

## Multiplicative Lie Triple Higher Derivation on Unital Algebra

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ABSTRACT. In this article, we show that under certain assumptions every multiplicative Lie triple higher derivation  $\mathfrak{L} = \{L_i\}_{i \in \mathbb{N}}$  on  $\mathfrak{U}$  is of standard form, i.e., each component  $L_i$  has the form  $L_i = \delta_i + \gamma_i$ , where  $\{\delta_i\}_{i \in \mathbb{N}}$  is an additive higher derivation on  $\mathfrak{U}$  and  $\{\gamma_i\}_{i \in \mathbb{N}}$  is a sequence of mappings  $\gamma_i : \mathfrak{U} \rightarrow \mathfrak{Z}(\mathfrak{U})$  vanishing at Lie triple products on  $\mathfrak{U}$ .

**Keywords:** Unital algebra, Lie triple derivation, Lie triple higher derivation.

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### 1. INTRODUCTION

Let  $\mathfrak{R}$  be a commutative ring with identity and  $\mathfrak{U}$  be a unital algebra over  $\mathfrak{R}$ . For any  $x, y \in \mathfrak{U}$ ,  $[x, y]$  will denote the commutator  $xy - yx$ . A map  $L : \mathfrak{U} \rightarrow \mathfrak{U}$  is called a multiplicative derivation on  $\mathfrak{U}$  if  $L(xy) = L(x)y + xL(y)$  holds for all  $x, y \in \mathfrak{U}$ . A map  $L : \mathfrak{U} \rightarrow \mathfrak{U}$  is called a multiplicative Lie derivation (resp. multiplicative Lie triple derivation) on  $\mathfrak{U}$  if  $L([x, y]) = [L(x), y] + [x, L(y)]$  (resp.  $L([[x, y], z]) = [[L(x), y], z] + [[x, L(y)], z] + [[x, y], L(z)]$ ) holds for all  $x, y, z \in \mathfrak{U}$ . The concept of derivations was extended to higher derivations. Let us recall the

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basic facts about higher derivations. Let  $\mathbb{N}$  be the set of all non negative integers and  $\mathfrak{L} = \{L_i\}_{i \in \mathbb{N}}$  be a family of maps  $L_i : \mathfrak{U} \rightarrow \mathfrak{U}$  such that  $L_0 = I_{\mathfrak{U}}$ . Then  $\mathfrak{L}$  is said to be a

- (i) *multiplicative higher derivation* on  $\mathfrak{U}$  if  $L_i(xy) = \sum_{r+s=i} L_r(x)L_s(y)$  for all  $x, y \in \mathfrak{U}$  and for each  $i \in \mathbb{N}$ ,  
(ii) *multiplicative Lie higher derivation* on  $\mathfrak{U}$  if

$$L_i([x, y]) = \sum_{r+s=i} [L_r(x), L_s(y)]$$

for all  $x, y \in \mathfrak{U}$  and for each  $i \in \mathbb{N}$ ,

- (iii) *multiplicative Lie triple higher derivation* on  $\mathfrak{U}$  if

$$L_i([[x, y], z]) = \sum_{r+s+t=i} [[L_r(x), L_s(y)], L_t(z)]$$

for all  $x, y, z \in \mathfrak{U}$  and for each  $i \in \mathbb{N}$ .

Particularly, if  $\mathfrak{L} = \{L_i\}_{i \in \mathbb{N}}$  is a family of linear maps, then the above maps are called higher derivation, Lie higher derivation and Lie triple higher derivation on  $\mathfrak{U}$  respectively. Obviously, every higher derivation is a Lie higher derivation and every Lie higher derivation is a Lie triple higher derivation. But the converse statements are not true in general (for a counterexample see [12]).

Lie triple (higher) derivation has been studied on several classes of rings and algebras [1, 4, 5, 6]. In the year 1978, Miers [11] initiated the investigation of Lie triple derivations on von Neumann algebras and proved that “if  $M$  is a von Neumann algebra with no central abelian summands then there exists an operator  $A \in M$  such that  $L(X) = [A, X] + \lambda(X)$ , where  $\lambda : M \rightarrow \mathfrak{Z}(M)$  is a linear map which annihilates brackets of operators in  $M$ .” Zhang et.al. [14] and Lu [9] investigated the Lie triple derivation on nest algebras. Xiao and Wei [12] obtained that every Lie triple derivation on a triangular algebra can be expressed as the sum of an additive derivation and a linear functional vanishing on all second commutators. Also, Ding and Li [3] considered the Lie  $n$ -derivation on unital algebra with a nontrivial idempotents and proved that every Lie  $n$ -derivation  $L$  on  $\mathfrak{U}$  is of the form  $L = d + \gamma$ , where  $d$  is a derivation on  $\mathfrak{U}$  and  $\gamma$  is a linear mapping from  $\mathfrak{U}$  into its centre  $\mathfrak{Z}(\mathfrak{U})$  that vanishes on  $[[\mathfrak{U}, \mathfrak{U}], \mathfrak{U}]$ . Apart from these, Ebrahimi [4, 5] studied Lie higher derivation on  $B(X)$  and Lie triple higher derivation on generalized matrix algebras respectively.

In last few decades, the multiplicative mappings on rings and algebras were studied by many authors [1, 7, 13]. Martindale [10] established a condition on a ring such that multiplicative bijective mappings on this ring are all additive. In particular, every multiplicative bijective mapping from a prime ring containing a nontrivial idempotent onto an arbitrary ring is additive. Xiao and Wei [13] considered the case of nonlinear Lie higher derivations on a triangular algebra. Let  $\mathfrak{L} = \{L_i\}_{i \in \mathbb{N}}$  be the Lie higher derivation on a triangular algebra. Then

$\mathfrak{L} = \{L_i\}_{i \in \mathbb{N}}$  is of the standard form, i.e.,  $L_i = d_i + \gamma_i$ , where  $\{d_i\}_{i \in \mathbb{N}}$  is an additive higher derivations and  $\{\gamma_i\}_{i \in \mathbb{N}}$  is sequence of a nonlinear functional vanishing on all commutators of the triangular algebra. Furthermore, Ashraf and Jabeen in [1] showed that every nonlinear Lie triple higher derivation on the triangular algebra has standard form. Han and Wei [7] studied multiplicative Lie higher derivations of a unital algebra and obtained similar conclusion as shown by Xiao and Wei [13]. In view of cited references, the main purpose of this paper is to prove that every multiplicative Lie triple higher derivation on a unital algebra has standard form under certain assumptions.

### 2. PRELIMINARIES

Throughout, this paper we shall use the following notions: Let  $\mathfrak{U} = p\mathfrak{U}p + p\mathfrak{U}q + q\mathfrak{U}p + q\mathfrak{U}q$  be unital algebra with nontrivial idempotents  $p$  and  $q = 1 - p$  satisfying (2.2). Let  $A = p\mathfrak{U}p, M = p\mathfrak{U}q, N = q\mathfrak{U}p$  and  $B = q\mathfrak{U}q$ . Then  $\mathfrak{U} = A + M + N + B$ . The center of  $\mathfrak{U}$  is

$$\mathfrak{Z}(\mathfrak{U}) = \{a + b \in A + B \mid am = mb, na = bn \text{ for all } m \in M, n \in N\}.$$

Define two natural projections  $\pi_A : \mathfrak{U} \rightarrow A$  and  $\pi_B : \mathfrak{U} \rightarrow B$  by  $\pi_A(a+m+n+b) = a$  and  $\pi_B(a + m + n + b) = b$ . Moreover,  $\pi_A(\mathfrak{Z}(\mathfrak{U})) \subseteq \mathfrak{Z}(A)$  and  $\pi_B(\mathfrak{Z}(\mathfrak{U})) \subseteq \mathfrak{Z}(B)$  and there exists a unique algebra isomorphism  $\tau : \pi_A(\mathfrak{Z}(\mathfrak{U})) \rightarrow \pi_B(\mathfrak{Z}(\mathfrak{U}))$  such that  $am = m\tau(a)$  and  $na = \tau(a)n$  for all  $a \in \pi_A(\mathfrak{Z}(\mathfrak{U})), m \in M, n \in N$ .

Let us assume  $\mathfrak{U}$  be an algebra with a nontrivial idempotent  $p$  and let  $q = 1 - p$  be also an idempotent. According to the well known Peirce decomposition,  $\mathfrak{U}$  can be represented in the following form:

$$\mathfrak{U} = p\mathfrak{U}p + p\mathfrak{U}q + q\mathfrak{U}p + q\mathfrak{U}q \tag{2.1}$$

where  $p\mathfrak{U}p$  and  $q\mathfrak{U}q$  are subalgebras with unital elements  $p$  and  $q$ , respectively,  $p\mathfrak{U}q$  is an  $(p\mathfrak{U}p, q\mathfrak{U}q)$ -bimodule and  $q\mathfrak{U}p$  is a  $(q\mathfrak{U}q, p\mathfrak{U}p)$ -bimodule. We will assume that  $\mathfrak{U}$  satisfies

$$\begin{aligned} pxp.p\mathfrak{U}q &= \{0\} = q\mathfrak{U}p.pxp \text{ implies } pxp = 0, \\ p\mathfrak{U}q.qxq &= \{0\} = qxq.q\mathfrak{U}p \text{ implies } qxq = 0 \end{aligned} \tag{2.2}$$

for all  $x \in \mathfrak{U}$ . Some specific examples of unital algebras with nontrivial idempotents having the property (2.2) are triangular algebras, matrix algebras and prime (and hence in particular simple) algebras with nontrivial idempotents.

