

Generalization of Ostrowski's Inequality for Differentiable Functions and its Applications

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ABSTRACT. We first establish weighted Ostrowski type inequalities for bounded differentiable functions. This inequality is also obtained for bounded above and bounded below differentiable functions. Some applications of the proposed results are presented to numerical standard and non standard quadrature rules. We recapture known results as well as obtain new results.

Keywords: Ostrowski's inequality, Numerical integration, Differentiable functions.

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1. INTRODUCTION

Our work mostly deals with integral inequalities. To highlight its importance we quote here from [20], "Among the many types of inequalities, integral

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inequalities are of supreme importance because over the last few decades this field has proven to be an extensively applicable field. The integral inequalities of various types have been widely studied in most subjects involving mathematical analysis. These inequalities are particularly useful in approximation theory and in numerical analysis where estimates of approximation errors are involved." Ostrowski inequality [25] is one of the most famous inequality, first presented by Alexander Markowich Ostrowski in 1938. It can be used to determine absolute deviation of functional value from its mean value. It is extremely important because of its wide range of applications in different areas of mathematics such as numerical integration, integral operator theory, probability theory and statistics. This inequality states that:

Proposition 1.1. *Let $\rho : I \rightarrow \mathbb{R}$ be a differentiable function on I^o such that $\rho \in L[j, k]$ where $j < k$ whose derivative ρ' is bounded on interior of I , i.e., $\|\rho'\|_\infty := \sup_{t \in (j, k)} |\rho'(t)| < \infty$. Then*

$$\left| \rho(\theta) - \frac{1}{k-j} \int_j^k \rho(t) dt \right| \leq (k-j) \left[\frac{1}{4} + \frac{\left(\theta - \frac{j+k}{2}\right)^2}{(k-j)^2} \right] \|\rho'(x)\|_\infty. \quad (1.1)$$

The constant $\frac{1}{4}$ is the best possible constant that it cannot be replaced by smaller one.

Ostrowski inequality for differentiable functions has been generalized in many times as stated in [6, 18, 19, 27]. For latest work related to Ostrowski inequality we refer the reader to following articles [1, 2, 3, 5, 7, 12, 13, 14, 15, 16, 17, 21, 22, 23, 24, 26]

To prove our main results we need the following two lemmas from [8] and [10].

Lemma 1.2. *Let $\rho : I \rightarrow \mathbb{R}$, be a differentiable function in interior I^o of interval I and also let $[j, k] \subset I^o$. Then the following identity holds*

$$\begin{aligned} \int_j^k K(\theta, t) \rho'(t) dt &= \rho(k) \int_\beta^k \omega(t) dt - \rho(j) \int_\alpha^j \omega(t) dt + \rho(\theta) \int_\alpha^{\frac{\alpha+\beta}{2}} \omega(t) dt \\ &+ \rho(j+k-\theta) \int_{\frac{\alpha+\beta}{2}}^\beta \omega(t) dt - \int_j^k \rho(t) \omega(t) dt, \end{aligned} \quad (1.2)$$

where $K(\theta, t)$ defined as:

$$K(\theta, t) = \begin{cases} \int_{\alpha}^t \omega(t) dt, & \text{if } t \in [j, \theta]; \\ \int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt, & \text{if } t \in (\theta, j+k-\theta]; \\ \int_{\beta}^t \omega(t) dt, & \text{if } t \in (j+k-\theta, k]; \end{cases} \quad (1.3)$$

where $\forall \theta \in [j, k]$, $\alpha = j + \lambda \frac{k-j}{2}$, $\beta = k - \lambda \frac{k-j}{2}$ and $\lambda \in [0, 1]$.

Lemma 1.3. Let $\rho : [j, k] \rightarrow \mathbb{R}$ be a differentiable function such that $\gamma(\theta) \leq \rho'(\theta) \leq \Gamma(\theta)$ for any $\gamma, \Gamma \in C[j, k]$ and $\theta \in [j, k]$. Then we have

$$\left| \rho'(t) - \frac{\gamma(t) + \Gamma(t)}{2} \right| \leq \frac{\Gamma(t) - \gamma(t)}{2}. \quad (1.4)$$

We are ready to present our main theorem which will be generalized in two ways: first, by adding weights that are actually probability density functions, and second, by adding a parameter. In this way, we will capture variety of results from various articles as special cases.

2. MAIN RESULTS

Theorem 2.1. Let $\rho : I \rightarrow \mathbb{R}$, be a differentiable function on I^0 and also let $\gamma(\theta) \leq \rho'(\theta) \leq \Gamma(\theta)$ for any $\gamma, \Gamma \in C[j, k]$ and $\theta \in [j, k]$. Then

$$\begin{aligned} m(\theta, \lambda) \leq & \rho(k) \int_{\beta}^k \omega(t) dt - \rho(j) \int_{\alpha}^j \omega(t) dt + \rho(\theta) \int_{\alpha}^{\frac{\alpha+\beta}{2}} \omega(t) dt \\ & + \rho(j+k-\theta) \int_{\frac{\alpha+\beta}{2}}^{\beta} \omega(t) dt - \int_j^k \rho(t) \omega(t) dt \leq M(\theta, \lambda), \end{aligned} \quad (2.1)$$

where

$$\begin{aligned} m(\theta, \lambda) = & \int_j^{\theta} \left(\left(\int_{\alpha}^t \omega(t) dt - \left| \int_{\alpha}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ & + \left. \left(\int_{\alpha}^t \omega(t) dt + \left| \int_{\alpha}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ & + \int_{\theta}^{j+k-\theta} \left(\left(\int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt - \left| \int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ & + \left. \left(\int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt + \left| \int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ & + \int_{j+k-\theta}^k \left(\left(\int_{\beta}^t \omega(t) dt - \left| \int_{\beta}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ & + \left. \left(\int_{\beta}^t \omega(t) dt + \left| \int_{\beta}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \end{aligned}$$

and

$$\begin{aligned}
 M(\theta, \lambda) &= \int_j^\theta \left(\left(\int_\alpha^t \omega(t) dt + \left| \int_\alpha^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
 &+ \left. \left(\int_\alpha^t \omega(t) dt - \left| \int_\alpha^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\
 &+ \int_\theta^{j+k-\theta} \left(\left(\int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt + \left| \int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
 &+ \left. \left(\int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt - \left| \int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\
 &+ \int_{j+k-\theta}^k \left(\left(\int_\beta^t \omega(t) dt + \left| \int_\beta^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
 &+ \left. \left(\int_\beta^t \omega(t) dt - \left| \int_\beta^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt.
 \end{aligned}$$

Proof. Using (1.2) and (1.4), we obtain

$$\begin{aligned}
 &\int_j^k K(\theta, t) \left(\rho'(t) - \frac{\gamma(t) + \Gamma(t)}{2} \right) dt \\
 &= \int_j^k K(\theta, t) \rho'(t) dt - \frac{1}{2} \left(\int_j^k K(\theta, t) (\gamma(t) + \Gamma(t)) dt \right) \\
 &= \rho(k) \int_\beta^k \omega(t) dt - \rho(j) \int_\alpha^j \omega(t) dt + \rho(\theta) \int_\alpha^{\frac{\alpha+\beta}{2}} \omega(t) dt \\
 &+ \rho(j+k-\theta) \int_{\frac{\alpha+\beta}{2}}^\beta \omega(t) dt - \int_j^k \omega(t) \rho(t) dt \\
 &- \frac{1}{2} \left[\int_j^\theta \int_\alpha^t \omega(t) dt (\gamma(t) + \Gamma(t)) dt + \int_\theta^{j+k-\theta} \int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt (\gamma(t) + \Gamma(t)) dt \right. \\
 &\left. + \int_{j+k-\theta}^k \int_\beta^t \omega(t) dt (\gamma(t) + \Gamma(t)) dt \right]. \tag{2.2}
 \end{aligned}$$

From (1.4) we have

$$\begin{aligned}
 &\left| \int_\beta^k \omega(t) dt \rho(k) - \rho(j) \int_\alpha^j \omega(t) dt + \rho(\theta) \int_\alpha^{\frac{\alpha+\beta}{2}} \omega(t) dt + \rho(j+k-\theta) \right. \\
 &\times \left. \int_{\frac{\alpha+\beta}{2}}^\beta \omega(t) dt - \int_j^k \omega(t) \rho(t) dt - \frac{1}{2} \left[\int_j^\theta \int_\alpha^t \omega(t) dt (\gamma(t) + \Gamma(t)) dt \right. \right. \\
 &+ \left. \int_\theta^{j+k-\theta} \int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt (\gamma(t) + \Gamma(t)) dt \right. \\
 &\left. \left. + \int_{j+k-\theta}^k \int_\beta^t \omega(t) dt (\gamma(t) + \Gamma(t)) dt \right] \right|
 \end{aligned}$$

$$\begin{aligned}
&= \left| \int_j^k K(\theta, t) \left(\rho'(t) - \frac{\gamma(t) + \Gamma(t)}{2} \right) dt \right| \\
&\leq \int_j^k |K(\theta, t)| \left| \left(\rho'(t) - \frac{\gamma(t) + \Gamma(t)}{2} \right) \right| dt \\
&\leq \int_j^k |K(\theta, t)| \left(\frac{\Gamma(t) - \gamma(t)}{2} \right) dt \\
&= \frac{1}{2} \left[\int_j^\theta \left| \int_\alpha^t \omega(t) dt \right| (\Gamma(t) - \gamma(t)) dt + \int_\theta^{j+k-\theta} \left| \int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt \right| \right. \\
&\quad \left. \times (\Gamma(t) - \gamma(t)) dt + \int_{j+k-\theta}^k \left| \int_\beta^t \omega(t) dt \right| (\Gamma(t) - \gamma(t)) dt \right]. \quad (2.3)
\end{aligned}$$

After rearranging (2.3), we get the required result. $\square$

Remark 2.2. It is worth mentioning that if we put $\omega(t) = \frac{1}{k-j}$ in our main result we will get the following result.

