 1, 2$. Thus we conclude from Lemma 2.1 that

$$\lim_{n\to\infty} d(z_n, w_n) = \lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(y_n, z_{n+1}) = 0. \tag{2.12}$$

Therefore the sequences $\{x_n\}$, $\{y_n\}$ and $\{w_n\}$ are bounded. Moreover, using (2.5) and (2.8), and then by taking limit, we can conclude that

$$\lim_{n\to\infty} f(y_n, z_{n+1}) = 0.$$

Theorem 2.3. *Assume that the bifunction f satisfies $B1, B2, B3$ and $B4$, and the multivalued mapping K satisfies $B5$, and $S^* \neq \emptyset$. Then the sequence $\{x_n\}$ generated by the algorithm, is Δ-convergent to a point of $S(f, K)$.*

Proof. Remark 2.2 shows that $\{x_n\}$ is bounded, hence there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $p \in C$ such that $x_{n_k} \xrightarrow{\Delta} p$. We first prove that $p \in \mathrm{Fix}(K)$. Note that $z_{n_k} \xrightarrow{\Delta} p$ and $w_{n_k} \xrightarrow{\Delta} p$ because $x_n = \beta_n z_n \oplus (1-\beta_n)w_n$ and $\lim_{k\to\infty} d(z_{n_k}, w_{n_k}) = 0$ by Remark 2.2. Since $\lim_{k\to\infty} d(z_{n_k}, w_{n_k}) = 0$, and K is demiclosed, thus $p \in K(p)$, i.e. p is a fixed point of $K(\cdot)$. Now we prove that $p \in S(f, K)$.

Note that z_{n+1} solves the minimization problem in (2.4). Let $z = tz_{n+1} \oplus (1-t)y$ such that $t \in [0, 1)$ and $y \in C$, then we get

$$\begin{aligned} & f(y_n, z_{n+1}) + \frac{1}{2\lambda_n}d^2(x_n, z_{n+1}) \leq f(y_n, z) + \frac{1}{2\lambda_n}d^2(x_n, z) \\ & \leq tf(y_n, z_{n+1}) + (1-t)f(y_n, y) \\ & + \frac{1}{2\lambda_n}\{td^2(x_n, z_{n+1}) + (1-t)d^2(x_n, y) - t(1-t)d^2(z_{n+1}, y)\}. \end{aligned}$$

The above inequality implies that

$$f(y_n, z_{n+1}) - f(y_n, y) \leq \frac{1}{2\lambda_n}\{d^2(x_n, y) - d^2(x_n, z_{n+1}) - td^2(z_{n+1}, y)\}.$$

Now, if $t \to 1^-$ we obtain

$$\frac{1}{2\lambda_n}\{d^2(x_n, z_{n+1}) + d^2(z_{n+1}, y) - d^2(x_n, y)\} \leq f(y_n, y) - f(y_n, z_{n+1}). \quad (2.13)$$

Therefore we have

$$\frac{-1}{2\lambda_n} d(x_n, z_{n+1})\{d(z_{n+1}, y) + d(x_n, y)\} \leq f(y_n, y) - f(y_n, z_{n+1}). \qquad (2.14)$$

Now since $x_{n_k} \xrightarrow{\Delta} p$, we have $y_{n_k} \xrightarrow{\Delta} p$ by (2.12). Replacing n by n_k in (2.14), taking limsup and using Remark 2.2, since $f(\cdot, y)$ is Δ-upper semicontinuous, we have:

$$0 \leq \limsup_{k\to\infty} f(y_{n_k}, y) \leq f(p, y), \qquad \forall y \in C.$$

Therefore, $p \in S(f, K)$. Finally, since $\lim_{n\to\infty} d(p, z_n)$ exists for each Δ-limit point of $\{x_n\}$ like p, therefore Opial's Lemma in Hadamard spaces (see Lemma 2.1 in [23]) implies that $\{x_n\}$ Δ-converges to a point of $S(f, K)$. □

3. Strong convergence of the extragradient method

In this section, we study the strong convergence of the sequence generated by the extragradient method to an element of $S(f, K)$ with some additional assumptions on the problem.