### 3. MULTIPLICATIVE LIE TRIPLE HIGHER DERIVATION

Following [8, Theorem 4.2.1], in this section we study the main result of this paper. In fact we obtain this result:

**Theorem 3.1.** *Let  $\mathfrak{U}$  be a 2-torsion free unital algebra with a nontrivial idempotent  $p$  satisfying (2.2) and  $\mathfrak{L} = \{L_i\}_{i \in \mathbb{N}}$  be a multiplicative Lie triple higher derivation on  $\mathfrak{U}$ . Let us assume that*

- (i)  $\pi_A(\mathfrak{Z}(\mathfrak{U})) = \mathfrak{Z}(A)$  and  $\pi_B(\mathfrak{Z}(\mathfrak{U})) = \mathfrak{Z}(B)$ ,
- (ii) either  $A$  or  $B$  does not contain nonzero central ideals,
- (iii)  $[x, \mathfrak{U}] \in \mathfrak{Z}(\mathfrak{U})$  implies that  $x \in \mathfrak{Z}(\mathfrak{U})$  for all  $x \in \mathfrak{U}$ .
- (iv) For each  $n \in \mathbb{N}$ , the condition  $nM = 0$  or  $Mn = 0$  implies  $n = 0$ ;
- (v) For each  $m \in \mathbb{M}$ , the condition  $mN = 0$  or  $Nm = 0$  implies  $m = 0$ ;
- (vi) For each  $i \in \mathbb{N}$ ,  $eL_i(e)f = 0$  and  $fL_i(e)e = 0$ ,

Then every multiplicative Lie triple higher derivation  $\mathfrak{L} = \{L_i\}_{i \in \mathbb{N}}$  is of the standard form, i.e., each component  $L_i$  has the form  $L_i = \delta_i + \gamma_i$ , where  $\{\delta_i\}_{i \in \mathbb{N}}$  is an additive higher derivation on  $\mathfrak{U}$  and  $\{\gamma_i\}_{i \in \mathbb{N}}$  is a sequence of mappings  $\gamma_i : \mathfrak{U} \rightarrow \mathfrak{Z}(\mathfrak{U})$  vanishing at Lie triple products in  $\mathfrak{U}$ , i.e.,  $\gamma_i([[x, y], z]) = 0$  for all  $x, y, z \in \mathfrak{U}$ .

In order to prove the theorem we will use the method of induction for the component index  $i$ . When  $i = 1$ ,  $L_1$  is a multiplicative Lie derivation on  $\mathfrak{U}$ . By [8, Theorem 4.2.1] it is easy to observe that there exist an additive derivation  $\delta_1$  and a map  $\gamma_1 : \mathfrak{U} \rightarrow \mathfrak{Z}(\mathfrak{U})$  vanishing at Lie triple products such that  $L_1(x) = \delta_1(x) + \gamma_1(x)$  for all  $x \in \mathfrak{U}$ . Now by [8, Lemma 4.2.1], we know that  $pL_1(q)q = 0 = qL_1(q)p$ . Following from the proof of [8, Theorem 4.2.1], It can be easily seen that  $L_1$  and  $\delta_1$  satisfy the following properties:

$$\begin{aligned} L_1(0) &= 0, & L_1(p), L_1(q) &\in \mathfrak{Z}(\mathfrak{U}), & \delta_1(p) &= 0, & \delta_1(q) &= 0 \\ L_1(A) &\subseteq A + \mathfrak{Z}(\mathfrak{U}), & L_1(B) &\subseteq B + \mathfrak{Z}(\mathfrak{U}), & \delta_1(A) &\subseteq A, & \delta_1(B) &\subseteq B \\ L_1(M) &\subseteq M, & L_1(N) &\subseteq N, & \delta_1(M) &\subseteq M, & \delta_1(N) &\subseteq N. \end{aligned}$$

Suppose that our result holds for all  $1 < r < i$ . It follows that there exist an additive higher derivation  $\{\delta_r\}_{r \in \mathbb{N}}$  of order  $r$  and a nonlinear mapping  $\{\gamma_r\}_{r \in \mathbb{N}}$  vanishing on all Lie triple product such that  $L_r(x) = \delta_r(x) + \gamma_r(x)$  for all  $x \in \mathfrak{U}$ . It can be easily seen that  $L_r$  and  $\delta_r$  satisfy the following properties:

$$\begin{aligned} L_r(0) &= 0, & L_r(p), L_r(q) &\in \mathfrak{Z}(\mathfrak{U}), & \delta_r(p) &= 0, & \delta_r(q) &= 0 \\ L_r(A) &\subseteq A + \mathfrak{Z}(\mathfrak{U}), & L_r(B) &\subseteq B + \mathfrak{Z}(\mathfrak{U}), & \delta_r(A) &\subseteq A, & \delta_r(B) &\subseteq B \\ L_r(M) &\subseteq M, & L_r(N) &\subseteq N, & \delta_r(M) &\subseteq M, & \delta_r(N) &\subseteq N. \end{aligned}$$

To prove our main result we begin with the following lemmas:

**Lemma 3.2.** For the index  $i \in \mathbb{N}$ , we have

- (i)  $L_i(0) = 0$ ,
- (ii)  $L_i(p), L_i(q) \in \mathfrak{Z}(\mathfrak{U})$ ,
- (iii)  $L_i(M) \subseteq M$ , and  $L_i(N) \subseteq N$ .

*Proof.* (i) On using induction hypothesis

$$\begin{aligned} L_i(0) &= [[L_i(0), 0], 0] + [[0, L_i(0)], 0] + [[0, 0], L_i(0)] \\ &+ \sum_{\substack{r+s+t=i \\ r,s,t < i}} [[L_r(0), L_s(0)], L_t(0)] = 0. \end{aligned} \tag{3.1}$$

(ii) For any  $m \in M$ , using induction hypothesis

$$\begin{aligned}
 L_i(m) &= L_i([m, q], q) \\
 &= [[L_i(m), q], q] + [[m, L_i(q)], q] + [[m, q], L_i(q)] \\
 &\quad + \sum_{\substack{r+s+t=i \\ r,s,t < i}} [[L_r(m), L_s(q)], L_t(q)] \\
 &= [[L_i(m), q], q] + [[m, L_i(q)], q] + [[m, q], L_i(q)] \\
 &= pL_i(m)q + qL_i(m)p + 2[m, L_i(q)]. \tag{3.2}
 \end{aligned}$$

Now left multiplying by  $p$  and right multiplying by  $q$  in above the expression yield  $2[m, L_i(q)] = 0$  and hence  $[m, L_i(q)] = 0$ . In a similar manner, we arrive at

$$L_i(n) = pL_i(n)q + qL_i(n)p + 2[n, L_i(q)].$$

Consequently,  $2[n, L_i(q)] = 0$  and hence  $[n, L_i(q)] = 0$ . Therefore, from  $[m, L_i(q)] = 0$  and  $[n, L_i(q)] = 0$ , we get  $L_i(q) \in \mathfrak{Z}(\mathfrak{U})$ . Similarly,  $L_i(p) \in \mathfrak{Z}(\mathfrak{U})$ .