Corollary 2.3. *Let all the assumptions of Theorem 2.1 be valid. Then*

$$\begin{aligned}
m_0(\theta, \lambda) &\leq \lambda \frac{\rho(j) + \rho(k)}{2} + (1 - \lambda) \frac{\rho(\theta) + \rho(j + k - \theta)}{2} - \int_j^k \rho(t) d(t) \\
&\leq M_0(\theta, \lambda) \quad (2.4)
\end{aligned}$$

$m_0(t, \lambda) =$

$$\begin{aligned}
&\frac{1}{k-j} \left[\int_{-\lambda \frac{k-j}{2}}^{t-(j+\lambda \frac{k-j}{2})} \left(\frac{\eta + |\eta|}{2} \mu \left(\eta + j + \lambda \frac{k-j}{2} \right) + \frac{\eta - |\eta|}{2} \nu \left(\eta + j + \lambda \frac{k-j}{2} \right) \right) d\eta \right. \\
&+ \int_{t-\frac{j+k}{2}}^{\frac{j+k}{2}-t} \left(\frac{\eta + |\eta|}{2} \mu \left(\eta + \frac{j+k}{2} \right) + \frac{\eta - |\eta|}{2} \nu \left(\eta + \frac{j+k}{2} \right) \right) d\eta \\
&\left. + \int_{j+\lambda \frac{k-j}{2}-t}^{\lambda \frac{k-j}{2}} \left(\frac{\eta + |\eta|}{2} \mu \left(\eta + j - \lambda \frac{k-j}{2} \right) + \frac{\eta - |\eta|}{2} \nu \left(\eta + j - \lambda \frac{k-j}{2} \right) \right) d\eta \right]
\end{aligned}$$

and

$$\begin{aligned}
M_0(t, \lambda) &= \frac{1}{k-j} \left[\int_{-\lambda \frac{k-j}{2}}^{t-(j+\lambda \frac{k-j}{2})} \left(\frac{\eta - |\eta|}{2} \mu \left(\eta + j + \lambda \frac{k-j}{2} \right) + \frac{\eta + |\eta|}{2} \nu \left(\eta + j + \lambda \frac{k-j}{2} \right) \right) d\eta \right. \\
&+ \int_{t-\frac{j+k}{2}}^{\frac{j+k}{2}-t} \left(\frac{\eta - |\eta|}{2} \mu \left(\eta + \frac{j+k}{2} \right) + \frac{\eta + |\eta|}{2} \nu \left(\eta + \frac{j+k}{2} \right) \right) d\eta \\
&\left. + \int_{j+\lambda \frac{k-j}{2}-t}^{\lambda \frac{k-j}{2}} \left(\frac{\eta - |\eta|}{2} \mu \left(\eta + j - \lambda \frac{k-j}{2} \right) + \frac{\eta + |\eta|}{2} \nu \left(\eta + j - \lambda \frac{k-j}{2} \right) \right) d\eta \right],
\end{aligned}$$

which is Theorem 2.3 of [7] and hence all its Corollaries and Remarks and further consequences would become our special cases. Throughout the section $\gamma_0, \gamma_1, \Gamma_0, \Gamma_1$ are real constants.

Remark 2.4. If we put $\lambda = 1$ in (2.1), we obtain following result.

Corollary 2.5. *Let all the assumptions of Theorem 2.1 be valid. Then*

$$m_1 \leq \rho(k) \int_{\frac{j+k}{2}}^k \omega(t) dt + \rho(j) \int_j^{\frac{j+k}{2}} \omega(t) dt - \int_j^k \rho(t) \omega(t) dt \leq M_1, \quad (2.5)$$

where

$$\begin{aligned} m_1 &= \int_j^k \left[\left(\int_{\frac{j+k}{2}}^t \omega(t) dt - \left| \int_{\frac{j+k}{2}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &\quad \left. + \left(\int_{\frac{j+k}{2}}^t \omega(t) dt + \left| \int_{\frac{j+k}{2}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right] dt \end{aligned}$$

and

$$\begin{aligned} M_1 &= \int_j^k \left[\left(\int_{\frac{j+k}{2}}^t \omega(t) dt + \left| \int_{\frac{j+k}{2}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &\quad \left. + \left(\int_{\frac{j+k}{2}}^t \omega(t) dt - \left| \int_{\frac{j+k}{2}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right] dt. \end{aligned}$$

Special Case 2.5.(a) If we take, $\gamma(t) = \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.5), then

$$\frac{(k-j)}{8}(\gamma_0 - \Gamma_0) \leq \frac{\rho(j) + \rho(k)}{2} - \frac{1}{k-j} \int_j^k \rho(t) dt \leq \frac{(k-j)}{8}(\Gamma_0 - \gamma_0),$$

which is the Corollary 2 of [27] and Special Case 2.4.1 of [7].

Special Case 2.5.(b) If we take, $\gamma(t) = \gamma_1 t + \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_1 t + \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.5), then

$$m_2 \leq \frac{\rho(j) + \rho(k)}{2} - \frac{1}{k-j} \int_j^k \rho(t) dt \leq M_2,$$

where

$$m_2 = \frac{(k-j)}{8} \left[\frac{(k-j)}{3}(\gamma_1 + \Gamma_1) + \frac{(j+k)}{2}(\gamma_1 - \Gamma_1) + \gamma_0 - \Gamma_0 \right]$$

and

$$M_2 = \frac{(k-j)}{8} \left[\frac{(k-j)}{3}(\gamma_1 + \Gamma_1) + \frac{(j+k)}{2}(\Gamma_1 - \gamma_1) + \Gamma_0 - \gamma_0 \right],$$

which is Special Case 2.4.2 of [7].

Remark 2.6. If we choose $\theta = \frac{j+k}{2}$ in (2.1), we obtain the following result.

Corollary 2.7. *Let all the assumptions of Theorem 2.1 be valid. Then*

$$\begin{aligned} m_3(\lambda) &\leq \rho(k) \int_{\beta}^k \omega(t) dt - \rho(j) \int_{\alpha}^j \omega(t) dt + \rho\left(\frac{k+j}{2}\right) \int_{\alpha}^{\beta} \omega(t) dt \\ &\quad - \int_j^k \rho(t) \omega(t) dt \leq M_3(\lambda), \quad (2.6) \end{aligned}$$

where

$$\begin{aligned} m_3(\lambda) &= \int_j^{\frac{j+k}{2}} \left(\left(\int_\alpha^t \omega(t) dt - \left| \int_\alpha^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &\quad + \left. \left(\int_\alpha^t \omega(t) dt + \left| \int_\alpha^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ &\quad + \int_{\frac{j+k}{2}}^k \left(\left(\int_\beta^t \omega(t) dt - \left| \int_\beta^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &\quad + \left. \left(\int_\beta^t \omega(t) dt + \left| \int_\beta^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \end{aligned}$$

and

$$\begin{aligned} M_3(\lambda) &= \int_j^{\frac{j+k}{2}} \left(\left(\int_\alpha^t \omega(t) dt + \left| \int_\alpha^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &\quad + \left. \left(\int_\alpha^t \omega(t) dt - \left| \int_\alpha^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ &\quad + \int_{\frac{j+k}{2}}^k \left(\left(\int_\beta^t \omega(t) dt + \left| \int_\beta^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &\quad + \left. \left(\int_\beta^t \omega(t) dt - \left| \int_\beta^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt. \end{aligned}$$

Remark 2.8. If we choose $\lambda = 0$ in (2.6), we obtain the following result.

Corollary 2.9. *Let all the assumptions of Theorem 2.1 be valid. Then*

$$m_4 \leq \rho \left(\frac{k+j}{2} \right) \int_j^k \omega(t) dt - \int_j^k \rho(t) \omega(t) dt \leq M_4(t)$$

where

$$\begin{aligned} m_4 &= \int_j^{\frac{j+k}{2}} \left(\left(\int_j^t \omega(t) dt - \left| \int_j^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &\quad + \left. \left(\int_j^t \omega(t) dt + \left| \int_j^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ &\quad + \int_{\frac{j+k}{2}}^k \left(\left(\int_k^t \omega(t) dt - \left| \int_k^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &\quad + \left. \left(\int_k^t \omega(t) dt + \left| \int_k^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \end{aligned}$$

and

$$\begin{aligned} M_4 &= \int_j^{\frac{j+k}{2}} \left(\left(\int_j^t \omega(t) dt + \left| \int_j^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &\quad \left. + \left(\int_j^t \omega(t) dt - \left| \int_j^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ &\quad + \int_{\frac{j+k}{2}}^k \left(\left(\int_k^t \omega(t) dt + \left| \int_k^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &\quad \left. + \left(\int_k^t \omega(t) dt - \left| \int_k^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt. \end{aligned}$$

Special Case 2.9.(a) If we take, $\gamma(\theta) = \gamma_0 \neq 0$, $\Gamma(\theta) = \Gamma_0 \neq 0$, $\theta = \frac{j+k}{2}$ and $\omega(t) = \frac{1}{k-j}$, in (2.7), then

$$\frac{(k-j)}{8}(\gamma_0 - \Gamma_0) \leq \rho \left(\frac{j+k}{2} \right) - \frac{1}{k-j} \int_j^k \rho(t) dt \leq \frac{(k-j)}{8}(\Gamma_0 - \gamma_0),$$

which is in fact the Special Case 1 of Theorem 1 presented in [18], Corollary 1 of [27] and Special Case 2.6.1 of [7].

Special Case 2.9.(b) If we take, $\gamma(\theta) = \gamma_1 \theta + \gamma_0 \neq 0$, $\Gamma(\theta) = \Gamma_1 \theta + \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.7), then

$$m_5 \leq \rho \left(\frac{j+k}{2} \right) - \frac{1}{k-j} \int_j^k \rho(t) dt \leq M_5,$$

where

$$m_5 = \frac{(k-j)}{8} \left(\frac{k-j}{3} (\gamma_1 + \Gamma_1) + j\gamma_1 - k\Gamma_1 + \gamma_0 - \Gamma_0 \right)$$

and

$$M_5 = \frac{(k-j)}{8} \left(\frac{k-j}{3} (\gamma_1 + \Gamma_1) + j\Gamma_1 - k\gamma_1 + \Gamma_0 - \gamma_0 \right),$$

which is example of Corollary 1 of [18] and Special Case 2.6.2 of [7].

Remark 2.10. By choosing $\lambda = \frac{1}{3}$ in (2.6), then we get the bounds for $\frac{1}{3}$ Simpson's rule.

Corollary 2.11. Let all the assumptions of Theorem 2.1 be valid. Then

$$\begin{aligned} m_6 &\leq \rho(k) \int_{\frac{j+5k}{6}}^k \omega(t) dt - \rho(j) \int_{\frac{5j+k}{6}}^j \omega(t) dt + \rho \left(\frac{k+j}{2} \right) \int_{\frac{5j+k}{6}}^{\frac{j+k}{2}} \omega(t) dt \\ &\quad + \rho \left(\frac{k+j}{2} \right) \int_{\frac{j+k}{2}}^{\frac{j+5k}{6}} \omega(t) dt - \int_j^k \rho(t) \omega(t) dt \leq M_6, \quad (2.7) \end{aligned}$$

where

$$\begin{aligned} m_6 &= \int_{-\frac{k-j}{6}}^{\frac{j+k}{2}} \left(\left(\int_{\frac{5j+k}{6}}^t \omega(t) dt - \left| \int_{\frac{5j+k}{6}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &+ \left. \left(\int_{\frac{5j+k}{6}}^t \omega(t) dt + \left| \int_{\frac{5j+k}{6}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ &+ \int_{\frac{j+k}{2}}^{\frac{k-j}{6}} \left(\left(\int_{\frac{j+5k}{6}}^t \omega(t) dt - \left| \int_{\frac{j+5k}{6}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &+ \left. \left(\int_{\frac{j+5k}{6}}^t \omega(t) dt + \left| \int_{\frac{j+5k}{6}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \end{aligned}$$

and

$$\begin{aligned} M_6 &= \int_{-\frac{k-j}{6}}^{\frac{j+k}{2}} \left(\left(\int_{\frac{5j+k}{6}}^t \omega(t) dt + \left| \int_{\frac{5j+k}{6}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &+ \left. \left(\int_{\frac{5j+k}{6}}^t \omega(t) dt - \left| \int_{\frac{5j+k}{6}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ &+ \int_{\frac{j+k}{2}}^{\frac{k-j}{6}} \left(\left(\int_{\frac{j+5k}{6}}^t \omega(t) dt + \left| \int_{\frac{j+5k}{6}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &+ \left. \left(\int_{\frac{j+5k}{6}}^t \omega(t) dt - \left| \int_{\frac{j+5k}{6}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt . \end{aligned}$$

Special Case 2.11.(a) If we take, $\gamma(t) = \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.7), then

$$\begin{aligned} \frac{(k-j)}{72}(\gamma_0 - \Gamma_0) &\leq \frac{1}{3} \left[\frac{\rho(j) + \rho(k)}{2} + 2\rho\left(\frac{j+k}{2}\right) \right] - \frac{1}{k-j} \int_j^k \rho(t) dt \\ &\leq \frac{(k-j)}{72}(\Gamma_0 - \gamma_0), \end{aligned}$$

which is Corollary 4 of [27] and Special Case 2.7.1 of [7].