Definition 3.1. A bifunction f is called strongly monotone, if there exists $\alpha > 0$ such that $f(x, y) + f(y, x) \leq -\alpha d^2(x, y)$ for all $x, y \in X$.
Also, a bifunction f is called strongly pseudo-monotone, if there exists $\beta > 0$ such that whenever $f(x, y) \geq 0$, then $f(y, x) \leq -\beta d^2(x, y)$ for all $x, y \in X$.

Definition 3.2. A function $h : X \to$ is called strongly convex, whenever for each pair $x, y \in X$ and each $\lambda \in [0, 1]$, we have

$$h(\lambda x \oplus (1-\lambda)y) \leq \lambda h(x) + (1-\lambda)h(y) - \lambda(1-\lambda)d^2(x, y).$$

We say that h is strongly concave whenever $-h$ is strongly convex.

Example 3.3. Suppose that $g : X \times X \to \mathbb{R}$ is monotone. If $\alpha : X \times X \to \mathbb{R}^+$, then $f(x, y) := \alpha(x, y)g(x, y)$ is pseudo-monotone but it is not necessarily monotone. Now if $\alpha(x, y) \geq c > 0$ and g is strongly monotone, then $f(x, y) = \alpha(x, y)g(x, y)$ is strongly pseudo-monotone.

Theorem 3.4. *Suppose that the assumptions of Theorem 2.3 are satisfied. If any of the following conditions is satisfied,*
i) f is strongly pseudo-monotone,
ii) $f(x, \cdot)$ is strongly convex for all $x \in X$,
iii) $f(\cdot, y)$ is strongly concave for all $y \in X$,

then the sequence $\{x_n\}$ *generated by the algorithm is strongly convergent to an element of* $S(f,K)$*.*

Proof. Note that by Theorem 2.3, the sequence $\{x_n\}$ is Δ-convergent to a point of $S(f,K)$. In the sequel, in each item, we prove that the sequence $\{x_n\}$ is strongly convergent to an element of $S(f,K)$. Therefore we suppose that $x_n \xrightarrow{\Delta} x^*$ and subsequently $y_n \xrightarrow{\Delta} x^*$ by Remark 2.2. Replacing y by x^* in (2.14), taking liminf from (2.14) and using Remark 2.2, we have

$$\liminf_{n\to\infty} f(y_n, x^*) \geq 0. \tag{3.1}$$

On the other hand, since $f(y_n, x^*) \leq 0$ for all n, therefore we get

$$\lim_{n\to\infty} f(y_n, x^*) = 0. \tag{3.2}$$

In the sequel, we continue to prove parts (i), (ii) and (iii) respectively:
i) Since $f(x^*, y_n) \geq 0$, by the definition of the strongly pseudo-monotone bifunction, there is $\beta > 0$ such that $f(y_n, x^*) \leq -\beta d^2(y_n, x^*)$. By taking liminf and using (3.1), we get

$$0 \leq \liminf_{n\to\infty} f(y_n, x^*) \leq \liminf_{n\to\infty}(-\beta d^2(y_n, x^*) \leq -\beta \limsup_{n\to\infty} d^2(y_n, x^*)).$$

Therefore $y_n \to x^*$ and subsequently $x_n \to x^*$ by Remark 2.2.
ii) By (2.14), we have

$$\frac{-1}{2\lambda_n} d(x_n, z_{n+1})\{d(z_{n+1}, y) + d(x_n, y)\} \leq f(y_n, y) - f(y_n, z_{n+1}). \tag{3.3}$$