(iii) Now from (3.2) and  $L_i(q) \in \mathfrak{Z}(\mathfrak{U})$ , we get that  $L_i(q) = pL_i(q)p + qL_i(q)q$ . For any  $x \in \mathfrak{U}$  and  $m_1, m_2 \in M$

$$\begin{aligned}
 0 &= L_i([m_1, m_2], x) \\
 &= [[L_i(m_1), m_2], x] + [[m_1, L_i(m_2)], x] + [[m_1, m_2], L_i(x)] \\
 &\quad + \sum_{\substack{r+s+t=i \\ r,s,t < i}} [[L_r(m_1), L_s(m_2)], L_t(x)] \\
 &= [[L_i(m_1), m_2], x] + [[m_1, L_i(m_2)], x]
 \end{aligned}$$

which implies that  $[L_i(m_1), m_2] + [m_1, L_i(m_2)] \in \mathfrak{Z}(\mathfrak{U})$ . Also,

$$\begin{aligned}
 [m_1, L_i(m_2)] &= [[p, m_1], L_i(m_2)] \\
 &= L_i([p, m_1], m_2) - [[L_i(p), m_1], m_2] - [[p, L_i(m_1)], m_2] \\
 &\quad - \sum_{\substack{r+s+t=i \\ r,s,t < i}} [[L_r(p), L_s(m_1)], L_t(m_2)] \\
 &= -[[p, L_i(m_1)], m_2].
 \end{aligned}$$

From (3.2), we find that

$$\begin{aligned}
 [L_i(m_1), m_2] + [m_1, L_i(m_2)] &= [qL_i(m_1)p, m_2] - [[p, L_i(m_1)], m_2] \\
 &= 2[qL_i(m_1)p, m_2].
 \end{aligned}$$

Since  $\mathfrak{U}$  is 2-torsion free,  $[qL_i(m_1)p, m_2] \in \mathfrak{Z}(\mathfrak{U})$ . By definition of  $\mathfrak{Z}(\mathfrak{U})$ , it follows that  $qL_i(m_1)p m_2 \in \mathfrak{Z}(\mathfrak{B})$  for all  $m_2 \in M$ . It can be easily seen that  $qL_i(m_1)pBq$  is an ideal of  $\mathfrak{B}$ . Now from assumption (ii),  $qL_i(m_1)pBq = \{0\}$  which implies that  $qL_i(m_1)p = 0$  for  $m_1 \in M$ . Therefore,  $L_i(m) = pL_i(m)q \in M$ . Similarly, we can obtain  $L_i(N) \subseteq N$ .  $\square$

**Lemma 3.3.** For any  $x \in \mathfrak{U}$ ,

- (i)  $L_i(p x q) = p L_i(x) q$ ,  
(ii)  $L_i(q x p) = q L_i(x) p$ .

*Proof.* On using  $L_i(q), L_i(p) \in \mathfrak{Z}(\mathfrak{U})$  and for any  $x \in \mathfrak{U}$

$$\begin{aligned} L_i(p x q) &= L_i([p, x], q) \\ &= [[L_i(p), x], q] + [[p, L_i(x)], q] + [[p, x], L_i(q)] \\ &\quad + \sum_{\substack{r+s+t=i \\ r,s,t < i}} [[L_r(p), L_s(x)], L_t(q)] \\ &= [[p, L_i(x)], q] \\ &= p L_i(x) q + q L_i(x) p. \end{aligned}$$

Since  $L_i(p x q) \in M$ , we have  $q L_i(x) p = 0$ . Hence  $L_i(p x q) = p L_i(x) q$  for all  $x \in \mathfrak{U}$ . Similarly, we can prove that  $L_i(q x p) = q L_i(x) p$  for all  $x \in \mathfrak{U}$ .  $\square$

**Lemma 3.4.**  $L_i(-m) = -L_i(m)$ , and  $L_i(-n) = -L_i(n)$  for all  $n \in N$ ,  $m \in M$ .

*Proof.* Since  $L_i(q), L_i(p) \in \mathfrak{Z}(\mathfrak{U})$  and  $L_i(M) \subseteq M$ , we have

$$\begin{aligned} L_i(-m) &= L_i([[m, p], q]) \\ &= [[L_i(m), p], q] + [[m, L_i(p)], q] + [[m, p], L_i(q)] \\ &\quad + \sum_{\substack{r+s+t=i \\ r,s,t < i}} [[L_r(m), L_s(p)], L_t(q)] \\ &= [[L_i(m), p], q] = -L_i(m) \end{aligned}$$

for all  $a \in A$  and  $m \in M$ . Similar proof for other case.  $\square$

**Lemma 3.5.**  $L_i(A) \subseteq A + \mathfrak{Z}(\mathfrak{U})$  and  $L_i(B) \subseteq B + \mathfrak{Z}(\mathfrak{U})$ .

*Proof.* Since for all  $a \in A, b \in B$  and  $m \in M$ , we have

$$\begin{aligned} 0 &= L_i([[a, b], m]) \\ &= [[L_i(a), b], m] + [[a, L_i(b)], m] + [[a, b], L_i(m)] \\ &\quad + \sum_{\substack{r+s+t=i \\ r,s,t < i}} [[L_r(a), L_s(b)], L_t(m)] \\ &= [[L_i(a), b], m] + [[a, L_i(b)], m] \\ &= -m[q L_i(a) q, b] + [a, p L_i(b) p] m. \end{aligned}$$

Similarly,  $n[q L_i(a) q, b] = [a, p L_i(b) p] n$  which leads to  $[a, p L_i(b) p] + [q L_i(a) q, b] \in \mathfrak{Z}(\mathfrak{U})$  for all  $a \in A$  and  $b \in B$ , i.e.,  $[a, p L_i(b) p] \in A$  and  $[q L_i(a) q, b] \in B$ . By assumption  $[a, p L_i(b) p] = 0$  and  $[q L_i(a) q, b] = 0$ . This implies that  $p L_i(b) p \in \mathfrak{Z}(A)$  and  $q L_i(a) q \in \mathfrak{Z}(B)$ .

We define a map  $\phi_{i_1} : A \rightarrow \mathfrak{Z}(\mathfrak{U})$  by  $\phi_{i_1}(a) = \tau^{-1}(q L_i(a) q) + q L_i(a) q$  where  $\tau$  is map defined in preliminaries. Therefore, on using  $L_i(p x q) = p L_i(x) q$  and  $L_i(q x p) = q L_i(x) p$ , we have

$$\begin{aligned} L_i(a) - \phi_{i_1}(a) &= p L_i(a) p + q L_i(a) q - \tau^{-1}(q L_i(a) q) - q L_i(a) q \\ &= p L_i(a) p - \tau^{-1}(q L_i(a) q) \in A. \end{aligned}$$

This implies that  $L_i(A) \subseteq A + \mathfrak{Z}(\mathfrak{U})$  for all  $a \in A$ . Similarly, we can define another map  $\phi_{i_2} : B \rightarrow \mathfrak{Z}(\mathfrak{U})$  by  $\phi_{i_2}(b) = \tau(pL_i(b)p) + pL_i(b)p$  such that  $L_i(B) \subseteq B + \mathfrak{Z}(\mathfrak{U})$  for all  $b \in B$ . □

*Remark 3.6.* Let us define the map  $\gamma_{i_1} : \mathfrak{U} \rightarrow \mathfrak{Z}(\mathfrak{U})$  by

$$\gamma_{i_1}(x) = qL_i(pxp)q + \tau^{-1}(qL_i(pxp)q) + pL_i(qxq)p + \tau(pL_i(qxq)p).$$

Obviously,  $\gamma_{i_1}(x) \in \mathfrak{Z}(\mathfrak{U})$  and  $\gamma_{i_1}([[x, y], z]) = 0$  for all  $x, y, z \in \mathfrak{U}$ .

Define another map  $\xi_i : \mathfrak{U} \rightarrow \mathfrak{U}$  as  $\xi_i(x) = L_i(x) - \gamma_{i_1}(x)$  for all  $x \in \mathfrak{U}$ . It is easy to see that

$$\begin{aligned} \xi_i(a) &= pL_i(a)p - \tau^{-1}(qL_i(a)q) \in A, \\ \xi_i(b) &= qL_i(b)q - \tau(pL_i(b)p) \in B, \\ \xi_i(m) &= L_i(m) \in M, \\ \xi_i(n) &= L_i(n) \in N \end{aligned}$$

for all  $a \in A, b \in B, m \in M$  and  $n \in N$ .

**Lemma 3.7.** *For any  $a \in A, b \in B, m \in M$  and  $n \in N$ , we have*

$$\begin{aligned} (i) \quad \xi_i(am) &= \xi_i(a)m + a\xi_i(m) + \sum_{\substack{r+s=i \\ r,s < i}} \delta_r(a)\delta_s(m), \\ (ii) \quad \xi_i(mb) &= \xi_i(m)b + m\xi_i(b) + \sum_{\substack{r+s=i \\ r,s < i}} \delta_r(m)\delta_s(b), \\ (iii) \quad \xi_i(bn) &= \xi_i(b)n + b\xi_i(n) + \sum_{\substack{r+s=i \\ r,s < i}} \delta_r(b)\delta_s(n), \\ (iv) \quad \xi_i(na) &= \xi_i(n)a + n\xi_i(a) + \sum_{\substack{r+s=i \\ r,s < i}} \delta_r(n)\delta_s(a). \end{aligned}$$

*Proof.* (i) For  $a \in A$  and  $m \in M$ , we have

$$\begin{aligned} \xi_i(am) &= \xi_i([[a, m], q]) \\ &= L_i([[a, m], q]) \\ &= [[L_i(a), m], q] + [[a, L_i(m)], q] + [[a, m], L_i(q)] \\ &\quad + \sum_{\substack{r+s+t=i \\ r,s,t < i}} [[L_r(a), L_s(m)], L_t(q)] \\ &= [[\xi_i(a), m], q] + [[a, \xi_i(m)], q] + \sum_{\substack{r+s+t=i \\ r,s,t < i}} [[\delta_r(a), \delta_s(m)], \delta_t(q)] \\ &= \xi_i(a)m + a\xi_i(m) + \sum_{\substack{r+s=i \\ r,s < i}} \delta_r(a)\delta_s(m). \end{aligned}$$

Similarly, we can prove (ii).