Special Case 2.11.(b) If we take, $\gamma(t) = \gamma_1 t + \gamma_0 \neq 0$ and $\Gamma(t) = \Gamma_1 t + \Gamma_0 \neq 0$, $\omega(t) = \frac{1}{k-j}$ in (2.7), then

$$m_7 \leq \frac{1}{3} \left[\frac{\rho(j) + \rho(k)}{2} + 2\rho\left(\frac{j+k}{2}\right) \right] - \frac{1}{k-j} \int_j^k \rho(t) dt \leq M_7,$$

where

$$m_7 = \frac{(k-j)}{72} \left[(k-j)(\gamma_1 + \Gamma_1) + \frac{j}{2}(7\gamma_1 - 3\Gamma_1) + \frac{k}{2}(3\gamma_1 - 7\Gamma_1) + 5(\gamma_0 - \Gamma_0) \right]$$

and

$$M_7 = \frac{(k-j)}{72} \left[(k-j)(\gamma_1 + \Gamma_1) + \frac{j}{2}(7\Gamma_1 - 3\gamma_1) + \frac{k}{2}(3\Gamma_1 - 7\gamma_1) + 5(\Gamma_0 - \gamma_0) \right],$$

which is Special Case 2.7.2 of [7].

Remark 2.12. If we choose $\lambda = \frac{1}{2}$ in (2.6), we obtain bound of average midpoint and trapezoidal inequality in the following result.

Corollary 2.13. *Let all the assumptions of Theorem 2.1 be valid. Then*

$$m_8 \leq \rho(k) \int_{\frac{j+3k}{4}}^k \omega(t) dt - \rho(j) \int_{\frac{3j+k}{4}}^j \omega(t) dt + \rho\left(\frac{k+j}{2}\right) \int_{\frac{3j+k}{4}}^{\frac{j+k}{2}} \omega(t) dt \\ + \rho\left(\frac{k+j}{2}\right) \int_{\frac{j+k}{2}}^{\frac{j+3k}{4}} \omega(t) dt - \int_j^k \rho(t) \omega(t) dt \leq M_8, \quad (2.8)$$

where

$$m_8 = \int_{-\frac{k-j}{4}}^{\frac{j+k}{2}} \left(\left(\int_{\frac{3j+k}{4}}^t \omega(t) dt - \left| \int_{\frac{3j+k}{4}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ + \left. \left(\int_{\frac{3j+k}{4}}^t \omega(t) dt + \left| \int_{\frac{3j+k}{4}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ + \int_{\frac{j+k}{2}}^{\frac{k-j}{4}} \left(\left(\int_{\frac{j+3k}{4}}^t \omega(t) dt - \left| \int_{\frac{j+3k}{4}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ + \left. \left(\int_{\frac{j+3k}{4}}^t \omega(t) dt + \left| \int_{\frac{j+3k}{4}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt$$

and

$$M_8 = \int_{-\frac{k-j}{4}}^{\frac{j+k}{2}} \left(\left(\int_{\frac{3j+k}{4}}^t \omega(t) dt + \left| \int_{\frac{3j+k}{4}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ + \left. \left(\int_{\frac{3j+k}{4}}^t \omega(t) dt - \left| \int_{\frac{3j+k}{4}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ + \int_{\frac{j+k}{2}}^{\frac{k-j}{4}} \left(\left(\int_{\frac{j+3k}{4}}^t \omega(t) dt + \left| \int_{\frac{j+3k}{4}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ + \left. \left(\int_{\frac{j+3k}{4}}^t \omega(t) dt - \left| \int_{\frac{j+3k}{4}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt.$$

Special Case 2.13.(a) If we take, $\gamma(t) = \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.8), then

$$\frac{(k-j)}{16} (\gamma_0 - \Gamma_0) \leq \frac{1}{2} \left[\frac{\rho(j) + \rho(k)}{2} + \rho\left(\frac{j+k}{2}\right) \right] - \frac{1}{k-j} \int_j^k \rho(t) dt \\ \leq \frac{(k-j)}{16} (\Gamma_0 - \gamma_0),$$

which is the Corollary 3 of [27] and Special Case 2.8.1 of [7].

Special Case 2.13.(b) If we take, $\gamma(t) = \gamma_1 t + \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_1 t + \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.8), then

$$m_9 \leq \frac{1}{2} \left[\frac{\rho(j) + \rho(k)}{2} + \rho \left(\frac{j+k}{2} \right) \right] - \frac{1}{k-j} \int_j^k \rho(t) dt \leq M_9,$$

where

$$m_9 = \frac{(k-j)}{16} \left[\frac{(k-j)}{6} (\gamma_1 + \Gamma_1) + \frac{j}{2} (\gamma_1 - \Gamma_1) + \frac{k}{2} (\gamma_1 - \Gamma_1) + \gamma_0 - \Gamma_0 \right]$$

and

$$M_9 = \frac{(k-j)}{16} \left[\frac{(k-j)}{6} (\gamma_1 + \Gamma_1) + \frac{j}{2} (\Gamma_1 - \gamma_1) + \frac{k}{2} (\Gamma_1 - \gamma_1) + \Gamma_0 - \gamma_0 \right],$$

which is Special Case 2.8.2 of [7].

Remark 2.14. If we choose $\theta = j$ in (2.1), for any value of $\lambda \in [0, 1]$ we obtain the following result.

Corollary 2.15. *Let all the assumptions of Theorem 2.1 be valid. Then*

$$\begin{aligned} m_{10}(\lambda) &\leq \rho(k) \int_{\beta}^k \omega(t) dt - \rho(j) \int_{\alpha}^j \omega(t) dt + \rho(p) \int_{\alpha}^{\frac{\alpha+\beta}{2}} \omega(t) dt \\ &\quad + \rho(k) \int_{\frac{\alpha+\beta}{2}}^{\beta} \omega(t) dt - \int_j^k \rho(t) \omega(t) dt \leq M_{10}(\lambda), \end{aligned} \quad (2.9)$$

where

$$\begin{aligned} m_{10}(\lambda) &= \int_j^k \left(\left(\int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt - \left| \int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &\quad \left. + \left(\int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt + \left| \int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \end{aligned}$$

and

$$\begin{aligned} M_{10}(\lambda) &= \int_j^k \left(\left(\int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt + \left| \int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &\quad \left. + \left(\int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt - \left| \int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt. \end{aligned}$$

Remark 2.16. If we take, $\gamma(t) = \gamma_0 \neq 0$ and $\Gamma(t) = \Gamma_0 \neq 0$, and $\gamma(t) = \gamma_1 t + \gamma_0 \neq 0$ and $\Gamma(t) = \Gamma_1 t + \Gamma_0 \neq 0$, and $\omega(t) = \frac{1}{k-j}$ in (2.9), then we obtain results similar to Special Cases 2.4 a and b respectively.

Remark 2.17. If we choose $\theta = k$, and $\lambda = 0$ in (2.1), we obtain the following result.

Corollary 2.18. *Let all suppositions of Theorem 2.1 be valid. Then*

$$m_{11} \leq \frac{\rho(j) + \rho(k)}{2} - \frac{1}{k-j} \int_j^k \rho(t) dt \leq M_{11}, \quad (2.10)$$

where

$$\begin{aligned} m_{11} &= \int_j^k \left(\left(\int_j^t \omega(t) dt - \left| \int_j^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &\quad + \left. \left(\int_j^t \omega(t) dt + \left| \int_j^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ &\quad + \int_k^j \left(\left(\int_{\frac{j+k}{2}}^t \omega(t) dt - \left| \int_{\frac{j+k}{2}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &\quad + \left. \left(\int_{\frac{j+k}{2}}^t \omega(t) dt + \left| \int_{\frac{j+k}{2}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ &\quad + \int_j^k \left(\left(\int_k^t \omega(t) dt - \left| \int_k^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &\quad + \left. \left(\int_k^t \omega(t) dt + \left| \int_k^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt. \end{aligned}$$

and

$$\begin{aligned} M_{11} &= \int_j^k \left(\left(\int_j^t \omega(t) dt + \left| \int_j^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &\quad + \left. \left(\int_j^t \omega(t) dt - \left| \int_j^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ &\quad + \int_k^j \left(\left(\int_{\frac{j+k}{2}}^t \omega(t) dt + \left| \int_{\frac{j+k}{2}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &\quad + \left. \left(\int_{\frac{j+k}{2}}^t \omega(t) dt - \left| \int_{\frac{j+k}{2}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ &\quad + \int_j^k \left(\left(\int_k^t \omega(t) dt + \left| \int_k^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &\quad + \left. \left(\int_k^t \omega(t) dt - \left| \int_k^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt. \end{aligned}$$

Special Case 2.18.(a) If we take, $\gamma(t) = \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.10), then

$$\frac{3(k-j)}{8}(\gamma_0 - \Gamma_0) \leq \left[\frac{\rho(j) + \rho(k)}{2} \right] - \frac{1}{k-j} \int_j^k \rho(t) dt \leq \frac{3(k-j)}{8}(\Gamma_0 - \gamma_0),$$

which is Special Case 2.10.1 of [7].

Special Case 2.18.(b) If we take, $\gamma(t) = \gamma_1 t + \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_1 t + \Gamma_0 \neq 0$

and $\omega(t) = \frac{1}{k-j}$ in (2.10), then

$$m_{12} \leq \left[\frac{\rho(j) + \rho(k)}{2} \right] - \frac{1}{k-j} \int_j^k \rho(t) dt \leq M_{12},$$

where

$$m_{12} = \frac{(k-j)}{2} \left[\frac{7(k-j)}{12} (\gamma_1 + \Gamma_1) + \frac{j}{8} (7\gamma_1 + \Gamma_1) - \frac{k}{8} (\gamma_1 + 7\Gamma_1) + \frac{3}{4} (\gamma_0 - \Gamma_0) \right]$$

and

$$M_{12} = \frac{(k-j)}{2} \left[\frac{7(k-j)}{12} (\gamma_1 + \Gamma_1) + \frac{j}{8} (\gamma_1 + 7\Gamma_1) - \frac{k}{8} (7\gamma_1 + \Gamma_1) + \frac{3}{4} (\Gamma_0 - \gamma_0) \right],$$

which is Special Case 2.10.2 of [7].

Remark 2.19. If we choose $\theta = k$ and $\lambda = \frac{1}{2}$ in (2.1), we obtain the following result.