Now, let $\lambda \in (0,1)$ and set $p_n = \lambda y_n \oplus (1-\lambda)x^*$ for all $n \in \mathbb{N}$. Since $f(y_n, \cdot)$ is strongly convex, we have

$$\begin{aligned}
\frac{-1}{2\lambda_n} d(x_n, z_{n+1})&\{d(z_{n+1}, p_n) + d(x_n, p_n)\} \leq f(y_n, p_n) - f(y_n, z_{n+1}) \\
&\leq \lambda f(y_n, y_n) + (1-\lambda) f(y_n, x^*) - \lambda(1-\lambda) d^2(y_n, x^*) - f(y_n, z_{n+1}) \\
&= (1-\lambda) f(y_n, x^*) - \lambda(1-\lambda) d^2(y_n, x^*) - f(y_n, z_{n+1}).
\end{aligned}$$

Note that $f(y_n, y_n) = 0$ by $B3$ and $B4$. Hence, we have

$$\begin{aligned}
\lambda(1-\lambda) d^2(y_n, x^*) \leq{}& \frac{1}{2\lambda_n} d(x_n, z_{n+1})\{d(z_{n+1}, p_n) + d(x_n, p_n)\} \\
&+ (1-\lambda) f(y_n, x^*) - f(y_n, z_{n+1}).
\end{aligned} \tag{3.4}$$

Taking limit from (3.4), we use (3.2) together with Remark 2.2 and the boundedness of $\{p_n\}$ in order to obtain that $d(y_n, x^*) \to 0$. Therefore we have $x_n \to x^*$.
iii) Let $\lambda \in (0,1)$ and set $p_n = \lambda y_n \oplus (1-\lambda)x^*$ for all $n \in \mathbb{N}$. Then since $f(\cdot, x^*)$ is strongly concave, we have

$$\lambda f(y_n, x^*) + (1-\lambda) f(x^*, x^*) + \lambda(1-\lambda) d^2(y_n, x^*) \leq f(p_n, x^*) \leq 0.$$

Therefore, we get $f(y_n, x^*) \le -(1-\lambda)d^2(y_n, x^*)$. Now, by taking liminf and (3.1), we get

$$0 \le \liminf_{n\to\infty} f(y_n, x^*) \le -(1-\lambda)\limsup_{i\to\infty} d^2(y_n, x^*).$$

Therefore $y_n \to x^*$. This implies that $x_n \to x^*$.

□

4. Halpern's regularization method

In this section, we modify our method in Section 2 by adding a step to the algorithm which ensures strong convergence of the generated sequence to a solution of QEP(f, K) (See (4.5)). We show that the generated sequence is strongly convergent to the projection of u onto the solution set S^*. Let $C \subset X$ be a nonempty, closed and convex set of an Hadamard space X, and $K : C \to 2^C$ be a multivalued quasi-nonexpansive mapping, and $f : X \times X \to$. We assume that the bifunction f satisfies $B1, B2, B3$, $B4$, the multivalued mapping K satisfies $B5$, and $S^* \neq \emptyset$. Next, we propose the following Halpern's regularization of the extragradient method for solving this problem.

Initialization: $v_0, u \in C$, $n := 0$, $0 < \alpha \le \lambda_k \le \beta < \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\}$ and $0 < \gamma < \beta_k < \delta < 1$ for all k. Take $\{\alpha_k\} \subset (0,1)$ such that $\lim_{k\to\infty} \alpha_k = 0$ and $\sum_{k=0}^{\infty} \alpha_k = +\infty$.
Iterative Step : Given v_n, define

$$w_n = P_{K(v_n)}(v_n), \tag{4.1}$$

$$x_n = \beta_n v_n \oplus (1-\beta_n) w_n, \tag{4.2}$$

$$y_n = \text{argmin}_{y\in C}\left\{f(x_n, y) + \frac{1}{2\lambda_n} d^2(x_n, y)\right\}, \tag{4.3}$$

$$z_n = \text{argmin}_{y\in C}\left\{f(y_n, y) + \frac{1}{2\lambda_n} d^2(x_n, y)\right\}, \tag{4.4}$$

$$v_{n+1} = \alpha_n u \oplus (1-\alpha_n) z_n. \tag{4.5}$$

Similar to the previous section, it is easy to see that the generated sequences are well defined. In order to prove the strong convergence result by our algorithm, we need the following lemmas.