(iii) For  $b \in B$  and  $n \in N$ , we have

$$\begin{aligned}
 \xi_i(bn) &= \xi_i([b, n], p) \\
 &= L_i([b, n], p) \\
 &= [[L_i(b), n], p] + [[b, L_i(n)], p] + [[b, n], L_i(p)] \\
 &\quad + \sum_{\substack{r+s+t=i \\ r,s,t < i}} [[L_r(b), L_s(n)], L_t(p)] \\
 &= [[\xi_i(b), n], p] + [[b, \xi_i(n)], p] + \sum_{\substack{r+s+t=i \\ r,s,t < i}} [[\delta_r(b), \delta_s(n)], \delta_t(p)] \\
 &= \xi_i(b)n + b\xi_i(n) + \sum_{\substack{r+s=i \\ r,s < i}} \delta_r(b)\delta_s(n).
 \end{aligned}$$

Similarly, we can prove (iv). □

**Lemma 3.8.** For any  $a_1, a_2 \in A$  and  $b_1, b_2 \in B$ , we have

$$\begin{aligned}
 (i) \quad \xi_i(a_1a_2) &= \xi_i(a_1)a_2 + a_1\xi_i(a_2) + \sum_{\substack{r+s=i \\ r,s < i}} \delta_r(a_1)\delta_s(a_2), \\
 (ii) \quad \xi_i(b_1b_2) &= \xi_i(b_1)b_2 + b_1\xi_i(b_2) + \sum_{\substack{r+s=i \\ r,s < i}} \delta_r(b_1)\delta_s(b_2).
 \end{aligned}$$

*Proof.* For any  $a_1, a_2 \in A$  and  $m \in M$ ,

$$\xi_i(a_1a_2m) = \xi_i(a_1a_2)m + a_1a_2\xi_i(m) + \sum_{\substack{r+s+t=i \\ r,s,t < i}} \delta_r(a_1)\delta_s(a_2)\delta_t(m).$$

On the other hand,

$$\begin{aligned}
 \xi_i(a_1a_2m) &= \xi_i(a_1)a_2m + a_1\xi_i(a_2m) + \sum_{\substack{r+s+t=i \\ r,s,t < i}} \delta_r(a_1)\delta_s(a_2)\delta_t(m) \\
 &= \xi_i(a_1)a_2m + a_1\xi_i(a_2)m + a_1a_2\xi_i(m) \\
 &\quad + a_1 \sum_{\substack{r+s=i \\ r,s < i}} \delta_r(a_2)\delta_s(m) + \sum_{\substack{r+s+t=i \\ r,s,t < i}} \delta_r(a_1)\delta_s(a_2)\delta_t(m). \quad (3.3)
 \end{aligned}$$

Above relations implies that

$$\{\xi_i(a_1a_2) - \xi_i(a_1)a_2 - a_1\xi_i(a_2) - \sum_{\substack{r+s=i \\ r,s < i}} \delta_r(a_1)\delta_s(a_2)\}m = 0. \quad (3.4)$$

In the similar manner, we obtain

$$n\{\xi_i(a_1a_2) - \xi_i(a_1)a_2 - a_1\xi_i(a_2) - \sum_{\substack{r+s=i \\ r,s < i}} \delta_r(a_1)\delta_s(a_2)\} = 0. \quad (3.5)$$

Comparing (3.4) and (3.5) yield that

$$\xi_i(a_1 a_2) = \xi_i(a_1) a_2 + a_1 \xi_i(a_2) + \sum_{\substack{r+s=i \\ r,s < i}} \delta_r(a_1) \delta_s(a_2)$$

for all  $a_1, a_2 \in A$ . Similarly, we can show the other part. □

**Lemma 3.9.** For any  $a \in A, m \in M$  and  $b \in B$ ,

- (i)  $\xi_i(a + m) - \xi_i(a) - \xi_i(m) \in \mathfrak{Z}(\mathfrak{U})$ ,
- (ii)  $\xi_i(m + b) - \xi_i(m) - \xi_i(b) \in \mathfrak{Z}(\mathfrak{U})$ .

*Proof.* (i) For any  $a \in A, m \in M$  and using  $L_i(q) \in \mathfrak{Z}(\mathfrak{U})$ , we have

$$\begin{aligned} \xi_i([p, m], q) &= L_i([p, a + m], q) \\ &= [[p, L_i(a + m)], q] + [[L_i(p), a + m], q] + [[p, a + m], L_i(q)] \\ &\quad + \sum_{\substack{r+s+t=i \\ r,s,t < i}} [[L_r(p), L_s(a + m)], L_t(q)] \\ &= [[p, \xi_i(a + m)], q]. \end{aligned} \tag{3.6}$$

Also,

$$\begin{aligned} \xi_i([p, m], q) &= [[p, L_i(m)], q] + [[L_i(p), m], q] + [[p, m], L_i(q)] \\ &\quad + \sum_{\substack{r+s+t=i \\ r,s,t < i}} [[L_r(p), L_s(m)], L_t(q)] \\ &= [[p, \xi_i(m)], q]. \end{aligned} \tag{3.7}$$

From (3.6) and (3.7) it follows that  $[[\xi_i(a + m) - \xi_i(a) - \xi_i(m), q], p] = 0$  which gives  $p(\xi_i(a + m) - \xi_i(a) - \xi_i(m))q + q(\xi_i(a + m) - \xi_i(a) - \xi_i(m))p = 0$ . Therefore, we obtain that  $\xi_i(a + m) - \xi_i(a) - \xi_i(m) \in A + B$ . Since  $[[a, m'], q] = [[a + m, m'], q]$  and  $L_i(M) \subseteq M, L_i(q) \in \mathfrak{Z}(\mathfrak{U})$ , for any  $a \in A$  and  $m, m' \in M$ .

$$\begin{aligned} \xi_i(am') &= L_i([a + m, m'], q) \\ &= [[L_i(a + m), m'], q] + [[a + m, L_i(m')], q] + [[a + m, m'], L_i(q)] \\ &\quad + \sum_{\substack{r+s+t=i \\ r,s,t < i}} [[L_r(a + m), L_s(m')], L_t(q)] \\ &= [[\xi_i(a + m), m'], q] + [[a + m, \xi_i(m')], q] \\ &\quad + \sum_{\substack{r+s+t=i \\ r,s,t < i}} [[\delta_r(a + m), \delta_s(m')], \delta_t(q)] \\ &= [[\xi_i(a + m), m'], q] + [[a, \xi_i(m')], q] + \sum_{\substack{r+s=i \\ r,s < i}} \delta_r(a) \delta_s(m'). \end{aligned} \tag{3.8}$$

On the other hand,

$$\begin{aligned}
 \xi_i(am') &= L_i([a, m'], q) \\
 &= [[L_i(a), m'], q] + [[a, L_i(m')], q] + [[a, m'], L_i(q)] \\
 &\quad + \sum_{\substack{r+s+t=i \\ r,s,t < i}} [[L_r(a), L_s(m')], L_t(q)] \\
 &= [[\xi_i(a), m'], q] + [[a, \xi_i(m')], q] + \sum_{\substack{r+s+t=i \\ r,s,t < i}} [[\delta_r(a), \delta_s(m')], \delta_t(q)] \\
 &= [[\xi_i(a), m'], q] + [[a, \xi_i(m')], q] + \sum_{\substack{r+s=i \\ r,s < i}} \delta_r(a)\delta_s(m'). \tag{3.9}
 \end{aligned}$$

Now from (3.8) and (3.9), we have  $[[\xi_i(a+m) - \xi_i(a) - \xi_i(m), m'], q] = 0$ . This together with  $\xi_i(a+m) - \xi_i(a) - \xi_i(m) \in A+B$  gives that

$$(\xi_i(a+m) - \xi_i(a) - \xi_i(m))m' = m'(\xi_i(a+m) - \xi_i(a) - \xi_i(m)) \tag{3.10}$$

for all  $a \in A$  and  $m, m' \in M$ . In the similar manner, from  $[[a+m, n'], q] = [[a, n'], q]$  for all  $a \in A, n' \in N$  and  $m \in M$ , we obtain that  $[[\xi_i(a+m) - \xi_i(a) - \xi_i(m), n'], q] = 0$  which together with  $\xi_i(a+m) - \xi_i(a) - \xi_i(m) \in A+B$  yields

$$(\xi_i(a+m) - \xi_i(a) - \xi_i(m))n' = n'(\xi_i(a+m) - \xi_i(a) - \xi_i(m)) \tag{3.11}$$

for all  $a \in A, n' \in N$  and  $m \in M$ . Now combining (3.10) and (3.11), we get  $\xi_i(a+m) - \xi_i(a) - \xi_i(m) \in \mathfrak{Z}(\mathfrak{U})$ . Similarly, we can find (ii).  $\square$

**Lemma 3.10.** For any  $a \in A, b \in B$  and  $n \in N$

- (i)  $\xi_i(a+n) - \xi_i(a) - \xi_i(n) \in \mathfrak{Z}(\mathfrak{U})$ ,
- (ii)  $\xi_i(n+b) - \xi_i(n) - \xi_i(b) \in \mathfrak{Z}(\mathfrak{U})$ .