Corollary 2.20. *Let all assumptions of Theorem 2.1 be valid. Then*

$$m_{13} \leq \rho(k) \int_{\frac{j+3k}{4}}^k \omega(t) dt - \rho(j) \int_{\frac{3j+k}{4}}^j \omega(t) dt + \rho(k) \int_{\frac{3j+k}{4}}^{\frac{j+k}{2}} \omega(t) dt + \rho(j) \int_{\frac{j+k}{2}}^{\frac{j+3k}{4}} \omega(t) dt - \int_j^k \rho(t) \omega(t) dt \leq M_{13}, \quad (2.11)$$

where

$$\begin{aligned} m_{13} &= \int_j^k \left(\left(\int_{\frac{3j+k}{4}}^t \omega(t) dt - \left| \int_{\frac{3j+k}{4}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &\quad + \left. \left(\int_{\frac{3j+k}{4}}^t \omega(t) dt + \left| \int_{\frac{3j+k}{4}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ &\quad + \int_k^j \left(\left(\int_{\frac{j+k}{2}}^t \omega(t) dt - \left| \int_{\frac{j+k}{2}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &\quad + \left. \left(\int_{\frac{j+k}{2}}^t \omega(t) dt + \left| \int_{\frac{j+k}{2}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ &\quad + \int_j^k \left(\left(\int_{\frac{j+3k}{4}}^t \omega(t) dt - \left| \int_{\frac{j+3k}{4}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &\quad + \left. \left(\int_{\frac{j+3k}{4}}^t \omega(t) dt + \left| \int_{\frac{j+3k}{4}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \end{aligned}$$

and

$$\begin{aligned}
 M_{13} = & \int_j^k \left(\left(\int_{\frac{3j+k}{4}}^t \omega(t) dt + \left| \int_{\frac{3j+k}{4}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
 & + \left. \left(\int_{\frac{3j+k}{4}}^t \omega(t) dt - \left| \int_{\frac{3j+k}{4}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\
 & + \int_k^j \left(\left(\int_{\frac{j+k}{2}}^t \omega(t) dt + \left| \int_{\frac{j+k}{2}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
 & + \left. \left(\int_{\frac{j+k}{2}}^t \omega(t) dt - \left| \int_{\frac{j+k}{2}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\
 & + \int_j^k \left(\left(\int_{\frac{j+3k}{4}}^t \omega(t) dt + \left| \int_{\frac{j+3k}{4}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
 & + \left. \left(\int_{\frac{j+3k}{4}}^t \omega(t) dt - \left| \int_{\frac{j+3k}{4}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt.
 \end{aligned}$$

Special Case 2.20.(a) If we take, $\gamma(t) = \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.10), then

$$\frac{9(k-j)}{32}(\gamma_0 - \Gamma_0) \leq \left[\frac{\rho(j) + \rho(k)}{2} \right] - \frac{1}{k-j} \int_j^k \rho(t) dt \leq \frac{9(k-j)}{32}(\Gamma_0 - \gamma_0),$$

which is Special Case 2.11.1 of [7].

Special Case 2.20.(b) If we take, $\gamma(t) = \gamma_1 t + \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_1 t + \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.10), then

$$m_{14} \leq \left[\frac{\rho(j) + \rho(k)}{2} \right] - \frac{1}{k-j} \int_j^k \rho(t) dt \leq M_{14},$$

where

$$\begin{aligned}
 m_{14} = & \frac{k-j}{16} \left[\frac{5(k-j)}{3}(\gamma_1 + \Gamma_1) + \frac{j}{2}(5\gamma_1 - \Gamma_1) - \frac{k}{2}(\gamma_1 - 5\Gamma_1) \right. \\
 & + \left. 3(\gamma_0 - \Gamma_0) \right]
 \end{aligned}$$

and

$$\begin{aligned}
 M_{14} = & \frac{k-j}{16} \left[\frac{5(k-j)}{3}(\gamma_1 + \Gamma_1) + \frac{j}{2}(5\Gamma_1 - \gamma_1) - \frac{k}{2}(\Gamma_1 - 5\gamma_1) \right. \\
 & + \left. 3(\Gamma_0 - \gamma_0) \right],
 \end{aligned}$$

which is Special Case 2.11.2 of [7].

Remark 2.21. If we choose $\theta = k$ and $\lambda = \frac{1}{3}$ in (2.1), then we get the following result.

Corollary 2.22. *Let the assumptions of Theorem 2.1 be valid. Then*

$$m_{15} \leq \rho(k) \int_{\frac{j+5k}{6}}^k \omega(t) dt - \rho(j) \int_{\frac{5j+k}{6}}^j \omega(t) dt + \rho(k) \int_{\frac{5j+k}{6}}^{\frac{j+k}{2}} \omega(t) dt \\ + \rho(j) \int_{\frac{j+k}{2}}^{\frac{j+5k}{6}} \omega(t) dt - \int_j^k \rho(t) \omega(t) dt \leq M_{15}, \quad (2.12)$$

where

$$m_{15} = \int_j^k \left(\left(\int_{\frac{5j+k}{6}}^t \omega(t) dt - \left| \int_{\frac{5j+k}{6}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ + \left. \left(\int_{\frac{5j+k}{6}}^t \omega(t) dt + \left| \int_{\frac{5j+k}{6}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ + \int_k^j \left(\left(\int_{\frac{j+k}{2}}^t \omega(t) dt - \left| \int_{\frac{j+k}{2}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ + \left. \left(\int_{\frac{j+k}{2}}^t \omega(t) dt + \left| \int_{\frac{j+k}{2}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ + \int_j^k \left(\left(\int_{\frac{j+5k}{6}}^t \omega(t) dt - \left| \int_{\frac{j+5k}{6}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ + \left. \left(\int_{\frac{j+5k}{6}}^t \omega(t) dt + \left| \int_{\frac{j+5k}{6}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt$$

and

$$M_{15} = \int_j^k \left(\left(\int_{\frac{5j+k}{6}}^t \omega(t) dt + \left| \int_{\frac{5j+k}{6}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ + \left. \left(\int_{\frac{5j+k}{6}}^t \omega(t) dt - \left| \int_{\frac{5j+k}{6}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ + \int_k^j \left(\left(\int_{\frac{j+k}{2}}^t \omega(t) dt + \left| \int_{\frac{j+k}{2}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ + \left. \left(\int_{\frac{j+k}{2}}^t \omega(t) dt - \left| \int_{\frac{j+k}{2}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ + \int_j^k \left(\left(\int_{\frac{j+5k}{6}}^t \omega(t) dt + \left| \int_{\frac{j+5k}{6}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ + \left. \left(\int_{\frac{j+5k}{6}}^t \omega(t) dt - \left| \int_{\frac{j+5k}{6}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt.$$

Special Case 2.22.(a) If we take, $\gamma(t) = \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.13), then

$$\frac{17(k-j)}{72}(\gamma_0 - \Gamma_0) \leq \left[\frac{\rho(j) + \rho(k)}{2} \right] - \frac{1}{k-j} \int_j^k \rho(t) dt \leq \frac{17(k-j)}{72}(\Gamma_0 - \gamma_0),$$

which is Special Case 2.12.1 of [7].

Special Case 2.22.(b) If we take, $\gamma(t) = \gamma_1 t + \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_1 t + \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.13), then

$$m_{16} \leq \left[\frac{\rho(j) + \rho(k)}{2} \right] - \frac{1}{k-j} \int_j^k \rho(t) dt \leq M_{16},$$

where

$$m_{16} = \frac{(k-j)}{72} \left[11(k-j)(\gamma_1 + \Gamma_1) + \frac{3j}{2}(33\gamma_1 - \Gamma_1) + \frac{3k}{2}(\gamma_1 - 33\Gamma_1) + 17(\gamma_0 - \Gamma_0) \right]$$

and

$$M_{16} = \frac{(k-j)}{72} \left[(k-j)(\gamma_1 + \Gamma_1) + \frac{3j}{2}(33\Gamma_1 - \gamma_1) + \frac{3k}{2}(\Gamma_1 - 33\gamma_1) + 17(\Gamma_0 - \gamma_0) \right],$$

which is Special Case 2.12.2 of [7].

Remark 2.23. If we choose $\theta = k$, and $\lambda = \frac{1}{4}$ in (2.1), we obtain a bound for trapezoidal rule in the following result.

Corollary 2.24. *Let all the assumptions of Theorem 2.1 be valid. Then*

$$\begin{aligned} m_{17} \leq & \rho(k) \int_{\frac{j+k}{8}}^k \omega(t) dt - \rho(j) \int_{\frac{j+k}{8}}^j \omega(t) dt + \rho(k) \int_{\frac{j+k}{8}}^{\frac{j+k}{2}} \omega(t) dt \\ & + \rho(j) \int_{\frac{j+k}{2}}^{\beta} \omega(t) dt - \int_j^k \rho(t) \omega(t) dt \leq M_{17}, \end{aligned} \quad (2.13)$$

where

$$\begin{aligned}
 m_{17} = & \int_j^k \left(\left(\int_{\frac{7j+k}{8}}^t \omega(t) dt - \left| \int_{\frac{7j+k}{8}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
 & + \left. \left(\int_{\frac{7j+k}{8}}^t \omega(t) dt + \left| \int_{\frac{7j+k}{8}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\
 & + \int_k^j \left(\left(\int_{\frac{j+k}{2}}^t \omega(t) dt - \left| \int_{\frac{j+k}{2}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
 & + \left. \left(\int_{\frac{j+k}{2}}^t \omega(t) dt + \left| \int_{\frac{j+k}{2}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\
 & + \int_j^k \left(\left(\int_{\beta}^t \omega(t) dt - \left| \int_{\beta}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
 & + \left. \left(\int_{\beta}^t \omega(t) dt + \left| \int_{\beta}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt.
 \end{aligned}$$

and

$$\begin{aligned}
 M_{17} = & \int_j^k \left(\left(\int_{\frac{7j+k}{8}}^t \omega(t) dt + \left| \int_{\frac{7j+k}{8}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
 & + \left. \left(\int_{\frac{7j+k}{8}}^t \omega(t) dt - \left| \int_{\frac{7j+k}{8}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\
 & + \int_k^j \left(\left(\int_{\frac{j+k}{2}}^t \omega(t) dt + \left| \int_{\frac{j+k}{2}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
 & + \left. \left(\int_{\frac{j+k}{2}}^t \omega(t) dt - \left| \int_{\frac{j+k}{2}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\
 & + \int_j^k \left(\left(\int_{\beta}^t \omega(t) dt + \left| \int_{\beta}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
 & + \left. \left(\int_{\beta}^t \omega(t) dt - \left| \int_{\beta}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt.
 \end{aligned}$$

Special Case 2.24.(a) If we take, $\gamma(t) = \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.13), then

$$\frac{17(k-j)}{64}(\gamma_0 - \Gamma_0) \leq \left[\frac{\rho(j) + \rho(k)}{2} \right] - \frac{1}{k-j} \int_j^k \rho(t) dt \leq \frac{17(k-j)}{64}(\Gamma_0 - \gamma_0),$$

which is Special Case 2.13.1 of [7].

Special Case 2.24.(b) If we take, $\gamma(t) = \gamma_1 t + \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_1 t + \Gamma_0 \neq 0$

and $\omega(t) = \frac{1}{k-j}$ in (2.13), then

$$m_{18} \leq \left[\frac{\rho(j) + \rho(k)}{2} \right] - \frac{1}{k-j} \int_j^k \rho(t) dt \leq M_{18},$$

where

$$m_{18} = \frac{(k-j)}{64} \left[\frac{35}{3}(k-j)(\gamma_1 + \Gamma_1) + \frac{j}{2}(35\gamma_1 + \Gamma_1) - \frac{k}{2}(\gamma_1 + 35\Gamma_1) + 17(\gamma_0 - \Gamma_0) \right]$$

and

$$M_{18} = \frac{(k-j)}{64} \left[\frac{35}{3}(k-j)(\gamma_1 + \Gamma_1) + \frac{j}{2}(35\Gamma_1 + \gamma_1) - \frac{k}{2}(\Gamma_1 + 35\gamma_1) + 17(\Gamma_0 - \gamma_0) \right],$$

which is Special Case 2.13.2 of [7].