Lemma 4.1. *Assume that the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, $\{v_n\}$ and $\{w_n\}$ are generated by the algorithm and $x^* \in S^*$, then*

$$\begin{aligned} d^2(z_n, x^*) \le\ & d^2(v_n, x^*) - \beta_n(1-\beta_n) d^2(v_n, w_n) \\ & - (1-2c_1\lambda_n) d^2(x_n, y_n) - (1-2c_2\lambda_n) d^2(y_n, z_n). \end{aligned}$$

Proof. Similar to the proof of Lemma 2.1 □

Lemma 4.2. [28] *Let $\{s_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ be a sequence of real numbers in $(0,1)$ with $\sum_{n=0}^{\infty}\alpha_n = +\infty$ and $\{t_n\}$ be a sequence of real numbers. Suppose that*

$$s_{n+1} \leq (1-\alpha_n)s_n + \alpha_n t_n \quad \textit{for all} \quad n \geq 1.$$

If $\limsup_{k\to\infty} t_{n_k} \leq 0$ for every subsequence $\{s_{n_k}\}$ of $\{s_n\}$ satisfying $\liminf_{k\to\infty}(s_{n_k+1} - s_{n_k}) \geq 0$, then $\lim_{n\to\infty} s_n = 0$.

Theorem 4.3. *Assume that the bifunction f satisfies $B1, B2, B3$ and $B4$, the multivalued mapping K satisfies $B5$, and $S^* \neq \emptyset$. Then the sequence $\{x_n\}$ generated by the algorithm converges strongly to $P_{S^*}u$.*

Proof. Let $x^* = P_{S^*}u$. Lemma 4.1 shows that

$$d(z_n, x^*) \leq d(v_n, x^*). \tag{4.6}$$

By (4.5) and (4.6), we obtain

$$\begin{aligned} d(v_{n+1}, x^*) &\leq \alpha_n d(u, x^*) + (1-\alpha_n)d(z_n, x^*) \\ &\leq \alpha_n d(u, x^*) + (1-\alpha_n)d(v_n, x^*) \\ &\leq \max\{d(u, x^*), d(v_n, x^*)\} \\ &\leq \cdots \leq \max\{d(u, x^*), d(v_0, x^*)\}, \end{aligned}$$

which implies that $\{v_n\}$ is bounded. Thus, by (4.6), $\{z_n\}$ is also bounded. On the other hand, (4.5) and (4.6) imply

$$\begin{aligned} d^2(v_{n+1}, x^*) &\leq (1-\alpha_n)d^2(z_n, x^*) + \alpha_n d^2(u, x^*) - \alpha_n(1-\alpha_n)d^2(u, z_n) \\ &\leq (1-\alpha_n)d^2(v_n, x^*) + \alpha_n d^2(u, x^*) - \alpha_n(1-\alpha_n)d^2(u, z_n). \end{aligned} \tag{4.7}$$

We are going to prove $d^2(v_n, x^*) \to 0$. By Lemma 4.2, it suffices to show that $\limsup_{k\to\infty}(d^2(u, x^*) - (1-\alpha_{n_k})d^2(u, z_{n_k})) \leq 0$ for every subsequence $\{d^2(v_{n_k}, x^*)\}$ of $\{d^2(v_n, x^*)\}$ satisfying $\liminf_{k\to\infty}(d^2(v_{n_k+1}, x^*) - d^2(v_{n_k}, x^*)) \geq 0$. Consider such a subsequence. We have