*Proof.* (i) For any  $a \in A, n \in N$  and using  $L_i(q) \in \mathfrak{Z}(\mathfrak{U})$ , we have

$$\begin{aligned}
 \xi_i([p, n], q) &= L_i([p, a+n], q) \\
 &= [[p, L_i(a+n)], q] + [[L_i(p), a+n], q] + [[p, a+n], L_i(q)] \\
 &\quad + \sum_{\substack{r+s+t=i \\ r,s,t < i}} [[L_r(p), L_s(a+n)], L_t(q)] \\
 &= [[p, \xi_i(a+n)], q]. \tag{3.12}
 \end{aligned}$$

Also,

$$\begin{aligned}
 \xi_i([p, n], q) &= [[p, L_i(n)], q] + [[L_i(p), n], q] + [[p, n], L_i(q)] \\
 &\quad + \sum_{\substack{r+s+t=i \\ r,s,t < i}} [[L_r(p), L_s(n)], L_t(q)] \\
 &= [[p, \xi_i(n)], q]. \tag{3.13}
 \end{aligned}$$

From (3.12) and (3.13) it follows that  $[[p, \xi_i(a+n) - \xi_i(a) - \xi_i(n)], q] = 0$  which yields that  $p(\xi_i(a+n) - \xi_i(a) - \xi_i(n))q + q(\xi_i(a+n) - \xi_i(a) - \xi_i(n))p = 0$ . Hence, we obtain that  $\xi_i(a+n) - \xi_i(a) - \xi_i(n) \in A + B$ . Since  $[[a, n'], q] = [[a+n, n'], q]$  and  $L_i(N) \subseteq N, L_i(q) \in \mathfrak{Z}(\mathfrak{U})$ , for any  $a \in A$  and  $n, n' \in N$

$$\begin{aligned} \xi_i(n'a) &= L_i([[a+n, n'], q]) \\ &= [[L_i(a+n), n'], q] + [[a+n, L_i(n')], q] + [[a+n, n'], L_i(q)] \\ &\quad + \sum_{\substack{r+s+t=i \\ r,s,t < i}} [[L_r(a+n), L_s(n')], L_t(q)] \\ &= [[\xi_i(a+n), n'], q] + [[a+n, \xi_i(n')], q] \\ &\quad + \sum_{\substack{r+s+t=i \\ r,s,t < i}} \delta_r(q)\delta_s(n')\delta_t(a+n) \\ &= [[\xi_i(a+n), n'], q] + [[a, \xi_i(n')], q] + \sum_{\substack{r+s+t=i \\ r,s,t < i}} \delta_r(q)\delta_s(n')\delta_t(a). \end{aligned} \tag{3.14}$$

On the other hand,

$$\begin{aligned} \xi_i(n'a) &= L_i([[a, n'], q]) \\ &= [[L_i(a), n'], q] + [[a, L_i(n')], q] + [[a, n'], L_i(q)] \\ &\quad + \sum_{\substack{r+s+t=i \\ r,s,t < i}} [[L_r(a), L_s(n')], L_t(q)] \\ &= [[\xi_i(a), n'], q] + [[a, \xi_i(n')], q] + \sum_{\substack{r+s=i \\ r,s < i}} \delta_r(q)\delta_s(n')\delta_t(a). \end{aligned} \tag{3.15}$$

Now combining (3.14) and (3.15), we have  $[[\xi_i(a+n) - \xi_i(a) - \xi_i(n), n'], q] = 0$  which together with  $\xi_i(a+n) - \xi_i(a) - \xi_i(n) \in A + B$  yields that

$$(\xi_i(a+n) - \xi_i(a) - \xi_i(n))n' = n'(\xi_i(a+n) - \xi_i(a) - \xi_i(n)) \tag{3.16}$$

for all  $a \in A$  and  $n, n' \in N$ . In the similar manner, from  $[[a+n, m'], q] = [[a, m'], q]$  for all  $a \in A, n \in N$  and  $m' \in M$ , we obtain that  $[[\xi_i(a+n) - \xi_i(a) - \xi_i(n), m'], q] = 0$ . combine this with  $\xi_i(a+n) - \xi_i(a) - \xi_i(n) \in A + B$  yields that

$$(\xi_i(a+n) - \xi_i(a) - \xi_i(n))m' = m'(\xi_i(a+n) - \xi_i(a) - \xi_i(n)) \tag{3.17}$$

for all  $a \in A, n \in N$  and  $m' \in M$ . Now combining (3.16) and (3.17), we get  $\xi_i(a+n) - \xi_i(a) - \xi_i(n) \in \mathfrak{Z}(\mathfrak{U})$ . Similarly we can find (ii).  $\square$

**Lemma 3.11.**  $\xi_i$  is additive on  $A, M, N$  and  $B$  respectively.

*Proof.* For any  $m_1, m_2 \in M$ , we have

$$\begin{aligned}
 \xi_i(m_1 + m_2) &= L_i([p + m_1, q + m_2], q) \\
 &= [[L_i(p + m_1), q + m_2], q] + [[p + m_1, L_i(q + m_2)], q] \\
 &\quad + [[p + m_1, q + m_2], L_i(q)] \\
 &\quad + \sum_{\substack{r+s+t=i \\ r,s,t < i}} [[L_r(p + m_1), L_s(q + m_2)], L_t(q)] \\
 &= [[\xi_i(p + m_1), q + m_2], q] + [[p + m_1, \xi_i(q + m_2)], q] \\
 &\quad + \sum_{\substack{r+s+t=i \\ r,s,t < i}} [[\delta_r(p + m_1), \delta_s(q + m_2)], \delta_t(q)] \\
 &= [[\xi_i(p) + \xi_i(m_1), q + m_2], q] + [[p + m_1, \xi_i(q) + \xi_i(m_2)], q] \\
 &= [[\xi_i(m_1), q + m_2], q] + [[p + m_1, \xi_i(m_2)], q] \\
 &= \xi_i(m_1) + \xi_i(m_2). \tag{3.18}
 \end{aligned}$$

which implies that  $\xi_i$  is additive on  $M$ . In similar manner, we can obtain that  $\xi_i$  is additive on  $N$ .

Now using Lemma 3.7 and (3.12), we find that

$$\begin{aligned}
 \xi_i((a_1 + a_2)m) &= \xi_i(a_1m) + \xi_i(a_2m) \\
 &= \xi_i(a_1)m + \xi_i(a_2)m + a_1\xi_i(m) + a_2\xi_i(m) \\
 &\quad + \sum_{\substack{r+s=i \\ r,s < i}} \delta_r(a_1)\delta_s(m) + \sum_{\substack{r+s=i \\ r,s < i}} \delta_r(a_2)\delta_s(m). \tag{3.19}
 \end{aligned}$$

On the other hand,

$$\begin{aligned}
 \xi_i((a_1 + a_2)m) &= \xi_i(a_1 + a_2)m + (a_1 + a_2)\xi_i(m) + \sum_{\substack{r+s=i \\ r,s < i}} \delta_r(a_1 + a_2)\delta_s(m) \\
 &= \xi_i(a_1 + a_2)m + a_1\xi_i(m) + a_2\xi_i(m) \\
 &\quad + \sum_{\substack{r+s=i \\ r,s < i}} \delta_r(a_1)\delta_s(m) + \sum_{\substack{r+s=i \\ r,s < i}} \delta_r(a_2)\delta_s(m). \tag{3.20}
 \end{aligned}$$

Above two expressions implies that

$$\xi_i(a_1 + a_2)m = \xi_i(a_1)m + \xi_i(a_2)m. \tag{3.21}$$

In the similar way,

$$n\xi_i(a_1 + a_2) = n\xi_i(a_1) + n\xi_i(a_2). \tag{3.22}$$

Last two equations together yield that  $\xi_i(a_1 + a_2) = \xi_i(a_1) + \xi_i(a_2)$  for all  $a_1, a_2 \in A$  which proves that  $\xi_i$  is additive on  $A$ . Similarly,  $\xi_i$  is additive on  $B$ .  $\square$

**Lemma 3.12.** For any  $a \in A, b \in B, n \in \mathbb{N}$  and  $m \in M$

- (i)  $\xi_i(a + m + b) - \xi_i(a) - \xi_i(m) - \xi_i(b) \in \mathfrak{Z}(\mathfrak{A})$ ,
- (ii)  $\xi_i(a + n + b) - \xi_i(a) - \xi_i(n) - \xi_i(b) \in \mathfrak{Z}(\mathfrak{A})$ .

