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Generalization of Ostrowski's Inequality for Differentiable Functions and its Applications

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Abstract. We first establish weighted Ostrowski type inequalities for bounded differentiable functions. This inequality is also obtained for bounded above and bounded below differentiable functions. Some applications of the proposed results are presented to numerical standard and non standard quadrature rules. We recapture known results as well as obtain new results.

Keywords: Ostrowski's inequality, Numerical integration, differentiable functions.

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1. Introduction

Our work mostly deals with integral inequalities. To highlight its importance we quote here from [20], "Among the many types of inequalities, integral

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inequalities are of supreme importance because over the last few decades this field has proven to be an extensively applicable field. The integral inequalities of various types have been widely studied in most subjects involving mathematical analysis. These inequalities are particularly useful in approximation theory and in numerical analysis where estimates of approximation errors are involved." Ostrowski inequality [25] is one of the most famous inequalities, first presented by Alexander Markowich Ostrowski in 1938. It can be used to determine absolute deviation of functional values from its mean values. It is extremely important because of its wide range of applications in different areas of mathematics such as numerical integration, integral operator theory, probability theory and statistics. This inequality states that:

Proposition 1.1. *Let $\rho : I \to \mathbb{R}$ be a differentiable function on I^o such that $\rho \in L[j,k]$ where $j < k$ whose derivative ρ' is bounded on interior of I, i.e., $\|\rho'\|_\infty := \sup_{t\in(j,k)} |\rho'(t)| < \infty$. Then*

$$\left|\rho(\theta) - \frac{1}{k-j}\int_j^k \rho(t)dt\right| \leq (k-j)\left[\frac{1}{4} + \frac{\left(\theta - \frac{j+k}{2}\right)^2}{(k-j)^2}\right]\|\rho'(x)\|_\infty. \tag{1.1}$$

The constant $\frac{1}{4}$ is the best possible constant that it cannot be replaced by smaller one.

Ostrowski inequality for differentiable functions has been generalized many times, as stated in [6, 18, 19, 27]. For latest work related to Ostrowski inequality, we refer the reader to following articles [1, 2, 3, 5, 8, 12, 13, 14, 15, 16, 17, 21, 22, 23, 24, 26]

To prove our main results, we need the following two lemmas from [7] and [10].

Lemma 1.2. *Let ρ: $I \to \mathbb{R}$, be a differentiable function in interior I^0 of interval I and also let $[j,k] \subset I^0$. Then the following identity holds*

$$\begin{aligned}\int_j^k K(\theta,t)\rho'(t)dt &= \rho(k)\int_\beta^k \omega(t)dt - \rho(j)\int_\alpha^j \omega(t)dt + \rho(\theta)\int_\alpha^{\frac{\alpha+\beta}{2}} \omega(t)dt \\ &+ \rho(j+k-\theta)\int_{\frac{\alpha+\beta}{2}}^\beta \omega(t)dt - \int_j^k \rho(t)\omega(t)dt, \end{aligned} \tag{1.2}$$

where $K(\theta,t)$ is defined as:

$$K(\theta,t) = \begin{cases} \displaystyle\int_\alpha^t \omega(t)dt, & \text{if} \quad t \in [j,\theta]; \\ \displaystyle\int_{\frac{\alpha+\beta}{2}}^t \omega(t)dt, & \text{if} \quad t \in (\theta, j+k-\theta]; \\ \displaystyle\int_\beta^t \omega(t)dt, & \text{if} \quad t \in (j+k-\theta, k]; \end{cases} \tag{1.3}$$

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where $\forall\ \theta \in [j,k]$, $\alpha = j + \lambda\frac{k-j}{2}$, $\beta = k - \lambda\frac{k-j}{2}$ *and* $\lambda \in [0,1]$.

Lemma 1.3. *Let* $\rho : [j,k] \to \mathbb{R}$ *be a differentiable function such that* $\gamma(\theta) \leq \rho'(\theta) \leq \Gamma(\theta)$ *for any* $\gamma, \Gamma \in C[j,k]$ *and* $\theta \in [j,k]$. *Then we have*

$$\left|\rho'(t) - \frac{\gamma(t) + \Gamma(t)}{2}\right| \leq \frac{\Gamma(t) - \gamma(t)}{2}. \tag{1.4}$$

We are ready to present our main theorem, which will be generalized in two ways: first, by adding weights that are probability density functions, and second, by adding a parameter. In this way, we will capture variety of results from various articles as special cases.

2. Main results

Theorem 2.1. *Let* ρ*:* $I \to \mathbb{R}$*, be a differentiable function in* I^0 *and also let* $\gamma(\theta) \leq \rho'(\theta) \leq \Gamma(\theta)$ *for any* $\gamma, \Gamma \in C[j,k]$ *and* $\theta \in [j,k]$. *Then*

$$\begin{aligned} m(\theta,\lambda) \leq \rho(k)\int_\beta^k \omega(t)dt - \rho(j)\int_\alpha^j \omega(t)dt + \rho(\theta)\int_\alpha^{\frac{\alpha+\beta}{2}} \omega(t)dt \\ + \rho(j+k-\theta)\int_{\frac{\alpha+\beta}{2}}^\beta \omega(t)dt - \int_j^k \rho(t)\omega(t)dt \leq M(\theta,\lambda), \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} m(\theta,\lambda) &= \int_j^\theta \left(\left(\int_\alpha^t \omega(t)dt - \left|\int_\alpha^t \omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right. \\ &+ \left.\left(\int_\alpha^t \omega(t)dt + \left|\int_\alpha^t \omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt \\ &+ \int_\theta^{j+k-\theta}\left(\left(\int_{\frac{\alpha+\beta}{2}}^t \omega(t)dt - \left|\int_{\frac{\alpha+\beta}{2}}^t \omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right. \\ &+ \left.\left(\int_{\frac{\alpha+\beta}{2}}^t \omega(t)dt + \left|\int_{\frac{\alpha+\beta}{2}}^t \omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt \\ &+ \int_{j+k-\theta}^k\left(\left(\int_\beta^t \omega(t)dt - \left|\int_\beta^t \omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right. \\ &+ \left.\left(\int_\beta^t \omega(t)dt + \left|\int_\beta^t \omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt \end{aligned}$$

and

$$\begin{aligned} M(\theta,\lambda) &= \int_j^\theta \left(\left(\int_\alpha^t \omega(t)dt + \left|\int_\alpha^t \omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right. \\ &+ \left.\left(\int_\alpha^t \omega(t)dt - \left|\int_\alpha^t \omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt \end{aligned}$$

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$$\begin{aligned}
&+ \int_{\theta}^{j+k-\theta}\left(\left(\int_{\frac{\alpha+\beta}{2}}^{t}\omega(t)dt+\left|\int_{\frac{\alpha+\beta}{2}}^{t}\omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right.\\
&+ \left.\left(\int_{\frac{\alpha+\beta}{2}}^{t}\omega(t)dt-\left|\int_{\frac{\alpha+\beta}{2}}^{t}\omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt\\
&+ \int_{j+k-\theta}^{k}\left(\left(\int_{\beta}^{t}\omega(t)dt+\left|\int_{\beta}^{t}\omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right.\\
&+ \left.\left(\int_{\beta}^{t}\omega(t)dt-\left|\int_{\beta}^{t}\omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt.
\end{aligned}$$

Proof. Using (1.2) and (1.3), we obtain

$$\begin{aligned}
&\int_{j}^{k}K(\theta,t)\left(\rho'(t)-\frac{\gamma(t)+\Gamma(t)}{2}\right)dt\\
&=\int_{j}^{k}K(\theta,t)\rho'(t)dt-\frac{1}{2}\left(\int_{j}^{k}K(\theta,t)\left(\gamma(t)+\Gamma(t)\right)dt\right)\\
&=\rho(k)\int_{\beta}^{k}\omega(t)dt-\rho(j)\int_{\alpha}^{j}\omega(t)dt+\rho(\theta)\int_{\alpha}^{\frac{\alpha+\beta}{2}}\omega(t)dt\\
&+\rho(j+k-\theta)\int_{\frac{\alpha+\beta}{2}}^{\beta}\omega(t)dt-\int_{j}^{k}\omega(t)\rho(t)dt\\
&-\frac{1}{2}\left[\int_{j}^{\theta}\int_{\alpha}^{t}\omega(t)dt\left(\gamma(t)+\Gamma(t)\right)dt+\int_{\theta}^{j+k-\theta}\int_{\frac{\alpha+\beta}{2}}^{t}\omega(t)dt\left(\gamma(t)+\Gamma(t)\right)dt\right.\\
&+\left.\int_{j+k-\theta}^{k}\int_{\beta}^{t}\omega(t)dt\left(\gamma(t)+\Gamma(t)\right)dt\right]. \qquad (2.2)
\end{aligned}$$

Applying absolute value and using (1.4) we get

$$\begin{aligned}
&\left|\int_{\beta}^{k}\omega(t)dt\rho(k)-\rho(j)\int_{\alpha}^{j}\omega(t)dt+\rho(\theta)\int_{\alpha}^{\frac{\alpha+\beta}{2}}\omega(t)dt+\rho(j+k-\theta)\right.\\
&\times\int_{\frac{\alpha+\beta}{2}}^{\beta}\omega(t)dt-\int_{j}^{k}\omega(t)\rho(t)dt-\frac{1}{2}\left[\int_{j}^{\theta}\int_{\alpha}^{t}\omega(t)dt\left(\gamma(t)+\Gamma(t)\right)dt\right.\\
&+\int_{\theta}^{j+k-\theta}\int_{\frac{\alpha+\beta}{2}}^{t}\omega(t)dt\left(\gamma(t)+\Gamma(t)\right)dt\\
&+\left.\left.\int_{j+k-\theta}^{k}\int_{\beta}^{t}\omega(t)dt\left(\gamma(t)+\Gamma(t)\right)dt\right]\right|\\
&=\left|\int_{j}^{k}K(\theta,t)\left(\rho'(t)-\frac{\gamma(t)+\Gamma(t)}{2}\right)dt\right|
\end{aligned}$$

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$$\leq \int_j^k |K(\theta,t)| \left| \left(\rho'(t) - \frac{\gamma(t)+\Gamma(t)}{2} \right) dt \right|$$
$$\leq \int_j^k |K(\theta,t)| \left(\frac{\Gamma(t)-\gamma(t)}{2} \right) dt$$
$$= \frac{1}{2} \left[\int_j^\theta \left| \int_\alpha^t \omega(t)dt \right| (\Gamma(t)-\gamma(t))\, dt + \int_\theta^{j+k-\theta} \left| \int_{\frac{\alpha+\beta}{2}}^t \omega(t)dt \right| \right.$$
$$\left. \times (\Gamma(t)-\gamma(t))\, dt + \int_{j+k-\theta}^k \left| \int_\beta^t \omega(t)dt \right| (\Gamma(t)-\gamma(t))\, dt \right]. \tag{2.3}$$

After rearranging (2.3), we get the required result. □

Remark 2.2. It is worth mentioning that if we put $\omega(t) = \frac{1}{k-j}$ in our main result we will get the following result.