Remark 2.25. if we choose $\theta = \frac{2j+k}{3}$ and $\lambda = \frac{1}{4}$ in (2.1), then we get the following result.

Corollary 2.26. *Let all the assumptions of Theorem 2.1 be valid. Then*

$$\begin{aligned} m_{19} &\leq \rho(k) \int_{\frac{j+7k}{8}}^k \omega(t) dt - \rho(j) \int_{\frac{7j+k}{8}}^j \omega(t) dt + \rho\left(\frac{2j+k}{3}\right) \int_{\frac{7j+k}{8}}^{\frac{j+k}{2}} \omega(t) dt \\ &+ \rho\left(\frac{j+2k}{3}\right) \int_{\frac{j+k}{2}}^{\frac{j+7k}{8}} \omega(t) dt - \int_j^k \rho(t) \omega(t) dt \leq M_{19}, \end{aligned} \quad (2.14)$$

where

$$\begin{aligned} m_{19} &= \int_j^{\frac{2j+k}{3}} \left(\left(\int_{\frac{7j+k}{8}}^t \omega(t) dt - \left| \int_{\frac{7j+k}{8}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &+ \left. \left(\int_{\frac{7j+k}{8}}^t \omega(t) dt + \left| \int_{\frac{7j+k}{8}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ &+ \int_{\frac{2j+k}{3}}^{\frac{j+2k}{3}} \left(\left(\int_{\frac{j+k}{2}}^t \omega(t) dt - \left| \int_{\frac{j+k}{2}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &+ \left. \left(\int_{\frac{j+k}{2}}^t \omega(t) dt + \left| \int_{\frac{j+k}{2}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\ &+ \int_{\frac{j+2k}{3}}^k \left(\left(\int_{\frac{j+7k}{8}}^t \omega(t) dt - \left| \int_{\frac{j+7k}{8}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\ &+ \left. \left(\int_{\frac{j+7k}{8}}^t \omega(t) dt + \left| \int_{\frac{j+7k}{8}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt. \end{aligned}$$

and

$$\begin{aligned}
 M_{19} &= \int_j^{\frac{2j+k}{3}} \left(\left(\int_{\frac{7j+k}{8}}^t \omega(t) dt + \left| \int_{\frac{7j+k}{8}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
 &+ \left. \left(\int_{\frac{7j+k}{8}}^t \omega(t) dt - \left| \int_{\frac{7j+k}{8}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\
 &+ \int_{\frac{2j+k}{3}}^{\frac{j+2k}{3}} \left(\left(\int_{\frac{j+k}{2}}^t \omega(t) dt + \left| \int_{\frac{j+k}{2}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
 &+ \left. \left(\int_{\frac{j+k}{2}}^t \omega(t) dt - \left| \int_{\frac{j+k}{2}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\
 &+ \int_{\frac{j+2k}{3}}^k \left(\left(\int_{\frac{j+7k}{8}}^t \omega(t) dt + \left| \int_{\frac{j+7k}{8}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
 &+ \left. \left(\int_{\frac{j+7k}{8}}^t \omega(t) dt - \left| \int_{\frac{j+7k}{8}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt.
 \end{aligned}$$

Special Case 2.26.(a) If we take, $\gamma(t) = \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.15), then

$$\begin{aligned}
 \frac{25(k-j)}{576}(\Gamma_0 - \gamma_0) &\leq \frac{3}{8} \left[\frac{\rho(j) + \rho(k)}{3} + \rho\left(\frac{2j+k}{3}\right) + \rho\left(\frac{j+2k}{3}\right) \right] \\
 &\quad - \frac{1}{k-j} \int_j^k \rho(t) dt \leq \frac{25(k-j)}{576}(\gamma_0 - \Gamma_0),
 \end{aligned}$$

which is Special Case 2.15.1 of [7].

Special Case 2.26.(b) If we take, $\gamma(t) = \gamma_1 t + \gamma_0 \neq 0$, $\Gamma(t) = \Gamma_1 t + \Gamma_0 \neq 0$ and $\omega(t) = \frac{1}{k-j}$, in (2.15) then

$$m_{20} \leq \frac{3}{8} \left[\frac{\rho(j) + \rho(k)}{3} + \rho\left(\frac{2j+k}{3}\right) + \rho\left(\frac{j+2k}{3}\right) \right] - \frac{1}{k-j} \int_j^k \rho(t) dt \leq M_{20},$$

where

$$m_{20} = \frac{(k-j)}{192} \left[(k-j)(\gamma_1 + \Gamma_1) + \frac{31}{6}(j\gamma_1 - k\Gamma_1) + \frac{19}{6}(k\gamma_1 - j\Gamma_1) + \frac{25}{3}(\gamma_0 - \Gamma_0) \right]$$

and

$$M_{20} = \frac{(k-j)}{192} \left[(k-j)(\gamma_1 + \Gamma_1) + \frac{31}{6}(k\Gamma_1 - j\gamma_1) + \frac{19}{6}(j\Gamma_1 - k\gamma_1) + \frac{25}{3}(\Gamma_0 - \gamma_0) \right],$$

which is Special Case 2.15.2 of [7].

Now we state two results with their consequences for function ρ whose first derivative is bounded below only and bounded above only respectively.

Theorem 2.27. Let $\rho : I \rightarrow \mathbb{R}$, be a differentiable function on I^0 of I , and let $[j, k] \subset I^0$. If ρ' is bounded below then $\gamma(\theta) \leq \rho'(\theta)$ for any $\gamma \in C[j, k]$, $\theta \in [j, k]$, then for all $\lambda \in [0, 1]$ we have

$$\begin{aligned} m_{21}(\theta, \lambda) &\leq \rho(k) \int_{\beta}^k \omega(t) dt - \rho(j) \int_{\alpha}^j \omega(t) dt + \rho(\theta) \int_{\alpha}^{\frac{\alpha+\beta}{2}} \omega(t) dt \\ &\quad + \rho(j+k-\theta) \int_{\frac{\alpha+\beta}{2}}^{\beta} \omega(t) dt - \int_j^k \rho(t) \omega(t) dt \leq M_{21}(\theta, \lambda), \quad (2.15) \end{aligned}$$

where

$$\begin{aligned} m_{21}(\theta, \lambda) &= \int_j^k \left(\int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt \right) \gamma(t) dt + \int_j^{\theta} \left(\int_{\alpha}^{\frac{\alpha+\beta}{2}} \omega(t) dt \right) \gamma(t) dt \\ &\quad - \int_{j+k-\theta}^k \left(\int_{\frac{\alpha+\beta}{2}}^{\beta} \omega(t) dt \right) \gamma(t) dt \\ &\quad - \max \left\{ \int_{\alpha}^{\theta} \omega(t) dt, \int_{\frac{\alpha+\beta}{2}}^{j+k-\theta} \omega(t) dt, \int_{\beta}^k \omega(t) dt \right\} \\ &\quad \times \left(\rho(k) - \rho(j) - \int_j^k \gamma(t) dt \right) \end{aligned}$$

and

$$\begin{aligned} M_{21}(\theta, \lambda) &= \int_j^k \left(\int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt \right) \gamma(t) dt + \int_j^{\theta} \left(\int_{\alpha}^{\frac{\alpha+\beta}{2}} \omega(t) dt \right) \gamma(t) dt \\ &\quad - \int_{j+k-\theta}^k \left(\int_{\frac{\alpha+\beta}{2}}^{\beta} \omega(t) dt \right) \gamma(t) dt \\ &\quad + \max \left\{ \int_{\alpha}^{\theta} \omega(t) dt, \int_{\frac{\alpha+\beta}{2}}^{j+k-\theta} \omega(t) dt, \int_{\beta}^k \omega(t) dt \right\} \\ &\quad \times \left(\rho(k) - \rho(j) - \int_j^k \gamma(t) dt \right). \end{aligned}$$

Proof. Since

$$\begin{aligned} &\int_j^k K(\theta, t) (\rho'(t) - \gamma(t)) dt \\ &= \rho(k) \int_{\beta}^k \omega(t) dt - \rho(j) \int_{\alpha}^j \omega(t) dt + \rho(\theta) \int_{\alpha}^{\frac{\alpha+\beta}{2}} \omega(t) dt \\ &\quad + \rho(j+k-\theta) \int_{\frac{\alpha+\beta}{2}}^{\beta} \omega(t) dt - \int_j^k \rho(t) \omega(t) dt - \int_j^k \omega(t) \rho(t) dt \\ &\quad - \int_j^k K(\theta, t) \gamma(t) dt \end{aligned}$$

$$\begin{aligned}
&= \rho(k) \int_{\beta}^k \omega(t) dt - \rho(j) \int_{\alpha}^j \omega(t) dt + \rho(\theta) \int_{\alpha}^{\frac{\alpha+\beta}{2}} \omega(t) dt \\
&+ \rho(j+k-\theta) \int_{\frac{\alpha+\beta}{2}}^{\beta} \omega(t) dt - \int_j^k \rho(t) \omega(t) dt \\
&- \left[\int_j^{\theta} \left(\int_{\alpha}^t \omega(t) dt \right) \gamma(t) dt + \int_{\theta}^{j+k-\theta} \left(\int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt \right) \gamma(t) dt \right. \\
&\left. + \int_{j+k-\theta}^k \left(\int_{\beta}^t \omega(t) dt \right) \gamma(t) dt \right].
\end{aligned}$$

Using modulus property, we have

$$\begin{aligned}
&\left| \rho(k) \int_{\beta}^k \omega(t) dt - \rho(j) \int_{\alpha}^j \omega(t) dt + \rho(\theta) \int_{\alpha}^{\frac{\alpha+\beta}{2}} \omega(t) dt \right. \\
&+ \rho(j+k-\theta) \int_{\frac{\alpha+\beta}{2}}^{\beta} \omega(t) dt - \int_j^k \rho(t) \omega(t) dt \\
&- \left[\int_j^{\theta} \left(\int_{\alpha}^t \omega(t) dt \right) \gamma(t) dt + \int_{\theta}^{j+k-\theta} \left(\int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt \right) \gamma(t) dt \right. \\
&\left. + \int_{j+k-\theta}^k \left(\int_{\beta}^t \omega(t) dt \right) \gamma(t) dt \right] \Bigg| \\
&= \left| \int_j^k K(\theta, t) (\rho'(t) - \gamma(t)) dt \right| \leq \int_j^k |K(\theta, t)| (\rho'(t) - \gamma(t)) dt \\
&\leq \max_{t \in [j, k]} |K(\theta, t)| \int_j^k (\rho'(t) - \gamma(t)) dt \\
&= \max \left\{ \int_{\alpha}^{\theta} \omega(t) dt, \int_{\frac{\alpha+\beta}{2}}^{j+k-\theta} \omega(t) dt, \int_{\beta}^k \omega(t) dt \right\} \\
&\times \left(\rho(k) - \rho(j) - \int_j^k \gamma(t) dt \right). \tag{2.16}
\end{aligned}$$

After rearrangement of (2.16) we get required result. $\square$

Remark 2.28. If we put $\omega(t) = \frac{1}{k-j}$ in inequality (2.16), then we will get the following result.