$$\begin{aligned} 0 &\leq \liminf_{k\to\infty}(d^2(v_{n_k+1}, x^*) - d^2(v_{n_k}, x^*)) \\ &\leq \liminf_{k\to\infty}(\alpha_{n_k} d^2(x^*, u) + (1-\alpha_{n_k})d^2(x^*, z_{n_k}) - d^2(x^*, v_{n_k})) \\ &= \liminf_{k\to\infty}(\alpha_{n_k}(d^2(x^*, u) - d^2(x^*, z_{n_k})) + d^2(x^*, z_{n_k}) - d^2(x^*, v_{n_k})) \\ &\leq \limsup_{k\to\infty} \alpha_{n_k}(d^2(x^*, u) - d^2(x^*, z_{n_k})) + \liminf_{k\to\infty}(d^2(x^*, z_{n_k}) - d^2(x^*, v_{n_k})) \\ &= \liminf_{k\to\infty}(d^2(x^*, z_{n_k}) - d^2(x^*, v_{n_k})) \\ &\leq \limsup_{k\to\infty}(d^2(x^*, z_{n_k}) - d^2(x^*, v_{n_k})) \leq 0. \end{aligned}$$

This shows that

$$\lim_{k\to\infty}(d^2(x^*, z_{n_k}) - d^2(x^*, v_{n_k})) = 0. \tag{4.8}$$

Since $\liminf_{n\to\infty}\beta_n(1-\beta_n)>0$ and $\liminf_{n\to\infty}(1-2c_i\lambda_n)>0$ for $i=1,2$, replacing n by n_k in Lemma 4.1, we conclude that

$$\lim_{k\to\infty} d^2(x_{n_k},y_{n_k})=\lim_{k\to\infty} d^2(y_{n_k},z_{n_k})=\lim_{k\to\infty} d^2(v_{n_k},w_{n_k})=0. \tag{4.9}$$

In the sequel, similar to (2.5) and (2.8), replacing z_{n+1} by z_n, we get

$$f(y_n,z_n)\le \frac{1}{2\lambda_n}\{d^2(x_n,x^*)-d^2(x_n,z_n)-d^2(z_n,x^*)\}. \tag{4.10}$$

and

$$(\frac{1}{2\lambda_n}-c_1)d^2(x_n,y_n)+(\frac{1}{2\lambda_n}-c_2)d^2(y_n,z_n)-\frac{1}{2\lambda_n}d^2(x_n,z_n)\le f(y_n,z_n). \tag{4.11}$$

Now since $d(x_n,x^*)\le d(v_n,x^*)$, replacing n by n_k in (4.10) and (4.11), taking limit and using (4.8), we get

$$\lim_{k\to\infty} f(y_{n_k},z_{n_k})=0. \tag{4.12}$$

On the other hand, there exists a subsequence $\{z_{n_{k_t}}\}$ of $\{z_{n_k}\}$ and $p\in C$ such that $z_{n_{k_t}}\xrightarrow{\Delta} p$ and

$$\limsup_{k\to\infty}(d^2(u,x^*)-(1-\alpha_{n_k})d^2(u,z_{n_k}))=\lim_{t\to\infty}(d^2(u,x^*)-(1-\alpha_{n_{k_t}})d^2(u,z_{n_{k_t}})).$$

By Δ-lower semicontinuity of $d^2(u,\cdot)$, we obtain

$$\begin{aligned}\limsup_{k\to\infty}(d^2(u,x^*)-(1-\alpha_{n_k})d^2(u,z_{n_k}))&=\lim_{t\to\infty}(d^2(u,x^*)-(1-\alpha_{n_{k_t}})d^2(u,z_{n_{k_t}}))\\&\le d^2(u,x^*)-d^2(u,p).\end{aligned} \tag{4.13}$$