Corollary 2.3. *Let all the assumptions of Theorem* 2.1 *be valid. Then*

$$m_0(\theta,\lambda) \leq \lambda \frac{\rho(j)+\rho(k)}{2} + (1-\lambda)\frac{\rho(\theta)+\rho(j+k-\theta)}{2} - \int_j^k \rho(t)d(t)$$
$$\leq M_0(\theta,\lambda),$$

where

$$m_0(\theta,\lambda) = \frac{1}{k-j} \left[\int_{-\lambda\frac{k-j}{2}}^{\theta-(j+\lambda\frac{k-j}{2})} \left(\frac{\eta+|\eta|}{2} \Gamma\left(\eta + j + \lambda\frac{k-j}{2} \right) \right. \right.$$
$$\left. + \frac{\eta-|\eta|}{2} \gamma\left(\eta + j + \lambda\frac{k-j}{2} \right) \right) d\eta$$
$$+ \int_{\theta-\frac{j+k}{2}}^{\frac{j+k}{2}-\theta} \left(\frac{\eta+|\eta|}{2} \Gamma\left(\eta + \frac{j+k}{2} \right) + \frac{\eta-|\eta|}{2} \gamma\left(\eta + \frac{j+k}{2} \right) \right) d\eta$$
$$+ \int_{j+\lambda\frac{k-j}{2}-\theta}^{\lambda\frac{k-j}{2}} \left(\frac{\eta+|\eta|}{2} \Gamma\left(\eta + j - \lambda\frac{k-j}{2} \right) \right.$$
$$\left.\left. + \frac{\eta-|\eta|}{2} \gamma\left(\eta + j - \lambda\frac{k-j}{2} \right) \right) d\eta \right]$$

and

$$M_0(\theta,\lambda) = \frac{1}{k-j} \left[\int_{-\lambda\frac{k-j}{2}}^{\theta-(j+\lambda\frac{k-j}{2})} \left(\frac{\eta-|\eta|}{2} \Gamma\left(\eta + j + \lambda\frac{k-j}{2} \right) \right. \right.$$
$$\left. + \frac{\eta+|\eta|}{2} \gamma\left(\eta + j + \lambda\frac{k-j}{2} \right) \right) d\eta$$

$$+\int_{\theta-\frac{j+k}{2}}^{\frac{j+k}{2}-\theta}\left(\frac{\eta-|\eta|}{2}\Gamma\left(\eta+\frac{j+k}{2}\right)+\frac{\eta+|\eta|}{2}\gamma\left(\eta+\frac{j+k}{2}\right)\right)d\eta$$
$$+\int_{j+\lambda\frac{k-j}{2}-\theta}^{\lambda\frac{k-j}{2}}\left(\frac{\eta-|\eta|}{2}\Gamma\left(\eta+j-\lambda\frac{k-j}{2}\right)\right.$$
$$\left.\left.+\frac{\eta-|\eta|}{2}\gamma\left(\eta+j-\lambda\frac{k-j}{2}\right)\right)d\eta\right],$$

which is Theorem 2.3 of [8] and hence all its Corollaries and Remarks and further consequences would become our special cases.

Throughout the section γ_0, γ_1, Γ_0, Γ_1 are real constants.

Remark 2.4. If we put $\lambda = 1$ in (2.1), we obtain following result.

Corollary 2.5. *Let all the assumptions of Theorem* 2.1 *be valid. Then*

$$m_1 \le \rho(k)\int_{\frac{j+k}{2}}^{k}\omega(t)dt+\rho(j)\int_{j}^{\frac{j+k}{2}}\omega(t)dt-\int_{j}^{k}\rho(t)\omega(t)dt \le M_1, \tag{2.4}$$

where

$$\begin{aligned} m_1 &= \int_j^k\left[\left(\left(\int_{\frac{j+k}{2}}^t\omega(t)dt-\left|\int_{\frac{j+k}{2}}^t\omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right.\right.\\ &+ \left.\left.\left(\int_{\frac{j+k}{2}}^t\omega(t)dt+\left|\int_{\frac{j+k}{2}}^t\omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)\right]dt \end{aligned}$$

and

$$\begin{aligned} M_1 &= \int_j^k\left[\left(\left(\int_{\frac{j+k}{2}}^t\omega(t)dt+\left|\int_{\frac{j+k}{2}}^t\omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right.\right.\\ &+ \left.\left.\left(\int_{\frac{j+k}{2}}^t\omega(t)dt-\left|\int_{\frac{j+k}{2}}^t\omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)\right]dt. \end{aligned}$$

Special Case 2.5.(a) If we take, $\gamma(t)=\gamma_0\neq 0$, $\Gamma(t)=\Gamma_0\neq 0$ and $\omega(t)=\frac{1}{k-j}$ in (2.4), then

$$\frac{(k-j)}{8}(\gamma_0-\Gamma_0)\le\frac{\rho(j)+\rho(k)}{2}-\frac{1}{k-j}\int_j^k\rho(t)dt$$
$$\le\frac{(k-j)}{8}(\Gamma_0-\gamma_0),$$

which is the Corollary 2 of [27] and Special Case 2.4.1 of [8].

Special Case 2.5.(b) If we take, $\gamma(t)=\gamma_1 t+\gamma_0\neq 0$, $\Gamma(t)=\Gamma_1 t+\Gamma_0\neq 0$ and $\omega(t)=\frac{1}{k-j}$ in (2.4), then

$$m_2\le\frac{\rho(j)+\rho(k)}{2}-\frac{1}{k-j}\int_j^k\rho(t)dt\le M_2,$$

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where

$$m_2 = \frac{(k-j)}{8}\left[\frac{(k-j)}{3}(\gamma_1+\Gamma_1)+\frac{(j+k)}{2}(\gamma_1-\Gamma_1)+\gamma_0-\Gamma_0\right]$$

and

$$M_2 = \frac{(k-j)}{8}\left[\frac{(k-j)}{3}(\gamma_1+\Gamma_1)+\frac{(j+k)}{2}(\Gamma_1-\gamma_1)+\Gamma_0-\gamma_0\right],$$

which is Special Case 2.4.2 of [8].

Remark 2.6. If we choose $\theta = \frac{j+k}{2}$ in (2.1), we obtain the following result.

Corollary 2.7. *Let all the assumptions of Theorem* 2.1 *be valid. Then*

$$m_3(\lambda) \le \rho(k)\int_\beta^k \omega(t)dt - \rho(j)\int_\alpha^j \omega(t)dt + \rho\left(\frac{k+j}{2}\right)\int_\alpha^\beta \omega(t)dt - \int_j^k \rho(t)\omega(t)dt \le M_3(\lambda), \quad (2.5)$$

where

$$\begin{aligned} m_3(\lambda) &= \int_j^{\frac{j+k}{2}} \left(\left(\int_\alpha^t \omega(t)dt - \left|\int_\alpha^t \omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right. \\ &+ \left.\left(\int_\alpha^t \omega(t)dt + \left|\int_\alpha^t \omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt \\ &+ \int_{\frac{j+k}{2}}^k \left(\left(\int_\beta^t \omega(t)dt - \left|\int_\beta^t \omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right. \\ &+ \left.\left(\int_\beta^t \omega(t)dt + \left|\int_\beta^t \omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt \end{aligned}$$

and

$$\begin{aligned} M_3(\lambda) &= \int_j^{\frac{j+k}{2}} \left(\left(\int_\alpha^t \omega(t)dt + \left|\int_\alpha^t \omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right. \\ &+ \left.\left(\int_\alpha^t \omega(t)dt - \left|\int_\alpha^t \omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt \\ &+ \int_{\frac{j+k}{2}}^k \left(\left(\int_\beta^t \omega(t)dt + \left|\int_\beta^t \omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right. \\ &+ \left.\left(\int_\beta^t \omega(t)dt - \left|\int_\beta^t \omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt. \end{aligned}$$

Remark 2.8. If we choose $\lambda = 0$ in (2.5), we obtain the following result.

Corollary 2.9. *Let all the assumptions of Theorem* 2.1 *be valid. Then*

$$m_4 \le \rho\left(\frac{k+j}{2}\right) - \int_j^k \rho(t)\omega(t)dt \le M_4(t) \quad (2.6)$$

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where

$$\begin{aligned}
m_4 &= \int_j^{\frac{j+k}{2}} \left(\left(\left(\int_j^t \omega(t)dt - \left| \int_j^t \omega(t)dt \right| \right) \frac{\Gamma(t)}{2} \right.\right. \\
&+ \left.\left(\int_j^t \omega(t)dt + \left| \int_j^t \omega(t)dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\
&+ \int_{\frac{j+k}{2}}^k \left(\left(\int_k^t \omega(t)dt - \left| \int_k^t \omega(t)dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
&+ \left. \left(\int_k^t \omega(t)dt + \left| \int_k^t \omega(t)dt \right| \right) \frac{\gamma(t)}{2} \right) dt
\end{aligned}$$

and

$$\begin{aligned}
M_4 &= \int_j^{\frac{j+k}{2}} \left(\left(\left(\int_j^t \omega(t)dt + \left| \int_j^t \omega(t)dt \right| \right) \frac{\Gamma(t)}{2} \right.\right. \\
&+ \left.\left(\int_j^t \omega(t)dt - \left| \int_j^t \omega(t)dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\
&+ \int_{\frac{j+k}{2}}^k \left(\left(\int_k^t \omega(t)dt + \left| \int_k^t \omega(t)dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
&+ \left. \left(\int_k^t \omega(t)dt - \left| \int_k^t \omega(t)dt \right| \right) \frac{\gamma(t)}{2} \right) dt.
\end{aligned}$$

Special Case 2.9.(a) If we take, $\gamma(\theta) = \gamma_0 \neq 0$, $\Gamma(\theta) = \Gamma_0 \neq 0$, $\theta = \frac{j+k}{2}$ and $\omega(t) = \frac{1}{k-j}$, in (2.5), then

$$\frac{(k-j)}{8}(\gamma_0 - \Gamma_0) \leq \rho\left(\frac{j+k}{2}\right) - \frac{1}{k-j}\int_j^k \rho(t)dt \leq \frac{(k-j)}{8}(\Gamma_0 - \gamma_0),$$

which is in fact the Special Case 1 of Theorem 1 presented in [18], Corollary 1 of [27] and Special Case 2.6.1 of [8].

Special Case 2.9.(b) If we take, $\gamma(\theta) = \gamma_1\theta + \gamma_0 \neq 0$, $\Gamma(\theta) = \Gamma_1\theta + \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.5), then

$$m_5 \leq \rho\left(\frac{j+k}{2}\right) - \frac{1}{k-j}\int_j^k \rho(t)dt \leq M_5,$$

where

$$m_5 = \frac{(k-j)}{8}\left(\frac{k-j}{3}(\gamma_1 + \Gamma_1) + j\gamma_1 - k\Gamma_1 + \gamma_0 - \Gamma_0\right)$$

and

$$M_5 = \frac{(k-j)}{8}\left(\frac{k-j}{3}(\gamma_1 + \Gamma_1) + j\Gamma_1 - k\gamma_1 + \Gamma_0 - \gamma_0\right),$$

which is example of Corollary 1 of [18] and Special Case 2.6.2 of [8].

Remark 2.10. By choosing $\lambda = \frac{1}{3}$ in (2.5), then we get the following corollary.