Corollary 2.29. *Let all the assumptions of Theorem 2.27 are valid. Then*

$$m_{22}(\theta, \lambda) \leq \lambda \frac{\rho(j) + \rho(k)}{2} + (1 - \lambda) \frac{\rho(\theta) + \rho(j+k-\theta)}{2} - \int_j^k \rho(t) d(t) \leq M_{22}(\theta, \lambda) \tag{2.17}$$

where

$$m_{22}(\lambda) = \frac{1}{k-j} \left[\int_j^k \left(t - \frac{j+k}{2} \right) \gamma(t) dt + \frac{k-j}{2} \left(\int_j^\theta (1-\lambda) \gamma(t) dt - \int_{j+k-\theta}^k (1-\lambda) \gamma(t) dt \right) - \max \left\{ \lambda \frac{k-j}{2}, \left(\theta - \frac{(2-\lambda)j + \lambda k}{2} \right), \left(\frac{j+k}{2} - \theta \right) \right\} \times \left(\rho(k) - \rho(j) - \int_j^k \gamma(t) dt \right) \right]$$

and

$$M_{22}(\lambda) = \frac{1}{k-j} \left[\int_j^k \left(t - \frac{j+k}{2} \right) \gamma(t) dt + \frac{k-j}{2} \left(\int_j^{\frac{j+k}{2}} (1-\lambda) \gamma(t) dt - \int_{\frac{j+k}{2}}^k (1-\lambda) \gamma(t) dt \right) + \max \left\{ \lambda \frac{k-j}{2}, \left(\theta - \frac{(2-\lambda)j + \lambda k}{2} \right), \left(\frac{j+k}{2} - \theta \right) \right\} \times \left(\rho(k) - \rho(j) - \int_j^k \gamma(t) dt \right) \right],$$

which is Theorem 2.8 of [7].

Remark 2.30. If we choose $\theta = \frac{j+k}{2}$ in (2.15), we get the following result.

Corollary 2.31. *Let all the assumptions of Theorem 2.1 be valid. Then*

$$m_{23}(\lambda) \leq \rho(k) \int_\beta^k \omega(t) dt - \rho(j) \int_\alpha^j \omega(t) dt + \rho \left(\frac{j+k}{2} \right) \int_\alpha^\beta \omega(t) dt - \int_j^k \rho(t) \omega(t) dt \leq M_{23}(\lambda), \quad (2.18)$$

where

$$m_{23}(\lambda) = \int_j^{\frac{j+k}{2}} \left(\int_\alpha^t \omega(t) dt \right) \gamma(t) dt + \int_{\frac{j+k}{2}}^k \left(\int_\beta^t \omega(t) dt \right) \gamma(t) dt - \max \left\{ \int_\alpha^{\frac{j+k}{2}} \omega(t) dt, \int_{\frac{\alpha+\beta}{2}}^{\frac{j+k}{2}} \omega(t) dt, \int_\beta^k \omega(t) dt \right\} \times \left(\rho(k) - \rho(j) - \int_j^k \gamma(t) dt \right)$$

and

$$M_{23}(\lambda) = \int_j^{\frac{j+k}{2}} \int_\alpha^t \omega(t) dt \gamma(t) dt + \int_{\frac{j+k}{2}}^k \int_\beta^t \omega(t) dt \gamma(t) dt \\ + \max \left\{ \int_\alpha^{\frac{j+k}{2}} \omega(t) dt, \int_{\frac{\alpha+\beta}{2}}^{\frac{j+k}{2}} \omega(t) dt, \int_\beta^k \omega(t) dt \right\} \\ \times \left(\rho(k) - \rho(j) - \int_j^k \gamma(t) dt \right).$$

Special Case 2.31. If we choose $\omega(t) = \frac{1}{k-j}$ in (2.18), we obtain the following result.

$$m_{24}(\lambda) \leq \left[\lambda \frac{\rho(j) + \rho(k)}{2} + (1-\lambda) \rho \left(\frac{j+k}{2} \right) \right] - \frac{1}{k-j} \int_j^k \rho(t) dt \\ \leq M_{24}(\lambda), \quad (2.19)$$

where

$$m_{24}(\lambda) = \frac{1}{k-j} \left[\int_j^k \left(t - \frac{j+k}{2} \right) \gamma(t) dt + \frac{k-j}{2} \left(\int_j^{\frac{j+k}{2}} (1-\lambda) \gamma(t) dt \right. \right. \\ \left. \left. - \int_{\frac{j+k}{2}}^k (1-\lambda) \gamma(t) dt \right) - \max \left\{ \lambda \frac{k-j}{2}, (1-\lambda) \frac{k-j}{2} \right\} \right. \\ \left. \times \left(\rho(k) - \rho(j) - \int_j^k \gamma(t) dt \right) \right]$$

and

$$M_{24}(\lambda) = \frac{1}{k-j} \left[\int_j^k \left(t - \frac{j+k}{2} \right) \gamma(t) dt + \frac{k-j}{2} \left(\int_j^{\frac{j+k}{2}} (1-\lambda) \gamma(t) dt \right. \right. \\ \left. \left. - \int_{\frac{j+k}{2}}^k (1-\lambda) \gamma(t) dt \right) + \max \left\{ \lambda \frac{k-j}{2}, (1-\lambda) \frac{k-j}{2} \right\} \right. \\ \left. \times \left(\rho(k) - \rho(j) - \int_j^k \gamma(t) dt \right) \right],$$

which is Corollary 2.9 of [7].

Theorem 2.32. Let $\rho : I \rightarrow \mathbb{R}$, be a differentiable function on I^0 of I , and let $[j, k] \subset I^0$. If ρ' is bounded above then $\rho'(\theta) \leq \Gamma(\theta)$ for any $\Gamma \in C[j, k]$,

$\theta \in [j, k]$, then for all $\lambda \in [0, 1]$ we have

$$\begin{aligned} m_{25}(\theta, \lambda) &\leq \rho(k) \int_{\beta}^k \omega(t) dt - \rho(j) \int_{\alpha}^j \omega(t) dt + \rho(\theta) \int_{\alpha}^{\frac{\alpha+\beta}{2}} \omega(t) dt \\ &\quad + \rho(j+k-\theta) \int_{\frac{\alpha+\beta}{2}}^{\beta} \omega(t) dt - \int_j^k \rho(t) \omega(t) dt \leq M_{25}(\theta, \lambda), \quad (2.20) \end{aligned}$$

where

$$\begin{aligned} m_{25}(\theta, \lambda) &= \int_j^k \left(\int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt \right) \Gamma(t) dt + \int_j^{\theta} \left(\int_{\alpha}^{\frac{\alpha+\beta}{2}} \omega(t) dt \right) \Gamma(t) dt \\ &\quad - \int_{j+k-\theta}^k \left(\int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt \right) \Gamma(t) dt \\ &\quad - \max \left\{ \int_{\alpha}^{\theta} \omega(t) dt, \int_{\frac{\alpha+\beta}{2}}^{j+k-\theta} \omega(t) dt, \int_{\beta}^k \omega(t) dt \right\} \\ &\quad \times \left(\int_j^k \Gamma(t) dt - \rho(k) + \rho(j) \right) \end{aligned}$$

and

$$\begin{aligned} M_{25}(\theta, \lambda) &= \int_j^k \left(\int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt \right) \Gamma(t) dt + \int_j^{\theta} \left(\int_{\alpha}^{\frac{\alpha+\beta}{2}} \omega(t) dt \right) \Gamma(t) dt \\ &\quad - \int_{j+k-\theta}^k \left(\int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt \right) \Gamma(t) dt \\ &\quad + \max \left\{ \int_{\alpha}^{\theta} \omega(t) dt, \int_{\frac{\alpha+\beta}{2}}^{j+k-\theta} \omega(t) dt, \int_{\beta}^k \omega(t) dt \right\} \\ &\quad \times \left(\int_j^k \Gamma(t) dt - \rho(k) + \rho(j) \right). \end{aligned}$$

Proof. Since

$$\begin{aligned}
 & \int_j^k K(\theta, t) (\rho'(t) - \Gamma(t)) dt \\
 &= \rho(k) \int_\beta^k \omega(t) dt - \rho(j) \int_\alpha^j \omega(t) dt + \rho(\theta) \int_\alpha^{\frac{\alpha+\beta}{2}} \omega(t) dt \\
 &+ \rho(j+k-\theta) \int_{\frac{\alpha+\beta}{2}}^\beta \omega(t) dt - \int_j^k \rho(t) \omega(t) dt - \int_j^k K(\theta, t) \Gamma(t) dt \\
 &= \rho(k) \int_\beta^k \omega(t) dt - \rho(j) \int_\alpha^j \omega(t) dt + \rho(\theta) \int_\alpha^{\frac{\alpha+\beta}{2}} \omega(t) dt \\
 &+ \rho(j+k-\theta) \int_{\frac{\alpha+\beta}{2}}^\beta \omega(t) dt - \int_j^k \rho(t) \omega(t) dt \\
 &- \left[\int_j^\theta \int_\alpha^t \omega(t) dt \Gamma(t) dt + \int_\theta^{j+k-\theta} \int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt \Gamma(t) dt \right. \\
 &\left. + \int_{j+k-\theta}^k \int_\beta^t \omega(t) dt \Gamma(t) dt \right],
 \end{aligned}$$

so we have

$$\begin{aligned}
 & \left| \rho(k) \int_\beta^k \omega(t) dt - \rho(j) \int_\alpha^j \omega(t) dt + \rho(\theta) \int_\alpha^{\frac{\alpha+\beta}{2}} \omega(t) dt \right. \\
 &+ \rho(j+k-\theta) \int_{\frac{\alpha+\beta}{2}}^\beta \omega(t) dt - \int_j^k \rho(t) \omega(t) dt \\
 &- \left[\int_j^\theta \int_\alpha^t \omega(t) dt \Gamma(t) dt + \int_\theta^{j+k-\theta} \int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt \Gamma(t) dt \right. \\
 &\left. + \int_{j+k-\theta}^k \int_\beta^t \omega(t) dt \Gamma(t) dt \right] \Big| \\
 &= \left| \int_j^k K(\theta, t) (\rho'(t) - \Gamma(t)) dt \right| \\
 &\leq \int_j^k |K(\theta, t)| (\Gamma(t) - \rho'(t)) dt \\
 &\leq \max_{t \in [j, k]} |K(\theta, t)| \int_j^k (\Gamma(t) - \rho'(t)) dt \\
 &= \max \left\{ \int_\alpha^\theta \omega(t) dt, \int_{\frac{\alpha+\beta}{2}}^{j+k-\theta} \omega(t) dt, \int_\beta^k \omega(t) dt \right\} \\
 &\times \left(\int_j^k \Gamma(t) dt - \rho(k) + \rho(j) \right), \tag{2.21}
 \end{aligned}$$

we get required result after some rearrangement. $\square$

Remark 2.33. If we put $\omega(t) = \frac{1}{k-j}$ in inequality (2.20) then we get Theorem 3 of [18] and Theorem 3 of [19].