Now let $z:=tz_n\oplus(1-t)y$ such that $t\in[0,1)$ and $y\in C$, then, by (4.4), we have

$$\begin{aligned}f(y_n,z_n)+\frac{1}{2\lambda_n}d^2(x_n,z_n)&\le f(y_n,z)+\frac{1}{2\lambda_n}d^2(x_n,z)\\&\le tf(y_n,z_n)+(1-t)f(y_n,y)\\&+\frac{1}{2\lambda_n}\{td^2(x_n,z_n)+(1-t)d^2(x_n,y)-t(1-t)d^2(z_n,y)\}.\end{aligned}$$

This implies that

$$f(y_n,z_n)-f(y_n,y)\le \frac{1}{2\lambda_n}\{d^2(x_n,y)-d^2(x_n,z_n)-td^2(z_n,y)\}.$$

Now, if $t\to 1^-$ we get

$$\frac{1}{2\lambda_n}\{d^2(x_n,z_n)+d^2(z_n,y)-d^2(x_n,y)\}\le f(y_n,y)-f(y_n,z_n). \tag{4.14}$$

Therefore we have

$$\frac{-1}{2\lambda_n}d(x_n,z_n)\{d(z_n,y)+d(x_n,y)\}\le f(y_n,y)-f(y_n,z_n). \tag{4.15}$$

Since $y_{n_{k_t}}\xrightarrow{\Delta} p$ by (4.9), replacing n by n_{k_t} in (4.15) and then taking limsup and using (4.9) and (4.12) we get

$$0\le \limsup_{t\to\infty} f(y_{n_{k_t}},y), \qquad \forall y\in C.$$

Now, since $f(\cdot, y)$ is Δ-upper semicontinuous, we get

$$f(p, y) \geq 0, \quad \forall y \in C. \tag{4.16}$$

Now, it remains to prove that $p \in \mathrm{Fix}(K)$. Note that $x_{n_{k_t}} \xrightarrow{\Delta} p$, $v_{n_{k_t}} \xrightarrow{\Delta} p$ and $w_{n_{k_t}} \xrightarrow{\Delta} p$ because $\lim_{t\to\infty} d(v_{n_{k_t}}, w_{n_{k_t}}) = 0$ by (4.9).
Since $\lim_{t\to\infty} d(v_{n_{k_t}}, w_{n_{k_t}}) = 0$, and K is demiclosed, thus $p \in K(p)$, i.e. p is a fixed point of $K(\cdot)$. Therefore we get $p \in S^*$ by (4.16).

Therefore we have $d(u, x^*) \leq d(u, p)$, thus (4.13) implies

$$\limsup_{k\to\infty}(d^2(u, x^*) - (1 - \alpha_{n_k})d^2(u, z_{n_k})) \leq 0. \tag{4.17}$$

Hence

$$d^2(v_n, x^*) \to 0$$

by Lemma 4.2. Now since $x_n = \beta_n v_n \oplus (1 - \beta_n)w_n$, Lemma 4.1 shows that

$$x_n \to x^* = P_{S^*}u.$$

$\square$

Now, we give an example to illustrate applications of Theorem 2.3 and Theorem 4.3, and we also do some numerical experiments.

Example 4.4. We define a metric on $\mathbb{R}^2$ as

$$d((x_1, x_2), (y_1, y_2)) = \sqrt{(x_1 - y_1)^2 + (x_1^2 - x_2 - y_1^2 + y^2)^2}.$$

Hence $(\mathbb{R}^2, d)$ is an Hadamard space with the geodesic

$$\gamma(t) = \Big((1-t)x_1 + ty_1, ((1-t)x_1 + ty_1)^2 - (1-t)(x_1^2 - x_2) - t(y_1^2 - y_2)\Big),$$

where the geodesic segment joining $x = (x_1, x_2)$ and $y = (y_1, y_2)$ and is contained in $\mathbb{R}^2$ (see [9]). Let $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be a bifunction which is defined by