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Corollary 2.11. *Let all the assumptions of Theorem* 2.1 *be valid. Then*

$$m_6 \le \rho(k)\int_{\frac{j+5k}{6}}^{k}\omega(t)dt - \rho(j)\int_{\frac{5j+k}{6}}^{j}\omega(t)dt + \rho\left(\frac{k+j}{2}\right)\int_{\frac{5j+k}{6}}^{\frac{j+k}{2}}\omega(t)dt$$
$$+\rho\left(\frac{k+j}{2}\right)\int_{\frac{j+k}{2}}^{\frac{j+5k}{6}}\omega(t)dt - \int_j^k \rho(t)\omega(t)dt \le M_6, \quad (2.7)$$

where

$$\begin{aligned} m_6 &= \int_{-\frac{k-j}{6}}^{\frac{j+k}{2}}\left(\left(\int_{\frac{5j+k}{6}}^{t}\omega(t)dt - \left|\int_{\frac{5j+k}{6}}^{t}\omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right. \\ &+ \left.\left(\int_{\frac{5j+k}{6}}^{t}\omega(t)dt + \left|\int_{\frac{5j+k}{6}}^{t}\omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt \\ &+ \int_{\frac{j+k}{2}}^{\frac{k-j}{6}}\left(\left(\int_{\frac{j+5k}{6}}^{t}\omega(t)dt - \left|\int_{\frac{j+5k}{6}}^{t}\omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right. \\ &+ \left.\left(\int_{\frac{j+5k}{6}}^{t}\omega(t)dt + \left|\int_{\frac{j+5k}{6}}^{t}\omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt \end{aligned}$$

and

$$\begin{aligned} M_6 &= \int_{-\frac{k-j}{6}}^{\frac{j+k}{2}}\left(\left(\int_{\frac{5j+k}{6}}^{t}\omega(t)dt + \left|\int_{\frac{5j+k}{6}}^{t}\omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right. \\ &+ \left.\left(\int_{\frac{5j+k}{6}}^{t}\omega(t)dt - \left|\int_{\frac{5j+k}{6}}^{t}\omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt \\ &+ \int_{\frac{j+k}{2}}^{\frac{k-j}{6}}\left(\left(\int_{\frac{j+5k}{6}}^{t}\omega(t)dt + \left|\int_{\frac{j+5k}{6}}^{t}\omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right. \\ &+ \left.\left(\int_{\frac{j+5k}{6}}^{t}\omega(t)dt - \left|\int_{\frac{j+5k}{6}}^{t}\omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt\ . \end{aligned}$$

Special Case 2.11.(a) If we take, $\gamma(t) = \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.7) , then

$$\frac{(k-j)}{72}(\gamma_0 - \Gamma_0) \le \frac{1}{3}\left[\frac{\rho(j)+\rho(k)}{2} + 2\rho\left(\frac{j+k}{2}\right)\right] - \frac{1}{k-j}\int_j^k \rho(t)dt$$
$$\le \frac{(k-j)}{72}(\Gamma_0 - \gamma_0),$$

which is $\frac{1}{3}$ Simpson's rule and Corollary 4 of [27] and Special Case 2.7.1 of [8].

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Special Case 2.11.(b) If we take, $\gamma(t)=\gamma_1 t+\gamma_0\neq 0$, $\Gamma(t)=\Gamma_1 t+\Gamma_0\neq 0$ and $\omega(t)=\frac{1}{k-j}$ in (2.7), then

$$m_7\leq \frac{1}{3}\left[\frac{\rho(j)+\rho(k)}{2}+2\rho\left(\frac{j+k}{2}\right)\right]-\frac{1}{k-j}\int_j^k\rho(t)dt\leq M_7,$$

where

$$m_7=\frac{(k-j)}{72}\left[(k-j)(\gamma_1+\Gamma_1)+\frac{j}{2}(7\gamma_1-3\Gamma_1)+\frac{k}{2}(3\gamma_1-7\Gamma_1)+5(\gamma_0-\Gamma_0)\right]$$

and

$$\begin{aligned}M_7 &= \frac{(k-j)}{72}\left[(k-j)(\gamma_1+\Gamma_1)+\frac{j}{2}(7\Gamma_1-3\gamma_1)+\frac{k}{2}(3\Gamma_1-7\gamma_1)\right.\\ &+ \left. 5(\Gamma_0-\gamma_0)\right],\end{aligned}$$

which is $\frac{1}{3}$ Simpson's rule and Special Case 2.7.2 of [8].

Remark 2.12. If we choose $\lambda=\frac{1}{2}$ in (2.5), we get the following corollary.

Corollary 2.13. *Let all the assumptions of Theorem* 2.1 *be valid. Then*

$$\begin{aligned}m_8\leq \rho(k)\int_{\frac{j+3k}{4}}^k\omega(t)dt-\rho(j)\int_{\frac{3j+k}{4}}^j\omega(t)dt+\rho\left(\frac{k+j}{2}\right)\int_{\frac{3j+k}{4}}^{\frac{j+k}{2}}\omega(t)dt\\ +\rho\left(\frac{k+j}{2}\right)\int_{\frac{j+k}{2}}^{\frac{j+3k}{4}}\omega(t)dt-\int_j^k\rho(t)\omega(t)dt\leq M_8,\quad (2.8)\end{aligned}$$

where

$$\begin{aligned}m_8 &= \int_{-\frac{k-j}{4}}^{\frac{j+k}{2}}\left(\left(\int_{\frac{3j+k}{4}}^t\omega(t)dt-\left|\int_{\frac{3j+k}{4}}^t\omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right.\\ &+ \left.\left(\int_{\frac{3j+k}{4}}^t\omega(t)dt+\left|\int_{\frac{3j+k}{4}}^t\omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt\\ &+ \int_{\frac{j+k}{2}}^{\frac{k-j}{4}}\left(\left(\int_{\frac{j+3k}{4}}^t\omega(t)dt-\left|\int_{\frac{j+3k}{4}}^t\omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right.\\ &+ \left.\left(\int_{\frac{j+3k}{4}}^t\omega(t)dt+\left|\int_{\frac{j+3k}{4}}^t\omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt\end{aligned}$$

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and

$$\begin{aligned} M_8 &= \int_{-\frac{k-j}{4}}^{\frac{j+k}{2}} \left(\left(\int_{\frac{3j+k}{4}}^{t} \omega(t)dt + \left|\int_{\frac{3j+k}{4}}^{t} \omega(t)dt\right|\right) \frac{\Gamma(t)}{2}\right. \\ &+ \left.\left(\int_{\frac{3j+k}{4}}^{t} \omega(t)dt - \left|\int_{\frac{3j+k}{4}}^{t} \omega(t)dt\right|\right) \frac{\gamma(t)}{2}\right) dt \\ &+ \int_{\frac{j+k}{2}}^{\frac{k-j}{4}} \left(\left(\int_{\frac{j+3k}{4}}^{t} \omega(t)dt + \left|\int_{\frac{j+3k}{4}}^{t} \omega(t)dt\right|\right) \frac{\Gamma(t)}{2}\right. \\ &+ \left.\left(\int_{\frac{j+3k}{4}}^{t} \omega(t)dt - \left|\int_{\frac{j+3k}{4}}^{t} \omega(t)dt\right|\right) \frac{\gamma(t)}{2}\right) dt. \end{aligned}$$

Special Case 2.13.(a) If we take, $\gamma(t) = \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.8), then

$$\begin{aligned} \frac{(k-j)}{16}(\gamma_0 - \Gamma_0) \leq \frac{1}{2}\left[\frac{\rho(j)+\rho(k)}{2} + \rho\left(\frac{j+k}{2}\right)\right] - \frac{1}{k-j}\int_j^k \rho(t)dt \\ \leq \frac{(k-j)}{16}(\Gamma_0 - \gamma_0), \end{aligned}$$

which is average midpoint and trapezoidal and Corollary 3 of [27] and Special Case 2.8.1 of [8].

Special Case 2.13.(b) If we take, $\gamma(t) = \gamma_1 t + \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_1 t + \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.8), then

$$m_9 \leq \frac{1}{2}\left[\frac{\rho(j)+\rho(k)}{2} + \rho\left(\frac{j+k}{2}\right)\right] - \frac{1}{k-j}\int_j^k \rho(t)dt \leq M_9,$$

where

$$m_9 = \frac{(k-j)}{16}\left[\frac{(k-j)}{6}(\gamma_1 + \Gamma_1) + \frac{j}{2}(\gamma_1 - \Gamma_1) + \frac{k}{2}(\gamma_1 - \Gamma_1) + \gamma_0 - \Gamma_0\right]$$

and

$$M_9 = \frac{(k-j)}{16}\left[\frac{(k-j)}{6}(\gamma_1 + \Gamma_1) + \frac{j}{2}(\Gamma_1 - \gamma_1) + \frac{k}{2}(\Gamma_1 - \gamma_1) + \Gamma_0 - \gamma_0\right],$$

which is midpoint and trapezoidal rule and Special Case 2.8.2 of [8].

Remark 2.14. If we choose $\theta = j$ in (2.1), for any value of $\lambda \in [0,1]$ we obtain the following result.

Corollary 2.15. *Let all the assumptions of Theorem* 2.1 *be valid. Then*

$$\begin{aligned} m_{10}(\lambda) \leq \rho(k)\int_\beta^k \omega(t)dt - \rho(j)\int_\alpha^j \omega(t)dt + \rho(p)\int_\alpha^{\frac{\alpha+\beta}{2}} \omega(t)dt \\ + \rho(k)\int_{\frac{\alpha+\beta}{2}}^{\beta} \omega(t)dt - \int_j^k \rho(t)\omega(t)dt \leq M_{10}(\lambda), \qquad (2.9) \end{aligned}$$

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where

$$\begin{aligned} m_{10}(\lambda) &= \int_j^k \left(\left(\int_{\frac{\alpha+\beta}{2}}^t \omega(t)dt - \left| \int_{\frac{\alpha+\beta}{2}}^t \omega(t)dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &+ \left. \left(\int_{\frac{\alpha+\beta}{2}}^t \omega(t)dt + \left| \int_{\frac{\alpha+\beta}{2}}^t \omega(t)dt \right| \right) \frac{\gamma(t)}{2} \right) dt \end{aligned}$$

and

$$\begin{aligned} M_{10}(\lambda) &= \int_j^k \left(\left(\int_{\frac{\alpha+\beta}{2}}^t \omega(t)dt + \left| \int_{\frac{\alpha+\beta}{2}}^t \omega(t)dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &+ \left. \left(\int_{\frac{\alpha+\beta}{2}}^t \omega(t)dt - \left| \int_{\frac{\alpha+\beta}{2}}^t \omega(t)dt \right| \right) \frac{\gamma(t)}{2} \right) dt. \end{aligned}$$

Remark 2.16.
(1) If we take $\Gamma(t) = \Gamma_0 \neq 0$, $\gamma(t) = \gamma_0 \neq 0$, and with $\omega(t) = \frac{1}{k-j}$ in (2.9), then we obtain results similar to **Special Case 2.5.(a)**.
(2) $\Gamma(t) = \Gamma_1 t + \Gamma_0 \neq 0$,$\gamma(t) = \gamma_1 t + \gamma_0 \neq 0$, and $\omega(t) = \frac{1}{k-j}$ in (2.9), then we obtain results similar to **Special Case 2.5.(b)**.

Remark 2.17. If we choose $\theta = k$, and $\lambda = 0$ in (2.1), we obtain the following result.

Corollary 2.18. *Let all the assumptions of Theorem* 2.1 *be valid. Then*

$$m_{11} \leq \frac{\rho(j)+\rho(k)}{2} - \frac{1}{k-j} \int_j^k \rho(t)dt \leq M_{11}, \tag{2.10}$$

where

$$\begin{aligned} m_{11} &= \int_j^k \left(\left(\int_j^t \omega(t)dt - \left| \int_j^t \omega(t)dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &+ \left. \left(\int_j^t \omega(t)dt + \left| \int_j^t \omega(t)dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ &+ \int_k^j \left(\left(\int_{\frac{j+k}{2}}^t \omega(t)dt - \left| \int_{\frac{j+k}{2}}^t \omega(t)dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &+ \left. \left(\int_{\frac{j+k}{2}}^t \omega(t)dt + \left| \int_{\frac{j+k}{2}}^t \omega(t)dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ &+ \int_j^k \left(\left(\int_k^t \omega(t)dt - \left| \int_k^t \omega(t)dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &+ \left. \left(\int_k^t \omega(t)dt + \left| \int_k^t \omega(t)dt \right| \right) \frac{\gamma(t)}{2} \right) dt. \end{aligned}$$

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and

$$\begin{aligned}
M_{11} &= \int_j^k \left(\left(\left(\int_j^t \omega(t)dt + \left|\int_j^t \omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right.\right.\\
&+ \left.\left(\int_j^t \omega(t)dt - \left|\int_j^t \omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt\\
&+ \int_k^j \left(\left(\left(\int_{\frac{j+k}{2}}^t \omega(t)dt + \left|\int_{\frac{j+k}{2}}^t \omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right.\right.\\
&+ \left.\left(\int_{\frac{j+k}{2}}^t \omega(t)dt - \left|\int_{\frac{j+k}{2}}^t \omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt\\
&+ \int_j^k \left(\left(\left(\int_k^t \omega(t)dt + \left|\int_k^t \omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right.\right.\\
&+ \left.\left(\int_k^t \omega(t)dt - \left|\int_k^t \omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt.
\end{aligned}$$

Special Case 2.18.(a) If we take, $\gamma(t) = \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.10), then

$$\frac{3(k-j)}{8}(\gamma_0 - \Gamma_0) \leq \left[\frac{\rho(j)+\rho(k)}{2}\right] - \frac{1}{k-j}\int_j^k \rho(t)dt \leq \frac{3(k-j)}{8}(\Gamma_0 - \gamma_0),$$

which is Special Case 2.10.1 of [8].