*Proof.* (i) For any  $a \in A, b \in B, m \in M$  and using  $L_i(p), L_i(q) \in \mathfrak{Z}(\mathfrak{U})$ , we have

$$\begin{aligned} & \xi_i([p, m], q) \\ &= L_i([p, a + m + b], q) \\ &= [[p, L_i(a + m + b)], q] + [[L_i(p), a + m + b], q] \\ &\quad + [[p, a + m + b], L_i(q)] + \sum_{\substack{r+s+t=i \\ r,s,t < i}} [[L_r(p), L_s(a + m + b)], L_t(q)] \\ &= [[p, \xi_i(a + m + b)], q]. \end{aligned} \tag{3.23}$$

Also,

$$\begin{aligned} \xi_i([p, m], q) &= [[p, L_i(m)], q] + [[L_i(p), m], q] + [[p, m], L_i(q)] \\ &\quad + \sum_{\substack{r+s+t=i \\ r,s,t < i}} [[L_r(p), L_s(m)], L_t(q)] \\ &= [[p, \xi_i(m)], q]. \end{aligned} \tag{3.24}$$

From (3.23) and (3.24) it follows that  $[[p, \xi_i(a + m + b) - \xi_i(a) - \xi_i(m) - \xi_i(b)], q] = 0$  which gives  $p(\xi_i(a + m + b) - \xi_i(a) - \xi_i(m) - \xi_i(b))q + q(\xi_i(a + m + b) - \xi_i(a) - \xi_i(m) - \xi_i(b))p = 0$ . Therefore, we obtain that  $\xi_i(a + m + b) - \xi_i(a) - \xi_i(m) - \xi_i(b) \in A + B$ . Since  $am' - m'b = [[a + m + b, m'], q]$  and  $L_i(M) \subseteq M, L_i(q) \in \mathfrak{Z}(\mathfrak{U})$ , for any  $a \in A, b \in B$  and  $m, m' \in M$

$$\begin{aligned} & \xi_i(am' - m'b) \\ &= L_i([a + m + b, m'], q) \\ &= [[L_i(a + m + b), m'], q] + [[a + m + b, L_i(m')], q] \\ &\quad + [[a + m + b, m'], L_i(q)] + \sum_{\substack{r+s+t=i \\ r,s,t < i}} [[L_r(a + m + b), L_s(m')], L_t(q)] \\ &= [[\xi_i(a + m + b), m'], q] + [[a + m + b, \xi_i(m')], q] \\ &\quad + \sum_{\substack{r+s+t=i \\ r,s,t < i}} [[\delta_r(a + m + b), \delta_s(m')], \delta_t(q)] \\ &= [[\xi_i(a + m + b), m'], q] + [[a + b, \xi_i(m')], q] \\ &\quad + \sum_{\substack{r+s=i \\ r,s < i}} \delta_r(a)\delta_s(m') - \sum_{\substack{r+s=i \\ r,s < i}} \delta_r(m')\delta_s(b). \end{aligned}$$

On the other hand,

$$\begin{aligned}
 \xi_i(am' - m'b) &= \xi_i(am') - \xi_i(m'b) \\
 &= \xi_i(a)m' - m'\xi_i(b) + a\xi_i(m') - \xi_i(m')b \\
 &\quad + \sum_{\substack{r+s=i \\ r,s < i}} \delta_r(a)\delta_s(m') - \sum_{\substack{r+s=i \\ r,s < i}} \delta_r(m')\delta_s(b) \\
 &= [[\xi_i(a), m'], q] + [[a, \xi_i(m')], q] + [[\xi_i(b), m'], q] \\
 &\quad + [[b, \xi_i(m')], q] + \sum_{\substack{r+s=i \\ r,s < i}} \delta_r(a)\delta_s(m') - \sum_{\substack{r+s=i \\ r,s < i}} \delta_r(m')\delta_s(b).
 \end{aligned} \tag{3.25}$$

Now from (3.25) and (3.25), we have  $[[\xi_i(a+m+b) - \xi_i(a) - \xi_i(m) - \xi_i(b), m'], q] = 0$  which together with  $\xi_i(a+m+b) - \xi_i(a) - \xi_i(m) - \xi_i(b) \in A+B$  gives that

$$(\xi_i(a+m+b) - \xi_i(a) - \xi_i(m) - \xi_i(b))m' = m'(\xi_i(a+m+b) - \xi_i(a) - \xi_i(m) - \xi_i(b)) \tag{3.26}$$

for all  $a \in A, b \in B$  and  $m, m' \in M$ . In the similar manner, from  $[[a+m+b, n'], q] = [[a+b, n'], q]$  for all  $a \in A, n' \in N$  and  $m \in M$ , we obtain that  $[[\xi_i(a+m+b) - \xi_i(a) - \xi_i(m) - \xi_i(b), n'], q] = 0$ , which together with  $\xi_i(a+m+b) - \xi_i(a) - \xi_i(m) - \xi_i(b) \in A+B$  yields that

$$(\xi_i(a+m+b) - \xi_i(a) - \xi_i(m) - \xi_i(b))n' = n'(\xi_i(a+m+b) - \xi_i(a) - \xi_i(m) - \xi_i(b)) \tag{3.27}$$

for all  $a \in A, b \in B, n' \in N$  and  $m \in M$ . Now combining (3.26) and (3.27), we get  $\xi_i(a+m+b) - \xi_i(a) - \xi_i(m) - \xi_i(b) \in \mathfrak{Z}(\mathfrak{U})$ . Similarly we can find other case.  $\square$

**Lemma 3.13.** For any  $n \in N$  and  $m \in M$ ,  $\xi_i(m+n) = \xi_i(m) + \xi_i(n)$ .

*Proof.* Using Lemmas 3.9 and 3.10, we have

$$\begin{aligned}
 \xi_i(m+n) &= L_i([p+m, p-n], p) \\
 &= [[L_i(p+m), p-n], p] + [[p+m, L_i(p-n)], p] \\
 &\quad + [[p+m, p-n], L_i(p)] \\
 &\quad + \sum_{\substack{r+s+t=i \\ r,s,t < i}} [[L_r(p+m), L_s(p-n)], L_t(p)] \\
 &= [[\xi_i(p+m), p-n], p] + [[p+m, \xi_i(p-n)], p] \\
 &\quad + \sum_{\substack{r+s+t=i \\ r,s,t < i}} [[\delta_r(p+m), \delta_s(p-n)], \delta_t(p)] \\
 &= [[\xi_i(p) + \xi_i(m), p-n], p] + [[p+m, \xi_i(p) - \xi_i(n)], p] \\
 &= [[\xi_i(m), p-n], p] + [[p+m, -\xi_i(n)], p] \\
 &= \xi_i(m) + \xi_i(n).
 \end{aligned}$$

$\square$

**Lemma 3.14.** For any  $a \in A, b \in B, n \in N$  and  $m \in M$ ,

$$\xi_i(a + m + n + b) - \xi_i(a) - \xi_i(m) - \xi_i(n) - \xi_i(b) \in \mathfrak{Z}(\mathfrak{U}).$$

*Proof.* Consider

$$\begin{aligned} \xi_i(m + n) &= L_i([p, a + m + n + b], q) \\ &= [[p, L_i(a + m + n + b)], q] + [[L_i(p), a + m + n + b], q] \\ &\quad + [[p, a + m + n + b], L_i(q)] \\ &\quad + \sum_{\substack{r+s+t=i \\ r,s,t < i}} [[L_r(p), L_s(a + m + n + b)], L_t(q)] \\ &= [[p, \xi_i(a + m + n + b)], q]. \end{aligned} \tag{3.28}$$

On the other hand,

$$\begin{aligned} \xi_i(m + n) &= \xi_i(m) + \xi_i(n) \\ &= \xi_i([p, a], q) + \xi_i([p, m], q) + \xi_i([p, n], q) + \xi_i([p, b], q) \\ &= [[p, L_i(a)], q] + [[p, L_i(m)], q] + [[p, L_i(n)], q] + [[p, L_i(b)], q] \\ &\quad + \sum_{\substack{r+s+t=i \\ r,s,t < i}} [[L_r(p), L_s(a)], L_t(q)] + \sum_{\substack{r+s+t=i \\ r,s,t < i}} [[L_r(p), L_s(m)], L_t(q)] \\ &\quad + \sum_{\substack{r+s+t=i \\ r,s,t < i}} [[L_r(p), L_s(n)], L_t(q)] + \sum_{\substack{r+s+t=i \\ r,s,t < i}} [[L_r(p), L_s(b)], L_t(q)] \\ &= [[p, \xi_i(a) + \xi_i(m) + \xi_i(n) + \xi_i(b)], q]. \end{aligned} \tag{3.29}$$