Corollary 2.34. Let all the assumptions of Theorem 2.32 be valid and if we choose $\theta = \frac{j+k}{2}$ in (2.20), we get

$$m_{26}(\lambda) \leq \rho(k) \int_{\beta}^k \omega(t) dt - \rho(j) \int_{\alpha}^j \omega(t) dt + \rho\left(\frac{j+k}{2}\right) \int_{\alpha}^{\beta} \omega(t) dt - \int_j^k \rho(t) \omega(t) dt \leq M_{26}(\lambda), \quad (2.22)$$

where

$$m_{26}(\lambda) = \int_j^{\frac{j+k}{2}} \int_{\alpha}^t \omega(t) dt \Gamma(t) dt + \int_{\frac{j+k}{2}}^k \int_{\beta}^t \omega(t) dt \Gamma(t) dt - \max \left\{ \int_{\alpha}^{\frac{j+k}{2}} \omega(t) dt, \int_{\frac{\alpha+\beta}{2}}^{\frac{j+k}{2}} \omega(t) dt, \int_{\beta}^k \omega(t) dt \right\} \left(\int_j^k \Gamma(t) dt - \rho(k) + \rho(j) \right)$$

and

$$M_{26}(\lambda) = \int_j^{\frac{j+k}{2}} \int_{\alpha}^t \omega(t) dt \Gamma(t) dt + \int_{\frac{j+k}{2}}^k \int_{\beta}^t \omega(t) dt \Gamma(t) dt + \max \left\{ \int_{\alpha}^{\frac{j+k}{2}} \omega(t) dt, \int_{\frac{\alpha+\beta}{2}}^{\frac{j+k}{2}} \omega(t) dt, \int_{\beta}^k \omega(t) dt \right\} \left(\int_j^k \Gamma(t) dt - \rho(k) + \rho(j) \right).$$

Special Case 2.34. If we choose $\omega(t) = \frac{1}{k-j}$ in (2.22), we obtain the following result.

$$m_{27}(\lambda) \leq \left[\lambda \frac{\rho(j) + \rho(k)}{2} + (1-\lambda) \rho\left(\frac{j+k}{2}\right) \right] - \frac{1}{k-j} \int_j^k \rho(t) dt \leq M_{27}(\lambda),$$

where

$$m_{27}(\lambda) = \frac{1}{k-j} \left[\int_j^k \left(t - \frac{j+k}{2} \right) \Gamma(t) dt + \frac{k-j}{2} \left(\int_j^{\frac{j+k}{2}} (1-\lambda) \Gamma(t) dt - \int_{\frac{j+k}{2}}^k (1-\lambda) \Gamma(t) dt \right) - \max \left\{ \lambda \frac{k-j}{2}, (1-\lambda) \frac{k-j}{2} \right\} \left(\int_j^k \Gamma(t) dt - \rho(k) + \rho(j) \right) \right]$$

and

$$\begin{aligned} M_{27}(\lambda) = & \frac{1}{k-j} \left[\int_j^k \left(t - \frac{j+k}{2} \right) \Gamma(t) dt \right. \\ & + \frac{k-j}{2} \left(\int_j^{\frac{j+k}{2}} (1-\lambda) \Gamma(t) dt - \int_{\frac{j+k}{2}}^k (1-\lambda) \Gamma(t) dt \right) \\ & \left. + \max \left\{ \lambda \frac{k-j}{2}, (1-\lambda) \frac{k-j}{2} \right\} \left(\int_j^k \Gamma(t) dt - \rho(k) + \rho(j) \right) \right], \end{aligned}$$

which is Corollary 2.11 of [7].

Remark 2.35. If we put $w(t) = \frac{1}{k-j}$ in inequality (2.20) is Special Case of Theorem 3 of [18] and Theorem 3 of [19].

Corollary 2.36. *Let all the assumptions of Theorem 2.32 be valid. Then*

$$\begin{aligned} m_{28}(\theta, \lambda) \leq & \left[\rho(k) \int_{\beta}^k \omega(t) dt - \rho(j) \int_{\alpha}^j \omega(t) dt + f \left(\frac{j+k}{2} \right) \int_{\alpha}^{\beta} \omega(t) dt \right. \\ & \left. - \int_j^k \rho(t) \omega(t) dt \right] \leq M_{28}(\theta, \lambda), \end{aligned} \quad (2.23)$$

where

$$\begin{aligned} m_{28}(\theta, \lambda) = & \int_j^{\theta} \left(\int_{\alpha}^t \omega(t) dt \right) \gamma(t) dt + \int_{\theta}^{j+k-\theta} \left(\int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt \right) \gamma(t) dt \\ & + \int_{j+k-\theta}^k \left(\int_{\beta}^t \omega(t) dt \right) \gamma(t) dt - \max \left\{ \int_{\alpha}^{\frac{j+k}{2}} \omega(t) dt, \int_{\frac{\alpha+\beta}{2}}^{\frac{j+k}{2}} \omega(t) dt, \int_{\beta}^k \omega(t) dt \right\} \\ & \times \left(\rho(k) - \rho(j) - \int_j^k \gamma(t) dt \right) \end{aligned}$$

and

$$\begin{aligned} M_{28}(\theta, \lambda) = & \int_j^{\theta} \left(\int_{\alpha}^t \omega(t) dt \right) \gamma(t) dt + \int_{\theta}^{j+k-\theta} \left(\int_{\frac{\alpha+\beta}{2}}^t \omega(t) dt \right) \gamma(t) dt \\ & + \int_{j+k-\theta}^k \left(\int_{\beta}^t \omega(t) dt \right) \gamma(t) dt + \max \left\{ \int_{\alpha}^{\frac{j+k}{2}} \omega(t) dt, \int_{\frac{\alpha+\beta}{2}}^{\frac{j+k}{2}} \omega(t) dt, \int_{\beta}^k \omega(t) dt \right\} \\ & \times \left(\rho(k) - \rho(j) - \int_j^k \gamma(t) dt \right). \end{aligned}$$

Remark 2.37. If $\gamma(\theta) \leq \rho'(\theta) \leq \Gamma(\theta)$ for any $\theta \in [j, k]$ and $\gamma, \Gamma \in C[j, k]$, and if one put $\lambda = 0$, then error bounds of non-standard quadrature $A_5(\rho)$ are given

as:

$$m_{29} \leq \frac{1}{2} \left[-\rho(j) + 2\rho\left(\frac{j+k}{2}\right) + \rho(k) \right] - \frac{1}{k-j} \int_j^k \rho(t) dt \leq M_{29}, \quad (2.24)$$

where

$$m_{29} = \frac{1}{k-j} \left[\int_j^{\frac{j+k}{2}} (t-j) \gamma(t) dt + \int_{\frac{j+k}{2}}^k (t-k) \gamma(t) dt + \frac{(k-j)}{2} \int_j^k \gamma(t) dt \right]$$

and

$$M_{29} = \frac{1}{k-j} \left[\int_j^{\frac{j+k}{2}} (t-j) \Gamma(t) dt + \int_{\frac{j+k}{2}}^k (t-k) \Gamma(t) dt + \frac{(k-j)}{2} \int_j^k \Gamma(t) dt \right],$$

which is Corollary 3 of [18], Corollary 4 of [19] and Corollary 2.11 of [7].

Proof. To prove (2.22), we use Corollary 2.31 and 2.34. First by putting $\lambda = 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.18), we get

$$\begin{aligned} & \frac{1}{k-j} \left[\int_j^k (t-j) \gamma(t) dt + \frac{k-j}{2} \left(\int_j^{\frac{j+k}{2}} \gamma(t) dt - \int_{\frac{j+k}{2}}^k \gamma(t) dt \right) \right] \\ & \leq \frac{1}{2} \left[-\rho(j) + 2\rho\left(\frac{j+k}{2}\right) + \rho(k) \right] - \frac{1}{k-j} \int_j^k \rho(t) dt \end{aligned} \quad (2.25)$$

provided that $\gamma(t) \leq \rho'(t) \forall t \in [j, k]$.

On the other hand, by assuming $\lambda = 0$ in (2.21), we obtain

$$\begin{aligned} & \frac{1}{2} \left[-\rho(j) + 2\rho\left(\frac{j+k}{2}\right) + \rho(k) \right] - \frac{1}{k-j} \int_j^k \rho(t) dt \\ & \leq \frac{1}{k-j} \left[\int_j^k (t-j) \Gamma(t) dt + \frac{k-j}{2} \left(\int_j^{\frac{j+k}{2}} \Gamma(t) dt - \int_{\frac{j+k}{2}}^k \Gamma(t) dt \right) \right] \end{aligned} \quad (2.26)$$

provided that $\rho'(t) \leq \Gamma(t) \forall t \in [j, k]$. Combining the above two inequalities obtain the required result. $\square$

Remark 2.38. If $\gamma(\theta) \leq \rho'(\theta) \leq \Gamma(\theta)$ for any $\theta \in [j, k]$ and $\gamma, \Gamma \in C[j, k]$, and if we choose $\lambda = 0$, then error bound of non-standard quadrature $A_6(\rho)$ would be

$$m_{30} \leq \frac{1}{2} \left[\rho(j) + 2\rho\left(\frac{j+k}{2}\right) - \rho(k) \right] - \frac{1}{k-j} \int_j^k \rho(t) dt \leq M_{30} \quad (2.27)$$

where

$$m_{30} = \frac{1}{k-j} \left[\int_j^{\frac{j+k}{2}} (t-j) \Gamma(t) dt + \int_{\frac{j+k}{2}}^k (t-k) \Gamma(t) dt - \frac{(k-j)}{2} \int_j^k \Gamma(t) dt \right]$$

and

$$M_{30} = \frac{1}{k-j} \left[\int_j^{\frac{j+k}{2}} (t-j) \gamma(t) dt + \int_{\frac{j+k}{2}}^k (t-k) \gamma(t) dt - \frac{(k-j)}{2} \int_j^k \gamma(t) dt \right],$$

which is Corollary 4 of [18], Corollary 5 of [19] and Corollary 2.11 of [7].

Proof. Proof of (2.25) is similar to that of Remark 2.37, if one replaces $\lambda = 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.18) and (2.21), respectively, and combines them together. $\square$

Remark 2.39. If $\gamma(\theta) \leq \rho'(\theta) \leq \Gamma(\theta)$ for any $\theta \in [j, k]$ and $\gamma, \Gamma \in C[j, k]$ then by replacing $\theta = k$, $\lambda = 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.18) and (2.21), respectively, then error of non-standard quadrature $A_7(\rho)$ is given as

$$m_{31} \leq \rho(j) - \frac{1}{(k-j)} \int_j^k \rho(t) dt \leq M_{31},$$

where

$$m_{31} = \frac{1}{(k-j)} \int_j^k (t-k)\Gamma(t) dt$$

and

$$M_{31} = \frac{1}{(k-j)} \int_j^k (t-k)\gamma(t) dt,$$

which is Corollary 5 of [18], Corollary 2 of [19] and Corollary 2.11.

Remark 2.40. If $\gamma(\theta) \leq \rho'(\theta) \leq \Gamma(\theta)$ for any $\theta \in [j, k]$ and $\gamma, \Gamma \in C[j, k]$ then by replacing $\theta = j$, $\lambda = 0$ and $\omega(t) = \frac{1}{k-j}$ in (2.18) and (2.21), respectively, one gets error bounds of non-standard quadrature $A_8(\rho)$ as follows:

$$m_{32} \leq \rho(k) - \frac{1}{(k-j)} \int_j^k \rho(t) dt \leq M_{32},$$

where

$$m_{32} = \frac{1}{(k-j)} \int_j^k (t-j)\gamma(t) dt \quad (2.28)$$

and

$$M_{32} = \frac{1}{(k-j)} \int_j^k (t-j)\Gamma(t) dt.$$

The inequality presented above is same as the Corollary 6 of [18], Corollary 3 of [19] and Corollary 2.11 of [7].