Special Case 2.18.(b) If we take, $\gamma(t) = \gamma_1 t + \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_1 t + \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.10), then

$$m_{12} \leq \left[\frac{\rho(j)+\rho(k)}{2}\right] - \frac{1}{k-j}\int_j^k \rho(t)dt \leq M_{12},$$

where

$$m_{12} = \frac{(k-j)}{2}\left[\frac{7(k-j)}{12}(\gamma_1 + \Gamma_1) + \frac{j}{8}(7\gamma_1 + \Gamma_1) - \frac{k}{8}(\gamma_1 + 7\Gamma_1) + \frac{3}{4}(\gamma_0 - \Gamma_0)\right]$$

and

$$M_{12} = \frac{(k-j)}{2}\left[\frac{7(k-j)}{12}(\gamma_1 + \Gamma_1) + \frac{j}{8}(\gamma_1 + 7\Gamma_1) - \frac{k}{8}(7\gamma_1 + \Gamma_1) + \frac{3}{4}(\Gamma_0 - \gamma_0)\right],$$

which is Special Case 2.10.2 of [8].

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Remark 2.19. If we choose $\theta = k$ and $\lambda = \frac{1}{2}$ in (2.1), we obtain the following result.

Corollary 2.20. *Let all the assumptions of Theorem* 2.1 *be valid. Then*

$$m_{13} \leq \rho(k)\int_{\frac{j+3k}{4}}^{k}\omega(t)dt - \rho(j)\int_{\frac{3j+k}{4}}^{j}\omega(t)dt + \rho(k)\int_{\frac{3j+k}{4}}^{\frac{j+k}{2}}\omega(t)dt + \rho(j)\int_{\frac{j+k}{2}}^{\frac{j+3k}{4}}\omega(t)dt - \int_{j}^{k}\rho(t)\omega(t)dt \leq M_{13}, \qquad (2.11)$$

where

$$\begin{aligned} m_{13} &= \int_j^k \left(\left(\int_{\frac{3j+k}{4}}^t \omega(t)dt - \left|\int_{\frac{3j+k}{4}}^t \omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right. \\ &+ \left.\left(\int_{\frac{3j+k}{4}}^t \omega(t)dt + \left|\int_{\frac{3j+k}{4}}^t \omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right) dt \\ &+ \int_k^j \left(\left(\int_{\frac{j+k}{2}}^t \omega(t)dt - \left|\int_{\frac{j+k}{2}}^t \omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right. \\ &+ \left.\left(\int_{\frac{j+k}{2}}^t \omega(t)dt + \left|\int_{\frac{j+k}{2}}^t \omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right) dt \\ &+ \int_j^k \left(\left(\int_{\frac{j+3k}{4}}^t \omega(t)dt - \left|\int_{\frac{j+3k}{4}}^t \omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right. \\ &+ \left.\left(\int_{\frac{j+3k}{4}}^t \omega(t)dt + \left|\int_{\frac{j+3k}{4}}^t \omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right) dt \end{aligned}$$

and

$$\begin{aligned} M_{13} &= \int_j^k \left(\left(\int_{\frac{3j+k}{4}}^t \omega(t)dt + \left|\int_{\frac{3j+k}{4}}^t \omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right. \\ &+ \left.\left(\int_{\frac{3j+k}{4}}^t \omega(t)dt - \left|\int_{\frac{3j+k}{4}}^t \omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right) dt \\ &+ \int_k^j \left(\left(\int_{\frac{j+k}{2}}^t \omega(t)dt + \left|\int_{\frac{j+k}{2}}^t \omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right. \\ &+ \left.\left(\int_{\frac{j+k}{2}}^t \omega(t)dt - \left|\int_{\frac{j+k}{2}}^t \omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right) dt \\ &+ \int_j^k \left(\left(\int_{\frac{j+3k}{4}}^t \omega(t)dt + \left|\int_{\frac{j+3k}{4}}^t \omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right. \\ &+ \left.\left(\int_{\frac{j+3k}{4}}^t \omega(t)dt - \left|\int_{\frac{j+3k}{4}}^t \omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right) dt\,. \end{aligned}$$

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Special Case 2.20.(a) If we take, $\gamma(t) = \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.11), then

$$\frac{9(k-j)}{32}(\gamma_0 - \Gamma_0) \leq \left[\frac{\rho(j)+\rho(k)}{2}\right] - \frac{1}{k-j}\int_j^k \rho(t)dt \leq \frac{9(k-j)}{32}(\Gamma_0 - \gamma_0),$$

which is Special Case 2.11.1 of [8].

Special Case 2.20.(b) If we take, $\gamma(t) = \gamma_1 t + \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_1 t + \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.11), then

$$m_{14} \leq \left[\frac{\rho(j)+\rho(k)}{2}\right] - \frac{1}{k-j}\int_j^k \rho(t)dt \leq M_{14},$$

where

$$\begin{aligned} m_{14} &= \frac{k-j}{16}\left[\frac{5(k-j)}{3}(\gamma_1 + \Gamma_1) + \frac{j}{2}(5\gamma_1 - \Gamma_1) - \frac{k}{2}(\gamma_1 - 5\Gamma_1)\right. \\ &+ \left. 3(\gamma_0 - \Gamma_0)\right] \end{aligned}$$

and

$$\begin{aligned} M_{14} &= \frac{k-j}{16}\left[\frac{5(k-j)}{3}(\gamma_1 + \Gamma_1) + \frac{j}{2}(5\Gamma_1 - \gamma_1) - \frac{k}{2}(\Gamma_1 - 5\gamma_1)\right. \\ &+ \left. 3(\Gamma_0 - \gamma_0)\right], \end{aligned}$$

which is Special Case 2.11.2 of [8].

Remark 2.21. If we choose $\theta = k$ and $\lambda = \frac{1}{3}$ in (2.1), then we get the following result.

Corollary 2.22. *Let all the assumptions of Theorem* 2.1 *be valid. Then*

$$\begin{aligned} m_{15} \leq \rho(k)\int_{\frac{j+5k}{6}}^{k} \omega(t)dt - \rho(j)\int_{\frac{5j+k}{6}}^{j} \omega(t)dt + \rho(k)\int_{\frac{5j+k}{6}}^{\frac{j+k}{2}} \omega(t)dt \\ +\rho(j)\int_{\frac{j+k}{2}}^{\frac{j+5k}{6}} \omega(t)dt - \int_j^k \rho(t)\omega(t)dt \leq M_{15}, \qquad (2.12) \end{aligned}$$

where

$$m_{15} = \int_j^k \left(\left(\int_{\frac{5j+k}{6}}^{t} \omega(t)dt - \left|\int_{\frac{5j+k}{6}}^{t} \omega(t)dt\right|\right) \frac{\Gamma(t)}{2}\right.$$

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$$
\begin{aligned}
&+ \left(\int_{\frac{5j+k}{6}}^{t} \omega(t)dt + \left|\int_{\frac{5j+k}{6}}^{t} \omega(t)dt\right|\right) \frac{\gamma(t)}{2}\Bigg) dt \\
&+ \int_{k}^{j} \left(\left(\int_{\frac{j+k}{2}}^{t} \omega(t)dt - \left|\int_{\frac{j+k}{2}}^{t} \omega(t)dt\right|\right) \frac{\Gamma(t)}{2}\right. \\
&+ \left(\int_{\frac{j+k}{2}}^{t} \omega(t)dt + \left|\int_{\frac{j+k}{2}}^{t} \omega(t)dt\right|\right) \frac{\gamma(t)}{2}\Bigg) dt \\
&+ \int_{j}^{k} \left(\left(\int_{\frac{j+5k}{6}}^{t} \omega(t)dt - \left|\int_{\frac{j+5k}{6}}^{t} \omega(t)dt\right|\right) \frac{\Gamma(t)}{2}\right. \\
&+ \left(\int_{\frac{j+5k}{6}}^{t} \omega(t)dt + \left|\int_{\frac{j+5k}{6}}^{t} \omega(t)dt\right|\right) \frac{\gamma(t)}{2}\Bigg) dt
\end{aligned}
$$

and

$$
\begin{aligned}
M_{15} &= \int_{j}^{k} \left(\left(\int_{\frac{5j+k}{6}}^{t} \omega(t)dt + \left|\int_{\frac{5j+k}{6}}^{t} \omega(t)dt\right|\right) \frac{\Gamma(t)}{2}\right. \\
&+ \left(\int_{\frac{5j+k}{6}}^{t} \omega(t)dt - \left|\int_{\frac{5j+k}{6}}^{t} \omega(t)dt\right|\right) \frac{\gamma(t)}{2}\Bigg) dt
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_{k}^{j} \left(\left(\int_{\frac{j+k}{2}}^{t} \omega(t)dt + \left|\int_{\frac{j+k}{2}}^{t} \omega(t)dt\right|\right) \frac{\Gamma(t)}{2}\right. \\
&+ \left(\int_{\frac{j+k}{2}}^{t} \omega(t)dt - \left|\int_{\frac{j+k}{2}}^{t} \omega(t)dt\right|\right) \frac{\gamma(t)}{2}\Bigg) dt \\
&+ \int_{j}^{k} \left(\left(\int_{\frac{j+5k}{6}}^{t} \omega(t)dt + \left|\int_{\frac{j+5k}{6}}^{t} \omega(t)dt\right|\right) \frac{\Gamma(t)}{2}\right. \\
&+ \left(\int_{\frac{j+5k}{6}}^{t} \omega(t)dt - \left|\int_{\frac{j+5k}{6}}^{t} \omega(t)dt\right|\right) \frac{\gamma(t)}{2}\Bigg) dt.
\end{aligned}
$$

Special Case 2.22.(a) If we take, $\gamma(t) = \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.12), then

$$
\begin{aligned}
\frac{17(k-j)}{72}(\gamma_0 - \Gamma_0) \leq \left[\frac{\rho(j)+\rho(k)}{2}\right] - \frac{1}{k-j}\int_{j}^{k} \rho(t)dt \\
\leq \frac{17(k-j)}{72}(\Gamma_0 - \gamma_0),
\end{aligned}
$$

which is Special Case 2.12.1 of [8].

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Special Case 2.22.(b) If we take, $\gamma(t)=\gamma_1 t+\gamma_0\neq 0$, $\Gamma(t)=\Gamma_1 t+\Gamma_0\neq 0$ and $\omega(t)=\frac{1}{k-j}$ in (2.12), then

$$m_{16}\leq\left[\frac{\rho(j)+\rho(k)}{2}\right]-\frac{1}{k-j}\int_j^k\rho(t)dt\leq M_{16},$$

where

$$m_{16}=\frac{(k-j)}{72}\left[11(k-j)(\gamma_1+\Gamma_1)+\frac{3j}{2}(33\gamma_1-\Gamma_1)+\frac{3k}{2}(\gamma_1-33\Gamma_1)\right.$$
$$\left.+17(\gamma_0-\Gamma_0)\right]$$

and

$$M_{16}=\frac{(k-j)}{72}\left[(k-j)(\gamma_1+\Gamma_1)+\frac{3j}{2}(33\Gamma_1-\gamma_1)+\frac{3k}{2}(\Gamma_1-33\gamma_1)\right.$$
$$\left.+17(\Gamma_0-\gamma_0)\right],$$

which is Special Case 2.12.2 of [8].