$$f(x, y) = a\Big(((y_2 + 1) - (y_1 + 1)^2)^2 - ((x_2 + 1) - (x_1 + 1)^2)^2\Big) + b(y_1^2 - x_1^2),$$

where $a, b \in \mathbb{R}_+$. Now, let $C = \{z = (z_1, z_2) \in \mathbb{R}^2 : z_1 \geq 0\}$, and $K(\cdot) : C \to 2^C$ be defined by $K(x) = \{z \in C | d(0, z) \leq 2 + \frac{1}{2}\sqrt{x_1^2 + (x_1^2 - x_2)^2}\}$ for each $x \in C$. It is easy to see that f satisfies B1–B4, and it can be shown that $K(\cdot) : C \to 2^C$ is a multivalued mapping with nonempty, closed and convex values, which is quasi nonexpansive and demiclosed. Hence B5 is satisfied too. Also, we have $S^* \neq \emptyset$. It is easy to check that for all $(a, b) \in \mathbb{R}_+^2$ the unique solution is $x^* = (0, 0)$. We take $\beta_n \equiv \frac{1}{2}$, $c_1 = c_2 = \frac{1}{4}$ and $\lambda_n \equiv \frac{1}{2}$. If $\{x_n\}$ is the sequence generated by the algorithm in Section 2 (see Theorem 2.3 and (2.12)), then the sequence $\{x_n\}$ converges to the solution of QEP(f, K) by Theorem 2.3. We performed some numerical experiments for this example. We chose randomly 100 random pairs $(a, b) \in [0, 100] \times [0, 100]$ and five starting points. Our stopping criterion is $d(x_n, x_{n+1}) < 10^{-8}$.

The numerical results are displayed in the following table, where the starting points, the average number of iterations and the average CPU times have been reported.

The numerical results obtained by performing the algorithm in Section 2		
Starting point: z_0	Average number of iterations	Average CPU time (Sec)
(7, 13)	15.69	1.429375
(9, -1)	15.66	1.166250
(17, 23)	5.64	0.277500
(43, -8)	5.41	0.353281
(62, -17)	5.62	0.274687

Also, all tests for the 100 problems corresponding to each starting point were successful, meaning that the sequence $\{x_n\}$ converges to $(0,0)$, which is the solution of QEP(f,K). This problem was solved by the Optimization Toolbox in Matlab R2020a and performed on a Laptop with Intel(R) Core(TM) i3-4005U CPU @ 1.70 GHz, 1700 Mhz, 2 Core(s), 4 Logical Processor(s), Ram 4.00 GB.

Finally, in order to implement the Halpern regularization method in Section 4 for this example, moreover we take $\alpha_n \equiv \frac{1}{n+1}$ and $u = (1,1)$. If $\{x_n\}$ is the sequence generated by our algorithm in Section 4, then $\{x_n\}$ converges to $P_{S^*}u$ by Theorem 4.3 where it is the projection of the point u onto the solution set S^*. We also performed some numerical experiments for this example. Again, we chose randomly 100 random pairs $(a,b) \in [0,100] \times [0,100]$ and five starting points. The numerical results are displayed in the following table, where the starting points, the average number of iterations and the average CPU times have been reported.

The numerical results obtained by performing the Halpern regularization method in Section 4		
Starting point: z_0	Average number of iterations	Average CPU time (Sec)
(23, 7)	111,17	7.337500
(4, -11)	103.51	7.020312
(7, -38)	101.13	6.814062
(11, -57)	102.00	6.634375
(69, 81)	101.24	6.756250

For this example, all tests for the 100 problems corresponding to each starting point were successful, meaning that the sequence $\{x_n\}$ converges to $(0,0)$, which is the solution of QEP(f,K). This problem was solved by the Optimization Toolbox in Matlab R2020a and performed on a Laptop with Intel(R) Core(TM) i3-4005U CPU @ 1.70 GHz, 1700 Mhz, 2 Core(s), 4 Logical Processor(s), Ram 4.00 GB.

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