3. APPLICATIONS TO NUMERICAL QUADRATURE RULES

Let $I_n : j = z_0 < z_1 < \dots < z_n = k$ be a partition of interval $[j, k]$ and let $h_i = z_{i+1} - z_i, i \in \{0, 1, 2, \dots, n-1\}$. Then

$$\int_j^k \omega(t) \rho(t) dt = Q_n(\rho, \omega, \lambda) + R_n(\rho, \omega, \lambda). \quad (3.1)$$

Consider a general quadrature rule

$$\begin{aligned}
 Q_n(\rho, \omega, \lambda) &:= \sum_{i=0}^{n-1} \left[\rho(z_{i+1}) \int_{\beta_i}^{z_{i+1}} \omega(t) dt - \rho(z_i) \int_{\alpha_i}^{z_i} \omega(t) dt + \rho(\theta_i) \int_{\alpha_i}^{\frac{\alpha_i + \beta_i}{2}} \omega(t) dt \right. \\
 &\quad \left. + \rho(z_i + z_{i+1} - \theta_i) \int_{\frac{\alpha_i + \beta_i}{2}}^{\beta_i} \omega(t) dt \right], \quad (3.2)
 \end{aligned}$$

where $\lambda \in [0, 1]$ and $\theta_i \in [z_i, z_{i+1}]$. Then we get following result:

Theorem 3.1. *Let all the assumptions of Theorem 2.1 be valid. Then (3.1) holds where $Q_n(\rho, \omega, \lambda)$ is given by formula (3.2) and remainder $R_n(I_n, \rho, \omega)$ satisfies estimates*

$$|R_n(\rho, \omega, \lambda)| \leq \sum_{i=0}^{n-1} \sup \{|R_1|, |R_2|\}, \quad (3.3)$$

where

$$\begin{aligned}
 R_1 &= \int_{z_i}^{\theta_i} \left(\left(\int_{\alpha_i}^t \omega(t) dt - \left| \int_{\alpha_i}^t \omega(t) dt \right| \right) \frac{\Gamma_i(t)}{2} \right. \\
 &\quad \left. + \left(\int_{\alpha_i}^t \omega(t) dt + \left| \int_{\alpha_i}^t \omega(t) dt \right| \right) \frac{\gamma_i(t)}{2} \right) dt \\
 &\quad + \int_{\theta_i}^{z_i + z_{i+1} - \theta_i} \left(\left(\int_{\frac{\alpha_i + \beta_i}{2}}^t \omega(t) dt - \left| \int_{\frac{\alpha_i + \beta_i}{2}}^t \omega(t) dt \right| \right) \frac{\Gamma_i(t)}{2} \right. \\
 &\quad \left. + \left(\int_{\frac{\alpha_i + \beta_i}{2}}^t \omega(t) dt + \left| \int_{\frac{\alpha_i + \beta_i}{2}}^t \omega(t) dt \right| \right) \frac{\gamma_i(t)}{2} \right) dt \\
 &\quad + \int_{z_i + z_{i+1} - \theta_i}^{z_{i+1}} \left(\left(\int_{\beta_i}^t \omega(t) dt - \left| \int_{\beta_i}^t \omega(t) dt \right| \right) \frac{\Gamma_i(t)}{2} \right. \\
 &\quad \left. + \left(\int_{\beta_i}^t \omega(t) dt + \left| \int_{\beta_i}^t \omega(t) dt \right| \right) \frac{\gamma_i(t)}{2} \right) dt
 \end{aligned}$$

and

$$\begin{aligned}
 R_2 &= \int_{z_i}^{\theta_i} \left(\left(\int_{\alpha_i}^t \omega(t) dt + \left| \int_{\alpha_i}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
 &+ \left. \left(\int_{\alpha_i}^t \omega(t) dt - \left| \int_{\alpha_i}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\
 &+ \int_{\theta_i}^{z_i+z_{i+1}-\theta_i} \left(\left(\int_{\frac{\alpha_i+\beta_i}{2}}^t \omega(t) dt + \left| \int_{\frac{\alpha_i+\beta_i}{2}}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
 &+ \left. \left(\int_{\frac{\alpha_i+\beta_i}{2}}^t \omega(t) dt - \left| \int_{\frac{\alpha_i+\beta_i}{2}}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt \\
 &+ \int_{z_i+z_{i+1}-\theta_i}^{z_{i+1}} \left(\left(\int_{\beta_i}^t \omega(t) dt + \left| \int_{\beta_i}^t \omega(t) dt \right| \right) \frac{\Gamma(t)}{2} \right. \\
 &+ \left. \left(\int_{\beta_i}^t \omega(t) dt - \left| \int_{\beta_i}^t \omega(t) dt \right| \right) \frac{\gamma(t)}{2} \right) dt,
 \end{aligned}$$

$$\forall \theta_i \in [z_i, z_{i+1}].$$

Proof. Applying inequality (2.1) on the intervals $[z_i, z_{i+1}]$ for $r \in \{1, 2, \dots, n-1\}$, and using (3.3), we get

$$\begin{aligned}
 R_i(\rho, \omega, \lambda) &= \int_{z_i}^{z_{i+1}} \omega(t) \rho(t) dt - \left[\rho(z_{i+1}) \int_{\beta_i}^{z_{i+1}} \omega(t) dt - \rho(z_i) \int_{\alpha_i}^{z_i} \omega(t) dt \right. \\
 &+ \left. \rho(\theta_i) \int_{\alpha_i}^{\frac{\alpha_i+\beta_i}{2}} \omega(t) dt + \rho(z_i + z_{i+1} - \theta_i) \int_{\frac{\alpha_i+\beta_i}{2}}^{\beta_i} \omega(t) dt \right].
 \end{aligned}$$

Summing it over i from 0 to $n-1$ we get

$$\begin{aligned}
 R_n(\rho, \omega, \lambda) &= \int_j^k \rho(t) \omega(t) dt - \sum_{i=0}^{n-1} \left[\rho(z_{i+1}) \int_{\beta_i}^{z_{i+1}} \omega(t) dt - \rho(z_i) \int_{\alpha_i}^{z_i} \omega(t) dt \right. \\
 &\times \left. \omega(t) dt + \rho(\theta_i) \int_{\alpha_i}^{\frac{\alpha_i+\beta_i}{2}} \omega(t) dt + \rho(z_i + z_{i+1} - \theta_i) \int_{\frac{\alpha_i+\beta_i}{2}}^{\beta_i} \omega(t) dt \right].
 \end{aligned}$$

It follows from (2.1) that

$$\begin{aligned}
 |R_n(\rho, \omega, \lambda)| &= \left| \int_j^k \rho(t) \omega(t) dt \right. \\
 &- \sum_{i=0}^{n-1} \left[\rho(z_{i+1}) \int_{\beta_i}^{z_{i+1}} \omega(t) dt - \rho(z_i) \int_{\alpha_i}^{z_i} \omega(t) dt \right. \\
 &+ \left. \left. \rho(\theta_i) \int_{\alpha_i}^{\frac{\alpha_i+\beta_i}{2}} \omega(t) dt + \rho(z_i + z_{i+1} - \theta_i) \int_{\frac{\alpha_i+\beta_i}{2}}^{\beta_i} \omega(t) dt \right] \right|
 \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{i=0}^{n-1} \sup \left\{ \left| \left[\int_{z_i}^{\theta_i} \left(\left(\int_{\alpha_i}^t \omega(t) dt - \left| \int_{\alpha_i}^t \omega(t) dt \right| \right) \frac{\Gamma_i(t)}{2} \right. \right. \right. \right. \\
&\quad + \left. \left(\int_{\alpha_i}^t \omega(t) dt + \left| \int_{\alpha_i}^t \omega(t) dt \right| \right) \frac{\gamma_i(t)}{2} \right) \right] dt \\
&\quad + \int_{\theta_i}^{z_i+z_{i+1}-\theta_i} \left(\left(\int_{\frac{\alpha_i+\beta_i}{2}}^t \omega(t) dt - \left| \int_{\frac{\alpha_i+\beta_i}{2}}^t \omega(t) dt \right| \right) \frac{\Gamma_i(t)}{2} \right. \\
&\quad + \left. \left(\int_{\frac{\alpha_i+\beta_i}{2}}^t \omega(t) dt + \left| \int_{\frac{\alpha_i+\beta_i}{2}}^t \omega(t) dt \right| \right) \frac{\gamma_i(t)}{2} \right) \right] dt \\
&\quad + \int_{z_i+z_{i+1}-\theta_i}^{z_{i+1}} \left(\left(\int_{\beta_i}^t \omega(t) dt - \left| \int_{\beta_i}^t \omega(t) dt \right| \right) \frac{\Gamma_i(t)}{2} \right. \\
&\quad + \left. \left(\int_{\beta_i}^t \omega(t) dt + \left| \int_{\beta_i}^t \omega(t) dt \right| \right) \frac{\gamma_i(t)}{2} \right) \right] dt \Big|, \\
&\quad \left| \int_{z_i}^{\theta_i} \left(\left(\int_{\alpha_i}^t \omega(t) dt + \left| \int_{\alpha_i}^t \omega(t) dt \right| \right) \frac{\Gamma_i(t)}{2} \right. \right. \\
&\quad + \left. \left(\int_{\alpha_i}^t \omega(t) dt - \left| \int_{\alpha_i}^t \omega(t) dt \right| \right) \frac{\gamma_i(t)}{2} \right) \right] dt \\
&\quad + \int_{\theta_i}^{z_i+z_{i+1}-\theta_i} \left(\left(\int_{\frac{\alpha_i+\beta_i}{2}}^t \omega(t) dt + \left| \int_{\frac{\alpha_i+\beta_i}{2}}^t \omega(t) dt \right| \right) \frac{\Gamma_i(t)}{2} \right. \\
&\quad + \left. \left(\int_{\frac{\alpha_i+\beta_i}{2}}^t \omega(t) dt - \left| \int_{\frac{\alpha_i+\beta_i}{2}}^t \omega(t) dt \right| \right) \frac{\gamma_i(t)}{2} \right) \right] dt \\
&\quad + \int_{z_i+z_{i+1}-\theta_i}^{z_{i+1}} \left(\left(\int_{\beta_i}^t \omega(t) dt + \left| \int_{\beta_i}^t \omega(t) dt \right| \right) \frac{\Gamma_i(t)}{2} \right. \\
&\quad + \left. \left(\int_{\beta_i}^t \omega(t) dt - \left| \int_{\beta_i}^t \omega(t) dt \right| \right) \frac{\gamma_i(t)}{2} \right) \right] dt \Big| \Big\}.
\end{aligned}$$

□

Remark 3.2. Similarly, we can state applications of other results and their cases as given in Section 2.

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5. CONCLUSION

In this paper, weighted Ostrowski type inequality is discussed for function differentiable with variable bounds. Applications to solve error bounds

of midpoint, trapezoidal, Simpson's and Simpson's quadrature and some non-standard quadrature rules are discussed. We also have many proven results as our special cases. In particular our results would generalize of [7, 18, 19, 27].

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