Remark 2.23. If we choose $\theta=k$, and $\lambda=\frac{1}{4}$ in (2.1), we obtain a bound for trapezoidal rule in the following result.

Corollary 2.24. *Let all the assumptions of Theorem* 2.1 *be valid. Then*

$$m_{17}\leq\rho(k)\int_{\frac{7j+k}{8}}^{k}\omega(t)dt-\rho(j)\int_{\frac{7j+k}{8}}^{j}\omega(t)dt+\rho(k)\int_{\frac{7j+k}{8}}^{\frac{j+k}{2}}\omega(t)dt$$
$$+\rho(j)\int_{\frac{j+k}{2}}^{\beta}\omega(t)dt-\int_j^k\rho(t)\omega(t)dt\leq M_{17},\qquad(2.13)$$

where

$$\begin{aligned}
m_{17} \;=\; & \int_j^k\left(\left(\int_{\frac{7j+k}{8}}^{t}\omega(t)dt-\left|\int_{\frac{7j+k}{8}}^{t}\omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right.\\
+ & \left.\left(\int_{\frac{7j+k}{8}}^{t}\omega(t)dt+\left|\int_{\frac{7j+k}{8}}^{t}\omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt\\
+ & \int_k^j\left(\left(\int_{\frac{j+k}{2}}^{t}\omega(t)dt-\left|\int_{\frac{j+k}{2}}^{t}\omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right.\\
+ & \left.\left(\int_{\frac{j+k}{2}}^{t}\omega(t)dt+\left|\int_{\frac{j+k}{2}}^{t}\omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt\\
+ & \int_j^k\left(\left(\int_{\beta}^{t}\omega(t)dt-\left|\int_{\beta}^{t}\omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right.\\
+ & \left.\left(\int_{\beta}^{t}\omega(t)dt+\left|\int_{\beta}^{t}\omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt.
\end{aligned}$$

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and

$$\begin{aligned} M_{17} &= \int_j^k \left(\left(\int_{\frac{7j+k}{8}}^t \omega(t)dt + \left| \int_{\frac{7j+k}{8}}^t \omega(t)dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &+ \left. \left(\int_{\frac{7j+k}{8}}^t \omega(t)dt - \left| \int_{\frac{7j+k}{8}}^t \omega(t)dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ &+ \int_k^j \left(\left(\int_{\frac{j+k}{2}}^t \omega(t)dt + \left| \int_{\frac{j+k}{2}}^t \omega(t)dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &+ \left. \left(\int_{\frac{j+k}{2}}^t \omega(t)dt - \left| \int_{\frac{j+k}{2}}^t \omega(t)dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ &+ \int_j^k \left(\left(\int_\beta^t \omega(t)dt + \left| \int_\beta^t \omega(t)dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &+ \left. \left(\int_\beta^t \omega(t)dt - \left| \int_\beta^t \omega(t)dt \right| \right) \frac{\gamma(t)}{2} \right) dt. \end{aligned}$$

Special Case 2.24.(a) If we take, $\gamma(t) = \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.13), then

$$\begin{aligned} \frac{17(k-j)}{64}(\gamma_0 - \Gamma_0) \leq \left[\frac{\rho(j)+\rho(k)}{2} \right] - \frac{1}{k-j} \int_j^k \rho(t)dt \\ \leq \frac{17(k-j)}{64}(\Gamma_0 - \gamma_0), \end{aligned}$$

which is Special Case 2.13.1 of [8].

Special Case 2.24.(b) If we take, $\gamma(t) = \gamma_1 t + \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_1 t + \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.13), then

$$m_{18} \leq \left[\frac{\rho(j)+\rho(k)}{2} \right] - \frac{1}{k-j} \int_j^k \rho(t)dt \leq M_{18},$$

where

$$\begin{aligned} m_{18} = \frac{(k-j)}{64} \left[\frac{35}{3}(k-j)(\gamma_1 + \Gamma_1) + \frac{j}{2}(35\gamma_1 + \Gamma_1) - \frac{k}{2}(\gamma_1 + 35\Gamma_1) \right. \\ \left. +17(\gamma_0 - \Gamma_0) \right] \end{aligned}$$

and

$$\begin{aligned} M_{18} = \frac{(k-j)}{64} \left[\frac{35}{3}(k-j)(\gamma_1 + \Gamma_1) + \frac{j}{2}(35\Gamma_1 + \gamma_1 +) - \frac{k}{2}(\Gamma_1 + 35\gamma_1) \right. \\ \left. +17(\Gamma_0 - \gamma_0) \right], \end{aligned}$$

which is Special Case 2.13.2 of [8].

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Remark 2.25. if we choose $\theta = \frac{2j+k}{3}$ and $\lambda = \frac{1}{4}$ in (2.1), then we get the following result.

Corollary 2.26. *Let all the assumptions of Theorem* 2.1 *be valid. Then*

$$m_{19} \leq \rho(k)\int_{\frac{j+7k}{8}}^{k}\omega(t)dt - \rho(j)\int_{\frac{7j+k}{8}}^{j}\omega(t)dt + \rho\left(\frac{2j+k}{3}\right)\int_{\frac{7j+k}{8}}^{\frac{j+k}{2}}\omega(t)dt$$
$$+\rho\left(\frac{j+2k}{3}\right)\int_{\frac{j+k}{2}}^{\frac{j+7k}{8}}\omega(t)dt - \int_{j}^{k}\rho(t)\omega(t)dt \leq M_{19}, \tag{2.14}$$

where

$$\begin{aligned} m_{19} &= \int_{j}^{\frac{2j+k}{3}}\left(\left(\int_{\frac{7j+k}{8}}^{t}\omega(t)dt - \left|\int_{\frac{7j+k}{8}}^{t}\omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right. \\ &+ \left.\left(\int_{\frac{7j+k}{8}}^{t}\omega(t)dt + \left|\int_{\frac{7j+k}{8}}^{t}\omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt \\ &+ \int_{\frac{2j+k}{3}}^{\frac{j+2k}{3}}\left(\left(\int_{\frac{j+k}{2}}^{t}\omega(t)dt - \left|\int_{\frac{j+k}{2}}^{t}\omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right. \\ &+ \left.\left(\int_{\frac{j+k}{2}}^{t}\omega(t)dt + \left|\int_{\frac{j+k}{2}}^{t}\omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt \end{aligned}$$

$$\begin{aligned} &+ \int_{\frac{j+2k}{3}}^{k}\left(\left(\int_{\frac{j+7k}{8}}^{t}\omega(t)dt - \left|\int_{\frac{j+7k}{8}}^{t}\omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right. \\ &+ \left.\left(\int_{\frac{j+7k}{8}}^{t}\omega(t)dt + \left|\int_{\frac{j+7k}{8}}^{t}\omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt. \end{aligned}$$

and

$$\begin{aligned} M_{19} &= \int_{j}^{\frac{2j+k}{3}}\left(\left(\int_{\frac{7j+k}{8}}^{t}\omega(t)dt + \left|\int_{\frac{7j+k}{8}}^{t}\omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right. \\ &+ \left.\left(\int_{\frac{7j+k}{8}}^{t}\omega(t)dt - \left|\int_{\frac{7j+k}{8}}^{t}\omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt \\ &+ \int_{\frac{2j+k}{3}}^{\frac{j+2k}{3}}\left(\left(\int_{\frac{j+k}{2}}^{t}\omega(t)dt + \left|\int_{\frac{j+k}{2}}^{t}\omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right. \\ &+ \left.\left(\int_{\frac{j+k}{2}}^{t}\omega(t)dt - \left|\int_{\frac{j+k}{2}}^{t}\omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt \\ &+ \int_{\frac{j+2k}{3}}^{k}\left(\left(\int_{\frac{j+7k}{8}}^{t}\omega(t)dt + \left|\int_{\frac{j+7k}{8}}^{t}\omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right. \\ &+ \left.\left(\int_{\frac{j+7k}{8}}^{t}\omega(t)dt - \left|\int_{\frac{j+7k}{8}}^{t}\omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt. \end{aligned}$$

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Special Case 2.26.(a) If we take, $\gamma(t) = \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.14), then

$$\frac{25(k-j)}{576}(\Gamma_0 - \gamma_0) \leq \frac{3}{8}\left[\frac{\rho(j)+\rho(k)}{3} + \rho\left(\frac{2j+k}{3}\right) + \rho\left(\frac{j+2k}{3}\right)\right] - \frac{1}{k-j}\int_j^k \rho(t)dt \leq \frac{25(k-j)}{576}(\gamma_0 - \Gamma_0),$$

which is Special Case 2.15.1 of [8].

Special Case 2.26.(b) If we take, $\gamma(t) = \gamma_1 t + \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_1 t + \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$, in (2.14) then

$$m_{20} \leq \frac{3}{8}\left[\frac{\rho(j)+\rho(k)}{3} + \rho\left(\frac{2j+k}{3}\right) + \rho\left(\frac{j+2k}{3}\right)\right] - \frac{1}{k-j}\int_j^k \rho(t)dt \leq M_{20},$$

where

$$m_{20} = \frac{(k-j)}{192}\left[(k-j)(\gamma_1+\Gamma_1) + \frac{31}{6}(j\gamma_1 - k\Gamma_1) + \frac{19}{6}(k\gamma_1 - j\Gamma_1) + \frac{25}{3}(\gamma_0 - \Gamma_0)\right]$$

and

$$M_{20} = \frac{(k-j)}{192}\left[(k-j)(\gamma_1+\Gamma_1) + \frac{31}{6}(k\Gamma_1 - j\gamma_1) + \frac{19}{6}(j\Gamma_1 - k\gamma_1) + \frac{25}{3}(\Gamma_0 - \gamma_0)\right],$$

which is Special Case 2.15.2 of [8].

Now we state two results with their consequences for function ρ whose first derivative is bounded below only and bounded above only respectively.