In view of (3.28) and (3.29), we have  $[[p, \xi_i(a + m + n + b) - \xi_i(a) - \xi_i(m) - \xi_i(n) - \xi_i(b)], q] = 0$  which gives  $p(\xi_i(a + m + n + b) - \xi_i(a) - \xi_i(m) - \xi_i(n) - \xi_i(b))q + q(\xi_i(a + m + n + b) - \xi_i(a) - \xi_i(m) - \xi_i(n) - \xi_i(b))p = 0$ . Therefore, we obtain that  $\xi_i(a + m + n + b) - \xi_i(a) - \xi_i(m) - \xi_i(n) - \xi_i(b) \in A + B$ . Since  $am' - m'b = [[a + m + n + b, m'], q]$  and  $L_i(M) \subseteq M, L_i(q) \in \mathfrak{Z}(\mathfrak{U})$ , for any  $a \in A, b \in B$  and  $m, m' \in M$

$$\begin{aligned} \xi_i(am' - m'b) &= L_i([a + m + n + b, m'], q) \\ &= [[L_i(a + m + n + b), m'], q] + [[a + m + n + b, L_i(m')], q] \\ &\quad + [[a + m + n + b, m'], L_i(q)] \\ &\quad + \sum_{\substack{r+s+t=i \\ r,s,t < i}} [[L_r(a + m + n + b), L_s(m')], L_t(q)] \end{aligned}$$

$$\begin{aligned}
&= [[\xi_i(a+m+n+b), m'], q] + [[a+m+n+b, \xi_i(m')], q] \\
&\quad + \sum_{\substack{r+s+t=i \\ r,s,t < i}} [[\delta_r(a+m+n+b), \delta_s(m')], \delta_t(q)] \\
&= [[\xi_i(a+m+n+b), m'], q] + [[a+b, \xi_i(m')], q] \\
&\quad + \sum_{\substack{r+s=i \\ r,s < i}} \delta_r(a)\delta_s(m') - \sum_{\substack{r+s=i \\ r,s < i}} \delta_r(m')\delta_s(b). \tag{3.30}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\xi_i(am' - m'b) &= \xi_i(am') - \xi_i(m'b) \\
&= \xi_i(a)m' - m'\xi_i(b) + a\xi_i(m') - \xi_i(m')b \\
&\quad + \sum_{\substack{r+s=i \\ r,s < i}} \delta_r(a)\delta_s(m') - \sum_{\substack{r+s=i \\ r,s < i}} \delta_r(m')\delta_s(b) \\
&= [[\xi_i(a), m'], q] + [[a, \xi_i(m')], q] + [[\xi_i(b), m'], q] \\
&\quad + [[b, \xi_i(m')], q] + \sum_{\substack{r+s=i \\ r,s < i}} \delta_r(a)\delta_s(m') - \sum_{\substack{r+s=i \\ r,s < i}} \delta_r(m')\delta_s(b). \tag{3.31}
\end{aligned}$$

Now from (3.30) and (3.31), we have  $[[\xi_i(a+m+n+b) - \xi_i(a) - \xi_i(m) - \xi_i(n) - \xi_i(b), m'], q] = 0$  which together with  $\xi_i(a+m+n+b) - \xi_i(a) - \xi_i(m) - \xi_i(n) - \xi_i(b) \in A + B$  gives that

$$\begin{aligned}
&(\xi_i(a+m+n+b) - \xi_i(a) - \xi_i(m) - \xi_i(n) - \xi_i(b))m' \\
&= m'(\xi_i(a+m+n+b) - \xi_i(a) - \xi_i(m) - \xi_i(n) - \xi_i(b)) \tag{3.32}
\end{aligned}$$

for all  $a \in A, b \in B$  and  $m, m' \in M$ . In the similar manner, from  $[[a+m+n+b, n'], q] = [[a+b, n'], q]$  for all  $a \in A, n' \in N$  and  $m \in M$ , we obtain that  $[[\xi_i(a+m+n+b) - \xi_i(a) - \xi_i(m) - \xi_i(n) - \xi_i(b), n'], q] = 0$  which together with  $\xi_i(a+m+n+b) - \xi_i(a) - \xi_i(m) - \xi_i(n) - \xi_i(b) \in A + B$  gives

$$\begin{aligned}
&(\xi_i(a+m+n+b) - \xi_i(a) - \xi_i(m) - \xi_i(n) - \xi_i(b))n' \\
&= n'(\xi_i(a+m+n+b) - \xi_i(a) - \xi_i(m) - \xi_i(n) - \xi_i(b)) \tag{3.33}
\end{aligned}$$

for all  $a \in A, b \in B, n' \in N$  and  $m \in M$ . Now combining (3.32) and (3.33), we get  $\xi_i(a+m+n+b) - \xi_i(a) - \xi_i(m) - \xi_i(n) - \xi_i(b) \in \mathfrak{Z}(\mathfrak{U})$ .  $\square$

**Lemma 3.15.** For any  $n \in N$  and  $m \in M$ , we have

$$\begin{aligned}
(i) \quad \xi_i(mn) &= \xi_i(m)n + m\xi_i(n) + \sum_{\substack{r+s=i \\ r,s < i}} \delta_r(m)\delta_s(n), \\
(ii) \quad \xi_i(nm) &= \xi_i(n)m + n\xi_i(m) + \sum_{\substack{r+s=i \\ r,s < i}} \delta_r(m)\delta_s(n).
\end{aligned}$$

*Proof.* For any  $m, m' \in M$  and  $n \in N$ , we have

$$\begin{aligned} \xi_i([[m, n], m']) &= [[L_i(m), n], m'] + [[m, L_i(n)], m'] + [[m, n], L_i(m')] \\ &\quad + \sum_{\substack{r+s+t=i \\ r,s,t < i}} [[L_r(m), L_s(n)], L_t(m')] \\ &= [[\xi_i(m), n], m'] + [[m, \xi_i(n)], m'] + [[m, n], \xi_i(m')] \\ &\quad + \sum_{\substack{r+s+t=i \\ r,s,t < i}} [[\delta_r(m), \delta_s(n)], \delta_t(m')]. \end{aligned} \tag{3.34}$$

On the other way,

$$\begin{aligned} \xi_i([[m, n], m']) &= \xi_i(mnm' + m'nm) \\ &= \xi_i(mn)m' + mn\xi_i(m') + m'\xi_i(nm) + \xi_i(m')nm \\ &\quad + \sum_{\substack{r+s=i \\ r,s < i}} \delta_r(mn)\delta_s(m') + \sum_{\substack{r+s=i \\ r,s < i}} \delta_r(m')\delta_s(nm). \end{aligned} \tag{3.35}$$

From the above two expressions, we get

$$[\xi_i(mn) - \xi_i(nm) - [\xi_i(m), n] - [m, \xi_i(n)] - \sum_{\substack{r+s=i \\ r,s < i}} [\delta_r(m), \delta_s(n)], m'] = 0. \tag{3.36}$$

Similarly, we have

$$[\xi_i(mn) - \xi_i(nm) - [\xi_i(m), n] - [m, \xi_i(n)] - \sum_{\substack{r+s=i \\ r,s < i}} [\delta_r(m), \delta_s(n)], n'] = 0. \tag{3.37}$$

From (3.36) and (3.37) it follows that

$$\begin{aligned} &\{\xi_i(mn) - \xi_i(m)n - m\xi_i(n) - \sum_{\substack{r+s=i \\ r,s < i}} \delta_r(m)\delta_s(n)\} \\ &+ \{-\xi_i(nm) + \xi_i(n)m + n\xi_i(m) + \sum_{\substack{r+s=i \\ r,s < i}} \delta_r(n)\delta_s(m)\} \in \mathfrak{Z}(\mathfrak{U}). \end{aligned} \tag{3.38}$$

From the assumption A does not contain nonzero central ideals. Assume that

$$\omega(m, n) = \xi_i(mn) - \xi_i(m)n - m\xi_i(n) + \sum_{\substack{r+s=i \\ r,s < i}} \delta_r(m)\delta_s(n).$$

Note that  $\omega(m, n) \in \mathfrak{Z}(A)$  for all  $m \in M$  and  $n \in N$ . On using Lemma 3.7, we obtain that

$$\begin{aligned} \omega(am, n) &= \xi_i(amn) - \xi_i(am)n - am\xi_i(n) + \sum_{\substack{r+s=i \\ r,s < i}} \delta_r(am)\delta_s(n) \\ &= \xi_i(a)mn + a\xi_i(mn) - \xi_i(a)mn - a\xi_i(m)n - am\xi_i(n) \\ &\quad + \sum_{\substack{r+s+t=i \\ r,s,t < i}} \delta_r(a)\delta_s(m)\delta_t(n) \\ &= a\xi_i(m)n + am\xi_i(n) + a\omega(m, n) - a\xi_i(m)n - am\xi_i(n) \\ &= a\omega(m, n). \end{aligned}$$

This implies  $A\omega(m, n)$  is a central ideal of  $A$ . Therefore,  $\omega(m, n) = 0$ , that is,

$$\xi_i(mn) = \xi_i(m)n + m\xi_i(n) + \sum_{\substack{r+s=i \\ r,s < i}} \delta_r(m)\delta_s(n)$$

for all  $m \in M$  and  $n \in N$ . Similarly, we have

$$\xi_i(nm) = \xi_i(n)m + n\xi_i(m) + \sum_{\substack{r+s=i \\ r,s < i}} \delta_r(n)\delta_s(m)$$

for all  $m \in M$  and  $n \in N$ . □

*Remark 3.16.* Now, in view of Lemma 3.14 we define a map  $\gamma_{i_2} : \mathfrak{U} \rightarrow \mathfrak{Z}(\mathfrak{U})$  by

$$\gamma_{i_2}(x) = \xi_i(x) - \xi_i(xpx) - \xi_i(xqx) - \xi_i(xqx) - \xi_i(xqx).$$

Clearly,  $\gamma_{i_2}(x) \in \mathfrak{Z}(\mathfrak{U})$  and  $\gamma_{i_2}(A) = \gamma_{i_2}(M) = \gamma_{i_2}(N) = \gamma_{i_2}(B) = 0$ . Now we are ready to define a map  $\delta_i : \mathfrak{U} \rightarrow \mathfrak{U}$  by  $\delta_i(x) = \xi_i(x) - \gamma_{i_2}(x)$ . Then  $L_i(x) = \delta_i(x) + \gamma_{i_1}(x) + \gamma_{i_2}(x) = \delta_i(x) + \gamma_i(x)$ , where  $\gamma_i(x) = \gamma_{i_1}(x) + \gamma_{i_2}(x)$  is a mapping from  $\mathfrak{U}$  to its center.