Theorem 2.27. *Let $\rho : I \to \mathbb{R}$, be a differentiable function on I^0 of I, and let $[j,k] \subset I^0$. If ρ' is bounded below then $\gamma(\theta) \leq \rho'(\theta)$ for any $\gamma \in C[j,k]$, $\theta \in [j,k]$, then for all $\lambda \in [0,1]$ we have*

$$m_{21}(\theta,\lambda) \leq \rho(k)\int_\beta^k \omega(t)dt - \rho(j)\int_\alpha^j \omega(t)dt + \rho(\theta)\int_\alpha^{\frac{\alpha+\beta}{2}} \omega(t)dt + \rho(j+k-\theta)\int_{\frac{\alpha+\beta}{2}}^\beta \omega(t)dt - \int_j^k \rho(t)\omega(t)dt \leq M_{21}(\theta,\lambda), \quad (2.15)$$

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where

$$m_{21}(\theta,\lambda)=\int_j^k\left(\int_{\frac{\alpha+\beta}{2}}^t\omega(t)dt\right)\gamma(t)dt+\int_j^\theta\left(\int_\alpha^{\frac{\alpha+\beta}{2}}\omega(t)dt\right)\gamma(t)dt$$
$$-\int_{j+k-\theta}^k\left(\int_{\frac{\alpha+\beta}{2}}^\beta\omega(t)dt\right)\gamma(t)dt$$
$$-\max\left\{\int_\alpha^\theta\omega(t)dt,\int_{\frac{\alpha+\beta}{2}}^{j+k-\theta}\omega(t)dt,\int_\beta^k\omega(t)dt\right\}$$
$$\times\left(\rho(k)-\rho(j)-\int_j^k\gamma(t)dt\right)$$

and

$$M_{21}(\theta,\lambda)=\int_j^k\left(\int_{\frac{\alpha+\beta}{2}}^t\omega(t)dt\right)\gamma(t)dt+\int_j^\theta\left(\int_\alpha^{\frac{\alpha+\beta}{2}}\omega(t)dt\right)\gamma(t)dt$$
$$-\int_{j+k-\theta}^k\left(\int_{\frac{\alpha+\beta}{2}}^\beta\omega(t)dt\right)\gamma(t)dt$$

$$+\max\left\{\int_\alpha^\theta\omega(t)dt,\int_{\frac{\alpha+\beta}{2}}^{j+k-\theta}\omega(t)dt,\int_\beta^k\omega(t)dt\right\}$$
$$\times\left(\rho(k)-\rho(j)-\int_j^k\gamma(t)dt\right).$$

Proof. Since

$$\int_j^k K(\theta,t)\left(\rho'(t)-\gamma(t)\right)dt$$
$$=\rho(k)\int_\beta^k\omega(t)dt-\rho(j)\int_\alpha^j\omega(t)dt+\rho(\theta)\int_\alpha^{\frac{\alpha+\beta}{2}}\omega(t)dt$$
$$+\rho(j+k-\theta)\int_{\frac{\alpha+\beta}{2}}^\beta\omega(t)dt-\int_j^k\rho(t)\omega(t)dt-\int_j^k\omega(t)\rho(t)dt$$
$$-\int_j^k K(\theta,t)\gamma(t)dt$$
$$=\rho(k)\int_\beta^k\omega(t)dt-\rho(j)\int_\alpha^j\omega(t)dt+\rho(\theta)\int_\alpha^{\frac{\alpha+\beta}{2}}\omega(t)dt$$

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$$+\rho(j+k-\theta)\int_{\frac{\alpha+\beta}{2}}^{\beta}\omega(t)dt-\int_{j}^{k}\rho(t)\omega(t)dt$$
$$-\left[\int_{j}^{\theta}\left(\int_{\alpha}^{t}\omega(t)dt\right)\gamma(t)dt+\int_{\theta}^{j+k-\theta}\left(\int_{\frac{\alpha+\beta}{2}}^{t}\omega(t)dt\right)\gamma(t)dt\right.$$
$$\left.+\int_{j+k-\theta}^{k}\left(\int_{\beta}^{t}\omega(t)dt\right)\gamma(t)dt\right].$$

Using modulus property, we have

$$\left|\rho(k)\int_{\beta}^{k}\omega(t)dt-\rho(j)\int_{\alpha}^{j}\omega(t)dt+\rho(\theta)\int_{\alpha}^{\frac{\alpha+\beta}{2}}\omega(t)dt\right.$$
$$+\rho(j+k-\theta)\int_{\frac{\alpha+\beta}{2}}^{\beta}\omega(t)dt-\int_{j}^{k}\rho(t)\omega(t)dt$$
$$-\left[\int_{j}^{\theta}\left(\int_{\alpha}^{t}\omega(t)dt\right)\gamma(t)dt+\int_{\theta}^{j+k-\theta}\left(\int_{\frac{\alpha+\beta}{2}}^{t}\omega(t)dt\right)\gamma(t)dt\right.$$
$$\left.\left.+\int_{j+k-\theta}^{k}\left(\int_{\beta}^{t}\omega(t)dt\right)\gamma(t)dt\right]\right|$$
$$=\left|\int_{j}^{k}K(\theta,t)\left(\rho'(t)-\gamma(t)\right)dt\right|\leq\int_{j}^{k}|K(\theta,t)|\left(\rho'(t)-\gamma(t)\right)dt$$
$$\leq\max_{t\in[j,k]}|K(\theta,t)|\int_{j}^{k}\left(\rho'(t)-\gamma(t)\right)dt$$
$$=\max\left\{\int_{\alpha}^{\theta}\omega(t)dt,\int_{\frac{\alpha+\beta}{2}}^{j+k-\theta}\omega(t)dt,\int_{\beta}^{k}\omega(t)dt\right\}$$
$$\times\left(\rho(k)-\rho(j)-\int_{j}^{k}\gamma(t)dt\right). \tag{2.16}$$

After rearrangement of (2.16) we get required result. □

Remark 2.28. If we put $\omega(t)=\frac{1}{k-j}$ in inequality (2.15), then we will get the following result.

Corollary 2.29. *Let all the assumptions of Theorem* 2.27 *be valid. Then*

$$m_{22}(\theta,\lambda)\leq\lambda\frac{\rho(j)+\rho(k)}{2}+(1-\lambda)\frac{\rho(\theta)+\rho(j+k-\theta)}{2}-\int_{j}^{k}\rho(t)d(t)$$
$$\leq M_{22}(\theta,\lambda) \tag{2.17}$$

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where

$$m_{22}(\lambda)=\frac{1}{k-j}\left[\int_j^k\left(t-\frac{j+k}{2}\right)\gamma(t)dt+\frac{k-j}{2}\left(\int_j^\theta(1-\lambda)\gamma(t)dt\right.\right.$$
$$\left.-\int_{j+k-\theta}^k(1-\lambda)\gamma(t)dt\right)-\max\left\{\lambda\frac{k-j}{2},\left(\theta-\frac{(2-\lambda)j+\lambda k}{2}\right),\left(\frac{j+k}{2}-\theta\right)\right\}$$
$$\left.\times\left(\rho(k)-\rho(j)-\int_j^k\gamma(t)dt\right)\right]$$

and

$$M_{22}(\lambda)=\frac{1}{k-j}\left[\int_j^k\left(t-\frac{j+k}{2}\right)\gamma(t)dt+\frac{k-j}{2}\left(\int_j^{\frac{j+k}{2}}(1-\lambda)\gamma(t)dt\right.\right.$$
$$\left.-\int_{\frac{j+k}{2}}^k(1-\lambda)\gamma(t)dt\right)+\max\left\{\lambda\frac{k-j}{2},\left(\theta-\frac{(2-\lambda)j+\lambda k}{2}\right),\left(\frac{j+k}{2}-\theta\right)\right\}$$
$$\left.\times\left(\rho(k)-\rho(j)-\int_j^k\gamma(t)dt\right)\right],$$

which is Theorem 2.8 of [8].

Remark 2.30. If we choose $\theta=\frac{j+k}{2}$ in (2.15), we get the following result.

Corollary 2.31. *Let all the assumptions of Theorem* 2.1 *be valid. Then*

$$m_{23}(\lambda)\leq\rho(k)\int_\beta^k\omega(t)dt-\rho(j)\int_\alpha^j\omega(t)dt+\rho\left(\frac{j+k}{2}\right)\int_\alpha^\beta\omega(t)dt$$
$$-\int_j^k\rho(t)\omega(t)dt\leq M_{23}(\lambda), \tag{2.18}$$

where

$$m_{23}(\lambda)=\int_j^{\frac{j+k}{2}}\left(\int_\alpha^t\omega(t)dt\right)\gamma(t)dt+\int_{\frac{j+k}{2}}^k\left(\int_\beta^t\omega(t)dt\right)\gamma(t)dt$$
$$-\max\left\{\int_\alpha^{\frac{j+k}{2}}\omega(t)dt,\int_{\frac{\alpha+\beta}{2}}^{\frac{j+k}{2}}\omega(t)dt,\int_\beta^k\omega(t)dt\right\}$$
$$\times\left(\rho(k)-\rho(j)-\int_j^k\gamma(t)dt\right)$$

and

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$$M_{23}(\lambda) = \int_j^{\frac{j+k}{2}} \int_\alpha^t \omega(t)dt\gamma(t)dt + \int_{\frac{j+k}{2}}^k \int_\beta^t \omega(t)dt\gamma(t)dt + \max\left\{\int_\alpha^{\frac{j+k}{2}} \omega(t)dt, \int_{\frac{\alpha+\beta}{2}}^{\frac{j+k}{2}} \omega(t)dt, \int_\beta^k \omega(t)dt\right\} \times \left(\rho(k) - \rho(j) - \int_j^k \gamma(t)dt\right).$$

Special Case 2.31. If we choose $\omega(t) = \frac{1}{k-j}$ in (2.18), we obtain the following result.

$$m_{24}(\lambda) \leq \left[\lambda\frac{\rho(j)+\rho(k)}{2} + (1-\lambda)\rho\left(\frac{j+k}{2}\right)\right] - \frac{1}{k-j}\int_j^k \rho(t)dt \leq M_{24}(\lambda), \quad (2.19)$$

where

$$m_{24}(\lambda) = \frac{1}{k-j}\left[\int_j^k \left(t - \frac{j+k}{2}\right)\gamma(t)dt + \frac{k-j}{2}\left(\int_j^{\frac{j+k}{2}} (1-\lambda)\,\gamma(t)dt - \int_{\frac{j+k}{2}}^k (1-\lambda)\,\gamma(t)dt\right) - \max\left\{\lambda\frac{k-j}{2}, (1-\lambda)\frac{k-j}{2}\right\} \times \left(\rho(k) - \rho(j) - \int_j^k \gamma(t)dt\right)\right]$$

and

$$M_{24}(\lambda) = \frac{1}{k-j}\left[\int_j^k \left(t - \frac{j+k}{2}\right)\gamma(t)dt + \frac{k-j}{2}\left(\int_j^{\frac{j+k}{2}} (1-\lambda)\,\gamma(t)dt - \int_{\frac{j+k}{2}}^k (1-\lambda)\,\gamma(t)dt\right) + \max\left\{\lambda\frac{k-j}{2}, (1-\lambda)\frac{k-j}{2}\right\} \times \left(\rho(k) - \rho(j) - \int_j^k \gamma(t)dt\right)\right],$$

which is Corollary 2.9 of [8].

Theorem 2.32. *Let $\rho : I \to \mathbb{R}$, be a differentiable function on I^0 of I, and let $[j,k] \subset I^0$. If ρ' is bounded above then $\rho'(\theta) \leq \Gamma(\theta)$ for any $\Gamma \in C[j,k]$, $\theta \in [j,k]$, then for all $\lambda \in [0,1]$ we have*

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$$m_{25}(\theta,\lambda) \leq \rho(k)\int_{\beta}^{k}\omega(t)dt - \rho(j)\int_{\alpha}^{j}\omega(t)dt + \rho(\theta)\int_{\alpha}^{\frac{\alpha+\beta}{2}}\omega(t)dt$$
$$+\rho(j+k-\theta)\int_{\frac{\alpha+\beta}{2}}^{\beta}\omega(t)dt - \int_{j}^{k}\rho(t)\omega(t)dt \leq M_{25}(\theta,\lambda), \quad (2.20)$$

where

$$m_{25}(\theta,\lambda) = \int_{j}^{k}\left(\int_{\frac{\alpha+\beta}{2}}^{t}\omega(t)dt\right)\Gamma(t)dt + \int_{j}^{\theta}\left(\int_{\alpha}^{\frac{\alpha+\beta}{2}}\omega(t)dt\right)\Gamma(t)dt$$
$$-\int_{j+k-\theta}^{k}\left(\int_{\frac{\alpha+\beta}{2}}^{t}\omega(t)dt\right)\Gamma(t)dt$$
$$-\max\left\{\int_{\alpha}^{\theta}\omega(t)_0dt, \int_{\frac{\alpha+\beta}{2}}^{j+k-\theta}\omega(t)dt, \int_{\beta}^{k}\omega(t)dt\right\}$$
$$\times\left(\int_{j}^{k}\Gamma(t)dt - \rho(k)+\rho(j)\right)$$

and

$$M_{25}(\theta,\lambda) = \int_{j}^{k}\left(\int_{\frac{\alpha+\beta}{2}}^{t}\omega(t)dt\right)\Gamma(t)dt + \int_{j}^{\theta}\left(\int_{\alpha}^{\frac{\alpha+\beta}{2}}\omega(t)dt\right)\Gamma(t)dt$$
$$-\int_{j+k-\theta}^{k}\left(\int_{\frac{\alpha+\beta}{2}}^{t}\omega(t)dt\right)\Gamma(t)dt$$
$$+\max\left\{\int_{\alpha}^{\theta}\omega(t)dt, \int_{\frac{\alpha+\beta}{2}}^{j+k-\theta}\omega(t)dt, \int_{\beta}^{k}\omega(t)dt\right\}$$
$$\times\left(\int_{j}^{k}\Gamma(t)dt - \rho(k)+\rho(j)\right).$$

Proof. Since

$$\int_{j}^{k}K(\theta,t)\left(\rho'(t)-\Gamma(t)\right)dt$$
$$= \rho(k)\int_{\beta}^{k}\omega(t)dt - \rho(j)\int_{\alpha}^{j}\omega(t)dt + \rho(\theta)\int_{\alpha}^{\frac{\alpha+\beta}{2}}\omega(t)dt$$
$$+\rho(j+k-\theta)\int_{\frac{\alpha+\beta}{2}}^{\beta}\omega(t)dt - \int_{j}^{k}\rho(t)\omega(t)dt - \int_{j}^{k}K(\theta,t)\Gamma(t)dt$$

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$$= \rho(k)\int_{\beta}^{k}\omega(t)dt - \rho(j)\int_{\alpha}^{j}\omega(t)dt + \rho(\theta)\int_{\alpha}^{\frac{\alpha+\beta}{2}}\omega(t)dt$$
$$+\rho(j+k-\theta)\int_{\frac{\alpha+\beta}{2}}^{\beta}\omega(t)dt - \int_{j}^{k}\rho(t)\omega(t)dt$$
$$-\left[\int_{j}^{\theta}\int_{\alpha}^{t}\omega(t)dt\Gamma(t)dt + \int_{\theta}^{j+k-\theta}\int_{\frac{\alpha+\beta}{2}}^{t}\omega(t)dt\Gamma(t)dt\right.$$
$$+\int_{j+k-\theta}^{k}\int_{\beta}^{t}\omega(t)dt\Gamma(t)dt.$$

so we have

$$\left|\rho(k)\int_{\beta}^{k}\omega(t)dt - \rho(j)\int_{\alpha}^{j}\omega(t)dt + \rho(\theta)\int_{\alpha}^{\frac{\alpha+\beta}{2}}\omega(t)dt\right.$$
$$+\rho(j+k-\theta)\int_{\frac{\alpha+\beta}{2}}^{\beta}\omega(t)dt - \int_{j}^{k}\rho(t)\omega(t)dt$$
$$-\left[\int_{j}^{\theta}\int_{\alpha}^{t}\omega(t)dt\Gamma(t)dt + \int_{\theta}^{j+k-\theta}\int_{\frac{\alpha+\beta}{2}}^{t}\omega(t)dt\Gamma(t)dt\right.$$
$$\left.\left.+\int_{j+k-\theta}^{k}\int_{\beta}^{t}\omega(t)dt\Gamma(t)dt\right]\right|$$
$$= \left|\int_{j}^{k}K(\theta,t)\left(\rho'(t)-\Gamma(t)\right)dt\right|$$
$$\leq \int_{j}^{k}|K(\theta,t)|\left(\Gamma(t)-\rho'(t)\right)dt$$
$$\leq \max_{t\in[j,k]}|K(\theta,t)|\int_{j}^{k}\left(\Gamma(t)-\rho'(t)\right)dt$$
$$= \max\left\{\int_{\alpha}^{\theta}\omega(t)dt, \int_{\frac{\alpha+\beta}{2}}^{j+k-\theta}\omega(t)dt, \int_{\beta}^{k}\omega(t)dt\right\}$$
$$\times\left(\int_{j}^{k}\Gamma(t)dt - \rho(k)+\rho(j)\right), \tag{2.21}$$

we get required result after some rearrangement. □

Remark 2.33. If we put $\omega(t) = \frac{1}{k-j}$ in inequality (2.20) then we get Theorem 3 of [18] and Theorem 3 of [19].

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Corollary 2.34. *Let all the assumptions of Theorem 2.32 be valid and if we choose* $\theta = \frac{j+k}{2}$ *in (2.20), then we get*

$$m_{26}(\lambda) \leq \rho(k) \int_{\beta}^{k} \omega(t)dt - \rho(j) \int_{\alpha}^{j} \omega(t)dt + \rho\left(\frac{j+k}{2}\right) \int_{\alpha}^{\beta} \omega(t)dt - \int_{j}^{k} \rho(t)\omega(t)dt \leq M_{26}(\lambda), \quad (2.22)$$

where

$$m_{26}(\lambda) = \int_{j}^{\frac{j+k}{2}} \int_{\alpha}^{t} \omega(t)dt\Gamma(t)dt + \int_{\frac{j+k}{2}}^{k} \int_{\beta}^{t} \omega(t)dt\Gamma(t)dt - \max\left\{\int_{\alpha}^{\frac{j+k}{2}} \omega(t)dt, \int_{\frac{\alpha+\beta}{2}}^{\frac{j+k}{2}} \omega(t)dt, \int_{\beta}^{k} \omega(t)dt\right\} \left(\int_{j}^{k} \Gamma(t)dt - \rho(k) + \rho(j)\right)$$

and

$$M_{26}(\lambda) = \int_{j}^{\frac{j+k}{2}} \int_{\alpha}^{t} \omega(t)dt\Gamma(t)dt + \int_{\frac{j+k}{2}}^{k} \int_{\beta}^{t} \omega(t)dt\Gamma(t)dt + \max\left\{\int_{\alpha}^{\frac{j+k}{2}} \omega(t)dt, \int_{\frac{\alpha+\beta}{2}}^{\frac{j+k}{2}} \omega(t)dt, \int_{\beta}^{k} \omega(t)dt\right\} \left(\int_{j}^{k} \Gamma(t)dt - \rho(k) + \rho(j)\right).$$

Special Case 2.34. If we choose $\omega(t) = \frac{1}{k-j}$ in (2.22), we obtain the following result.

$$m_{27}(\lambda) \leq \left[\lambda \frac{\rho(j) + \rho(k)}{2} + (1-\lambda)\rho\left(\frac{j+k}{2}\right)\right] - \frac{1}{k-j} \int_{j}^{k} \rho(t)dt \leq M_{27}(\lambda),$$

where

$$m_{27}(\lambda) = \frac{1}{k-j} \left[\int_{j}^{k} \left(t - \frac{j+k}{2}\right) \Gamma(t)dt + \frac{k-j}{2} \left(\int_{j}^{\frac{j+k}{2}} (1-\lambda)\Gamma(t)dt - \int_{\frac{j+k}{2}}^{k} (1-\lambda)\Gamma(t)dt\right) - \max\left\{\lambda \frac{k-j}{2}, (1-\lambda)\frac{k-j}{2}\right\} \left(\int_{j}^{k} \Gamma(t)dt - \rho(k) + \rho(j)\right)\right]$$

and

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$$M_{27}(\lambda) = \frac{1}{k-j}\left[\int_j^k \left(t - \frac{j+k}{2}\right)\Gamma(t)dt \right.$$
$$+ \frac{k-j}{2}\left(\int_j^{\frac{j+k}{2}} (1-\lambda)\Gamma(t)dt - \int_{\frac{j+k}{2}}^k (1-\lambda)\Gamma(t)dt\right)$$
$$\left. + \max\left\{\lambda\frac{k-j}{2}, (1-\lambda)\frac{k-j}{2}\right\}\left(\int_j^k \Gamma(t)dt - \rho(k) + \rho(j)\right)\right],$$

which is Corollary 2.11 of [8].

Remark 2.35. If $\gamma(\theta) \leq \rho'(\theta) \leq \Gamma(\theta)$ for any $\theta \in [j,k]$ and $\gamma,\Gamma \in C[j,k]$, and if one put $\lambda = 0$, then error bounds of non-standard quadrature are given as:

$$m_{29} \leq \frac{1}{2}\left[-\rho(j) + 2\rho\left(\frac{j+k}{2}\right) + \rho(k)\right] - \frac{1}{k-j}\int_j^k \rho(t)dt \leq M_{29}, \quad (2.23)$$

where

$$m_{29} = \frac{1}{k-j}\left[\int_j^{\frac{j+k}{2}} (t-j)\gamma(t)dt + \int_{\frac{j+k}{2}}^k (t-k)\gamma(t)dt + \frac{(k-j)}{2}\int_j^k \gamma(t)dt\right]$$

and

$$M_{29} = \frac{1}{k-j}\left[\int_j^{\frac{j+k}{2}} (t-j)\Gamma(t)dt + \int_{\frac{j+k}{2}}^k (t-k)\Gamma(t)dt + \frac{(k-j)}{2}\int_j^k \Gamma(t)dt\right],$$

which is Corollary 3 of [18], Corollary 4 of [19] and Corollary 2.11 of [8].

Proof. To prove (2.23), we use Corollaries 2.31 and 2.34. First by putting $\lambda = 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.22), we get

$$\frac{1}{k-j}\left[\int_j^k (t-j)\gamma(t)dt + \frac{k-j}{2}\left(\int_j^{\frac{j+k}{2}} \gamma(t)dt - \int_{\frac{j+k}{2}}^k \gamma(t)dt\right)\right]$$
$$\leq \frac{1}{2}\left[-\rho(j) + 2\rho\left(\frac{j+k}{2}\right) + \rho(k)\right] - \frac{1}{k-j}\int_j^k \rho(t)dt \quad (2.24)$$

provided that $\gamma(t) \leq \rho'(t)\ \forall\ t \in [j,k]$.

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On the other hand, by assuming $\lambda = 0$ and $\omega(t) = \frac{1}{b-a}$ in (2.22), we obtain

$$\frac{1}{2}\left[-\rho(j) + 2\rho\left(\frac{j+k}{2}\right) + \rho(k)\right] - \frac{1}{k-j}\int_j^k \rho(t)dt$$
$$\leq \frac{1}{k-j}\left[\int_j^k (t-j)\,\Gamma(t)dt + \frac{k-j}{2}\left(\int_j^{\frac{j+k}{2}} \Gamma(t)dt - \int_{\frac{j+k}{2}}^k \Gamma(t)dt\right)\right], \quad (2.25)$$

provided that $\rho'(t) \leq \Gamma(t)\ \forall\ t \in [j,k]$. Combining the above two inequalities obtain the required result. □

Remark 2.36. If $\gamma(\theta) \leq \rho'(\theta) \leq \Gamma(\theta)$ for any $\theta \in [j,k]$ and $\gamma, \Gamma \in C[j,k]$, and if we choose $\lambda = 0$, then error bound of non-standard quadrature would be

$$m_{30} \leq \frac{1}{2}\left[\rho(j) + 2\rho\left(\frac{j+k}{2}\right) - \rho(k)\right] - \frac{1}{k-j}\int_j^k \rho(t)dt \leq M_{30} \qquad (2.26)$$

where

$$\begin{aligned} m_{30} &= \frac{1}{k-j}\left[\int_j^{\frac{j+k}{2}} (t-j)\,\Gamma(t)dt + \int_{\frac{j+k}{2}}^k (t-k)\Gamma(t)dt\right. \\ &- \left.\frac{(k-j)}{2}\int_j^k \Gamma(t)dt\right] \end{aligned}$$

and

$$\begin{aligned} M_{30} &= \frac{1}{k-j}\left[\int_j^{\frac{j+k}{2}} (t-j)\,\gamma(t)dt\right. \\ &+ \left.\int_{\frac{j+k}{2}}^k (t-k)\gamma(t)dt - \frac{(k-j)}{2}\int_j^k \gamma(t)dt\right], \end{aligned}$$

which is Corollary 4 of [18], Corollary 5 of [19] and Corollary 2.11 of [8].