**Lemma 3.17.**  $\delta_i(a + m + n + b) = \delta_i(a) + \delta_i(m) + \delta_i(n) + \delta_i(b)$  for all  $a \in A, b \in B, m \in M$  and  $n \in N$ .

*Proof.* It is clear that

$$\begin{aligned} \delta_i(a + m + n + b) &= \xi_i(a + m + n + b) - \gamma_{i_2}(a + m + n + b) \\ &= \xi_i(a + m + n + b) - \xi_i(a + m + n + b) \\ &\quad + \xi_i(a) + \xi_i(m) + \xi_i(n) + \xi_i(b) \\ &= \xi_i(a) + \xi_i(m) + \xi_i(n) + \xi_i(b) \\ &= \delta_i(a) + \delta_i(m) + \delta_i(n) + \delta_i(b) \end{aligned}$$

for all  $a \in A, b \in B, m \in M$  and  $n \in N$ . □

Now we are well equipped to prove our main theorem.

*Proof of Theorem 3.1.* For any  $x, y \in \mathfrak{U}$ . Suppose that  $x = a_1 + m_1 + n_1 + b_1$  and  $x = a_2 + m_2 + n_2 + b_2$  where  $a_1, a_2 \in \mathbf{A}, b_1, b_2 \in \mathbf{B}, n_1, n_2 \in \mathbf{N}$  and  $m_1, m_2 \in \mathbf{M}$ .

$$\begin{aligned} \delta_i(x + y) &= \delta_i(a_1 + m_1 + n_1 + b_1 + a_2 + m_2 + n_2 + b_2) \\ &= \delta_i(a_1 + a_2) + \delta_i(m_1 + m_2) + \delta_i(n_1 + n_2) + \delta_i(b_1 + b_2) \\ &= \delta_i(a_1) + \delta_i(m_1) + \delta_i(n_1) + \delta_i(b_1) \\ &\quad + \delta_i(a_2) + \delta_i(m_2) + \delta_i(n_2) + \delta_i(b_2) \\ &= \delta_i(a_1 + m_1 + n_1 + b_1) + \delta_i(a_2 + m_2 + n_2 + b_2) \\ &= \delta_i(x) + \delta_i(y). \end{aligned}$$

Thus,  $\delta_i$  is additive on  $\mathfrak{U}$ . Also, from Lemmas 3.7, 3.8 and 3.17

$$\begin{aligned} \delta_i(xy) &= \delta_i(a_1a_2 + a_1m_2 + m_1n_2 + m_1b_2 + n_1a_2 + n_1m_2 + b_1n_2 + b_1b_2) \\ &= \delta_i(a_1a_2 + m_1n_2) + \delta_i(a_1m_2 + m_1b_2) \\ &\quad + \delta_i(n_1a_2 + b_1n_2) + \delta_i(n_1m_2 + b_1b_2) \\ &= \delta_i(a_1a_2) + \delta_i(a_1m_2) + \delta_i(n_1a_2) + \delta_i(n_1m_2) \\ &\quad + \delta_i(m_1n_2) + \delta_i(m_1b_2) + \delta_i(b_1n_2) + \delta_i(b_1b_2) \\ &= \delta_i(a_1)a_2 + a_1\delta_i(a_2) + \sum_{\substack{r+s=i \\ r,s < i}} \delta_r(a_1)\delta_s(a_2) + \delta_i(a_1)m_2 + a_1\delta_i(m_2) \\ &\quad + \sum_{\substack{r+s=i \\ r,s < i}} \delta_r(a_1)\delta_s(m_2) + \delta_i(n_1)a_2 + n_1\delta_i(a_2) + \sum_{\substack{r+s=i \\ r,s < i}} \delta_r(n_1)\delta_s(a_2) \\ &\quad + \delta_i(n_1)m_2 + n_1\delta_i(m_2) + \sum_{\substack{r+s=i \\ r,s < i}} \delta_r(n_1)\delta_s(m_2) + \delta_i(m_1)n_2 \\ &\quad + m_1\delta_i(n_2) + \sum_{\substack{r+s=i \\ r,s < i}} \delta_r(m_1)\delta_s(n_2) + \delta_i(m_1)b_2 + m_1\delta_i(b_2) \\ &\quad + \sum_{\substack{r+s=i \\ r,s < i}} \delta_r(m_1)\delta_s(b_2) + \delta_i(b_1)n_2 + b_1\delta_i(n_2) + \sum_{\substack{r+s=i \\ r,s < i}} \delta_r(b_1)\delta_s(n_2) \\ &\quad + \delta_i(b_1)b_2 + b_1\delta_i(b_2) + \sum_{\substack{r+s=i \\ r,s < i}} \delta_r(b_1)\delta_s(b_2) \\ &= (\delta_i(a_1) + \delta_i(m_1) + \delta_i(n_1) + \delta_i(b_1))(a_2 + m_2 + n_2 + b_2) \\ &\quad + (a_1 + m_1 + n_1 + b_1)(\delta_i(a_2) + \delta_i(m_2) + \delta_i(n_2) + \delta_i(b_2)) \\ &\quad + \sum_{\substack{r+s=i \\ r,s < i}} \{\delta_r(a_1) + \delta_r(m_1) + \delta_r(n_1) + \delta_r(b_1)\} \\ &\quad \{\delta_s(a_2) + \delta_s(m_2) + \delta_s(n_2) + \delta_s(b_2)\} \\ &= \delta_i(x)y + x\delta_i(y) + \sum_{\substack{r+s=i \\ r,s < i}} \delta_r(x)\delta_s(y). \end{aligned}$$

This shows that  $\{\delta_i\}_{i \in \mathbb{N}}$  is an additive higher derivation. Now, we have to prove that  $\gamma_i$  vanishes on all Lie triple products of  $\mathfrak{U}$ . It is easy to see that  $\gamma_i(x) \in \mathfrak{Z}(\mathfrak{U})$  for all  $x \in \mathfrak{U}$ . Further,

$$\begin{aligned} \gamma_i([[x, y], z]) &= L_i([[x, y], z]) - \delta_i([[x, y], z]) \\ &= [[L_i(x), y], z] + [[x, L_i(y)], z] + [[x, y], L_i(z)] \\ &\quad - [[\delta_i(x), y], z] - [[x, \delta_i(y)], z] - [[x, y], \delta_i(z)] \\ &= 0. \end{aligned}$$

for all  $x, y, z \in \mathfrak{U}$ . Hence, this proves the required result.  $\square$

In particular, we have the following corollary.

**Corollary 3.18.** [1, Theorem 3.1] *Let  $\mathfrak{T}$  be a 2-torsion free triangular algebra and  $\mathfrak{L} = \{L_i\}_{i \in \mathbb{N}}$  be a multiplicative Lie triple higher derivation on  $\mathfrak{T}$ . Let us assume that*

- (i)  $\pi_A(\mathfrak{Z}(\mathfrak{T})) = \mathfrak{Z}(A)$  and  $\pi_B(\mathfrak{Z}(\mathfrak{T})) = \mathfrak{Z}(B)$ ,
- (ii) either A or B does not contain nonzero central ideals,

*Then every multiplicative Lie triple higher derivation  $\mathfrak{L} = \{L_i\}_{i \in \mathbb{N}}$  is of the standard form, i.e., each component  $L_i$  has the form  $L_i = \delta_i + \gamma_i$ , where  $\{\delta_i\}_{i \in \mathbb{N}}$  is an additive higher derivation on  $\mathfrak{T}$  and  $\{\gamma_i\}_{i \in \mathbb{N}}$  is a sequence of mapping  $\gamma_i : \mathfrak{T} \rightarrow \mathfrak{Z}(\mathfrak{T})$  vanishing at Lie triple products in  $\mathfrak{T}$ .*

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