Proof. Proof of (2.26) is similar to that of Remark 2.35, if one replaces $\lambda = 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.18) and (2.22), respectively, and combines them together. □

Remark 2.37. If $\gamma(\theta) \leq \rho'(\theta) \leq \Gamma(\theta)$ for any $\theta \in [j,k]$ and $\gamma, \Gamma \in C[j,k]$ then by replacing $\theta = k$, $\lambda = 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.18) and (2.22), respectively, then error of non-standard quadrature is given as

$$m_{31} \leq \rho(j) - \frac{1}{(k-j)}\int_j^k \rho(t)dt \leq M_{31},$$

where

$$m_{31} = \frac{1}{(k-j)}\int_j^k (t-k)\Gamma(t)dt$$

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and

$$M_{31} = \frac{1}{(k-j)} \int_j^k (t-k)\gamma(t)dt,$$

which is Corollary 5 of [18], Corollary 2 of [19] and Corollary 2.11.

Remark 2.38. If $\gamma(\theta) \leq \rho'(\theta) \leq \Gamma(\theta)$ for any $\theta \in [j,k]$ and $\gamma, \Gamma \in C[j,k]$ then by replacing $\theta = j$, $\lambda = 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.18) and (2.22), respectively, one gets error bounds of non-standard quadrature as follows:

$$m_{32} \leq \rho(k) - \frac{1}{(k-j)} \int_j^k \rho(t)dt \leq M_{32},$$

where

$$m_{32} = \frac{1}{(k-j)} \int_j^k (t-j)\gamma(t)dt \tag{2.27}$$

and

$$M_{32} = \frac{1}{(k-j)} \int_j^k (t-j)\Gamma(t)dt.$$

The inequality presented above is same as the Corollary 6 of [18], Corollary 3 of [19] and Corollary 2.11 of [8].

3. Applications to Numerical Quadrature Rules

Let $I_n : j = z_0 < z_1 < \cdots < z_n = k$ be a partition of interval $[j,k]$ and let $h_i = z_{i+1} - z_i, i \in \{0, 1, 2, \ldots, n-1\}$. Then

$$\int_j^k \omega(t)\rho(t)dt = Q_n(\rho, \omega, \lambda) + R_n(\rho, \omega, \lambda). \tag{3.1}$$

Consider a general quadrature rule

$$\begin{aligned} Q_n(\rho, \omega, \lambda) &= \sum_{i=0}^{n-1} \left[\rho(z_{i+1}) \int_{\beta_i}^{z_{i+1}} \omega(t)dt - \rho(z_i) \int_{\alpha_i}^{z_i} \omega(t)dt \right. \\ &+ \left. \rho(\theta_i) \int_{\alpha_i}^{\frac{\alpha_i+\beta_i}{2}} \omega(t)dt + \rho(z_i + z_{i+1} - \theta_i) \int_{\frac{\alpha_i+\beta_i}{2}}^{\beta_i} \omega(t)dt \right] \end{aligned} \tag{3.2}$$

where $\lambda \in [0,1]$ and $\theta_i \in [z_i, z_{i+1}]$. Then we get following result:

Theorem 3.1. *Let all the assumptions of Theorem* 2.1 *be valid. Then* (3.1) *holds where* $Q_n(\rho, \omega, \lambda)$ *is given by formula* (3.2) *and remainder* $R_n(I_n, \rho, \omega)$ *satisfies estimates*

$$|R_n(\rho, \omega, \lambda)| \leq \sum_{i=0}^{n-1} \sup \{|R_1|, |R_2|\}, \tag{3.3}$$

where

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$$\begin{aligned}
R_1 &= \int_{z_i}^{\theta_i} \left(\left(\int_{\alpha_i}^{t} \omega(t)dt - \left| \int_{\alpha_i}^{t} \omega(t)dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
&\quad \left. + \left(\int_{\alpha_i}^{t} \omega(t)dt + \left| \int_{\alpha_i}^{t} \omega(t)dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\
&\quad + \int_{\theta_i}^{z_i+z_{i+1}-\theta_i} \left(\left(\int_{\frac{\alpha_i+\beta_i}{2}}^{t} \omega(t)dt - \left| \int_{\frac{\alpha_i+\beta_i}{2}}^{t} \omega(t)dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
&\quad + \left. \left(\int_{\frac{\alpha_i+\beta_i}{2}}^{t} \omega(t)dt + \left| \int_{\frac{\alpha_i+\beta_i}{2}}^{t} \omega(t)dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\
&\quad + \int_{z_i+z_{i+1}-\theta_i}^{z_{i+1}} \left(\left(\int_{\beta_i}^{t} \omega(t)dt - \left| \int_{\beta_i}^{t} \omega(t)dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
&\quad + \left. \left(\int_{\beta_i}^{t} \omega(t)dt + \left| \int_{\beta_i}^{t} \omega(t)dt \right| \right) \frac{\gamma(t)}{2} \right) dt
\end{aligned}$$

and

$$\begin{aligned}
R_2 &= \int_{z_i}^{\theta_i} \left(\left(\int_{\alpha_i}^{t} \omega(t)dt + \left| \int_{\alpha_i}^{t} \omega(t)dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
&+ \left. \left(\int_{\alpha_i}^{t} \omega(t)dt - \left| \int_{\alpha_i}^{t} \omega(t)dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\
&+ \int_{\theta_i}^{z_i+z_{i+1}-\theta_i} \left(\left(\int_{\frac{\alpha_i+\beta_i}{2}}^{t} \omega(t)dt + \left| \int_{\frac{\alpha_i+\beta_i}{2}}^{t} \omega(t)dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
&+ \left. \left(\int_{\frac{\alpha_i+\beta_i}{2}}^{t} \omega(t)dt - \left| \int_{\frac{\alpha_i+\beta_i}{2}}^{t} \omega(t)dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\
&+ \int_{z_i+z_{i+1}-\theta_i}^{z_{i+1}} \left(\left(\int_{\beta_i}^{t} \omega(t)dt + \left| \int_{\beta_i}^{t} \omega(t)dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
&+ \left. \left(\int_{\beta_i}^{t} \omega(t)dt - \left| \int_{\beta_i}^{t} \omega(t)dt \right| \right) \frac{\gamma(t)}{2} \right) dt,
\end{aligned}$$

$\forall\ \theta_i \in [z_i, z_{i+1}]$.

Proof. Applying inequality (2.1) on the intervals $[z_i, z_{i+1}]$ for $r \in \{1, 2, \ldots, n-1\}$, and using (3.3), we get

$$\begin{aligned}
R_i(\rho, \omega, \lambda) &= \int_{z_i}^{z_{i+1}} \omega(t)\rho(t)dt - \left[\rho(z_{i+1}) \int_{\beta_i}^{z_{i+1}} \omega(t)dt - \rho(z_i) \int_{\alpha_i}^{z_i} \omega(t)dt \right. \\
&+ \left. \rho(\theta_i) \int_{\alpha_i}^{\frac{\alpha_i+\beta_i}{2}} \omega(t)dt + \rho(z_i + z_{i+1} - \theta_i) \int_{\frac{\alpha_i+\beta_i}{2}}^{\beta_i} \omega(t)dt \right].
\end{aligned}$$

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Summing it over i from 0 to $n-1$ we get

$$R_n(\rho,\omega,\lambda)=\int_j^k \rho(t)\omega(t)dt-\sum_{i=0}^{n-1}\left[\rho(z_{i+1})\int_{\beta_i}^{z_{i+1}}\omega(t)dt-\right.$$
$$\left.\rho(z_i)\int_{\alpha_i}^{z_i}\omega(t)dt+\rho(\theta_i)\int_{\alpha_i}^{\frac{\alpha_i+\beta_i}{2}}\omega(t)dt+\rho(z_i+z_{i+1}-\theta_i)\int_{\frac{\alpha_i+\beta_i}{2}}^{\beta_i}\omega(t)dt\right].$$

It follows from (2.1) that

$$\begin{aligned}
|R_n(\rho,\omega,\lambda)| &= \left|\int_j^k \rho(t)\omega(t)dt\right.\\
&-\sum_{i=0}^{n-1}\left[\rho(z_{i+1})\int_{\beta_i}^{z_{i+1}}\omega(t)dt-\rho(z_i)\int_{\alpha_i}^{z_i}\omega(t)dt\right.\\
&\left.\left.+\ \rho(\theta_i)\int_{\alpha_i}^{\frac{\alpha_i+\beta_i}{2}}\omega(t)dt+\rho(z_i+z_{i+1}-\theta_i)\int_{\frac{\alpha_i+\beta_i}{2}}^{\beta_i}\omega(t)dt\right]\right|\\
&\leq \sum_{i=0}^{n-1}\sup\left\{\left|\left[\int_{z_i}^{\theta_i}\left(\left(\int_{\alpha_i}^{t}\omega(t)dt-\left|\int_{\alpha_i}^{t}\omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right.\right.\right.\right.\\
&\left.+\left(\int_{\alpha_i}^{t}\omega(t)dt+\left|\int_{\alpha_i}^{t}\omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt\\
&+\int_{\theta_i}^{z_i+z_{i+1}-\theta_i}\left(\left(\int_{\frac{\alpha_i+\beta_i}{2}}^{t}\omega(t)dt-\left|\int_{\frac{\alpha_i+\beta_i}{2}}^{t}\omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right.\\
&\left.+\ \left(\int_{\frac{\alpha_i+\beta_i}{2}}^{t}\omega(t)dt+\left|\int_{\frac{\alpha_i+\beta_i}{2}}^{t}\omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt\\
&+\int_{z_i+z_{i+1}-\theta_i}^{z_{i+1}}\left(\left(\int_{\beta_i}^{t}\omega(t)dt-\left|\int_{\beta_i}^{t}\omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right.\\
&\left.\left.\left.+\ \left(\int_{\beta_i}^{t}\omega(t)dt+\left|\int_{\beta_i}^{t}\omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt\right]\right|,\\
&\left|\int_{z_i}^{\theta_i}\left(\left(\int_{\alpha_i}^{t}\omega(t)dt+\left|\int_{\alpha_i}^{t}\omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right.\right.\\
&\left.+\left(\int_{\alpha_i}^{t}\omega(t)dt-\left|\int_{\alpha_i}^{t}\omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt\\
&+\int_{\theta_i}^{z_i+z_{i+1}-\theta_i}\left(\left(\int_{\frac{\alpha_i+\beta_i}{2}}^{t}\omega(t)dt+\left|\int_{\frac{\alpha_i+\beta_i}{2}}^{t}\omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right.\\
&\left.+\left(\int_{\frac{\alpha_i+\beta_i}{2}}^{t}\omega(t)dt-\left|\int_{\frac{\alpha_i+\beta_i}{2}}^{t}\omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)dt
\end{aligned}$$

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$$+\int_{z_i+z_{i+1}-\theta_i}^{z_{i+1}}\left(\left(\left(\int_{\beta_i}^{t}\omega(t)dt+\left|\int_{\beta_i}^{t}\omega(t)dt\right|\right)\frac{\Gamma(t)}{2}\right.\right.$$
$$\left.\left.\left.\left.+\quad\left(\int_{\beta_i}^{t}\omega(t)dt-\left|\int_{\beta_i}^{t}\omega(t)dt\right|\right)\frac{\gamma(t)}{2}\right)\right)dt\right]\right|\right\}.$$

□

Remark 3.2. Similarly, we can state applications of other results and their cases as given in Section 2.

4. Conclusion

In this paper, weighted Ostrowski type inequality is discussed for function differentiable functions with variable bounds. Applications to solve error bounds of midpoint, trapezoidal, Simpson's and Simpson's quadrature and some non-standard quadrature rules are presented. We also have many proven results as our special cases. In particularly our results would generalization of [8, 18, 19, 27].

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