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The Cozero-divisor Graph of a Commutative Ring: A Survey

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Abstract. In 2011, M. Afkhami and K. Khashyarmanesh introduced the cozero-divisor graph. Let R be a commutative ring with identity and let $W^*(R)$ be the set of all non-zero non-unit elements of R. The cozero-divisor graph $\Gamma'(R)$ of R is a simple graph with the vertex set $W^*(R)$, and two distinct vertices a and b are adjacent if and only if $a \notin bR$ and $b \notin aR$. In this paper, we offer a survey of results on cozero-divisor graph of commutative rings.

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1. Introduction

Beck (1988) was the pioneer in the field of zero-divisor graphs. For a commutative ring R with identity, the zero-divisor graph of R, denoted $\Gamma(R)$, is

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the graph whose vertices are the non-zero zero-divisors of R with two distinct vertices joined by an edge when the product of the vertices is zero. He was mainly interested in coloring and then this investigation of the coloring of a commutative ring was continued by Anderson and Naseer in [15]. Anderson and Livingston [13], studied the zero-divisor graph whose vertices are the non-zero zero-divisors. The zero-divisor graph helps us to study the algebraic properties of rings using graph theoretical tools. We can translate some algebraic properties of a ring to graph theory language and then the geometric properties of graphs help us explore some interesting results in the algebraic structures of rings. We refer to the reader the papers [10, 11, 13, 14] and [23] for the history of this topic and the basic properties of zero-divisor graphs. Let G be a graph with vertex set $V(G)$. A graph G is said to be *connected* if there is a path between every two distinct vertices. For $a, b \in V(G)$ with $a \neq b$, $d(a,b)$ denotes the length of the shortest path from a to b. If there is no such path, then we will make the convention $d(a,b) = \infty$. For any $a \in V(G)$, the degree of $a, d(a)$, denotes the number of edges incident with a. Also for two distinct vertices x and y in G, the notation $x - y$ means that x and y are adjacent. A graph G is *complete* if each pair of distinct vertices is joined by an edge. For a positive integer n, we use K_n to denote the complete graph with n vertices. Also, we say G is totally disconnected if no two vertices of G are adjacent. We denote $K_{m,n}$ for the complete bipartite graph, with part sizes m and n. A *cycle* is a path that begins and ends at the same vertex in which no edge is repeated and all vertices other than the starting and ending vertices are distinct. We use C_n to denote the cycle with n vertices, where $n \geq 3$. If a graph G has a cycle, then the girth of G(notated $gr(G)$) is defined as the length of the shortest cycle of G; otherwise, $gr(G) = \infty$. Suppose that H is a nonempty subset of $V(G)$. The subgraph of a graph G whose vertex set is H and whose edge set is the set of those edges of G with both ends in H is called the subgraph of G induced by H and denoted by $\langle H \rangle$. The disjoint union of graphs G_1 and G_2, which is denoted by $G_1 \cup G_2$, where G_1 and G_2 are two vertex-disjoint graphs, is a graph with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. A graph G may be expressed uniquely as a disjoint union of connected graphs. These graphs are called the connected components, or simply the components, of G. General references for ring theory [17] and for graph theory [20, 22].

The goal of this survey article is to enclose many of the main results on the cozero-divisor graph of a commutative ring.

2. Cozero-divisor graph of a commutative ring

For any element a in R, let $f_a : R \to R$ be an R-module endomorphism on R defined by multiplication by a. So the authors in [3] considered the set of zero-divisors of R as follows:

$$Z(R) := \{x \in R : \text{the endomorphism } f_x \text{ is not injective}\}.$$

In a dual manner, set

$$\begin{aligned} W(R) &:= \{x \in R : \text{the endomorphism } f_x \text{ is not surjective}\} \\ &= \{x \in R : xR \neq R\}. \end{aligned}$$

Clearly, for two distinct elements x and y in $Z^*(R)$, the vertices x and y are adjacent if and only if x is a non-zero element in $\mathrm{Ker}(f_y)$ or y is a non-zero element in $\mathrm{Ker}(f_x)$. Concerning this, they were led to introduce the following graph, denoted by $\Gamma'(R)$, which is a dual of $\Gamma(R)$ 'in some sense'. For two distinct elements x and y in $W^*(R) := W(R) \setminus \{0\}$, the vertices x and y are adjacent if and only if $x + yR$ is a non-zero element in $\mathrm{Coker}(f_y)$ and $y + xR$ is a non-zero element in $\mathrm{Coker}(f_x)$. This means that x and y are adjacent, in $\Gamma'(R)$, if and only if $x \notin yR$ and $y \notin xR$. Thus the definition of cozero-divisor graph $\Gamma'(R)$ is given as follows.

Definition 2.1. [3] Let R be a commutative ring with identity and let $W^*(R)$ be the set of all non-zero non-unit elements of R. The cozero-divisor graph $\Gamma'(R)$ of R is a simple graph with the vertex set $W^*(R)$, and two distinct vertices a and b are adjacent if and only if $a \notin bR$ and $b \notin aR$.

Note that $W(R)$ consists of all the non-unit elements of R and every non-unit element of R belongs to a maximal ideal. Thus, clearly, $W(R) = \cup_{\mathfrak{m} \in \mathrm{Max}(R)} \mathfrak{m}$.

Let R be a ring. Then R is a field if and only if $\Gamma'(R)$ is an empty graph. If $R = \mathbb{Z}_{p^n}$, where p is prime and $n \in \mathbb{N}$, then $\Gamma'(R) \cong \overline{K}_m$, where $m = p^n - p^{n-1}(p-1) - 1$.

We recall the following basic results from [3].

Theorem 2.2. [3, Theorem 2.1] *Let R be a commutative ring with identity. Then $\Gamma'(R)$ is not complete if and only if there exists an element $a \in W^*(R)$ such that $|aR| > 2$.*

Corollary 2.3. [3, Corollary 2.2] *Let R be a commutative ring with identity. Then $\Gamma'(R)$ is complete if and only if $aR = \{0, a\}$ for all a in $W^*(R)$.*

Theorem 2.4. [3, Theorem 2.3] *Let R be a commutative ring with identity. Then*
(i) $\Gamma'(R) - J(R)$ is connected.
(ii) If R is a non-local ring, then $diam(\Gamma'(R) - J(R)) \leq 2$.

If R is a local ring with $J(R) = 0$, then $\Gamma'(R)$ is connected and $\mathrm{diam}(\Gamma'(R) \leq 2$.

Theorem 2.5. [3, Theorem 2.5] *Let R be a non-local ring such that, for every element $a \in J(R)$, there exists $\mathfrak{m} \in Max(R)$ and $b \in \mathfrak{m} - J(R)$ with $a \notin bR$. Then $\Gamma'(R)$ is connected and $diam(\Gamma'(R)) \leq 3$.*

Suppose R is a local ring with maximal ideal $\mathfrak{m}$. If $\Gamma'(R)$ is connected, then either $\mathfrak{m}$ is a cyclic ideal with two elements or $\mathfrak{m}$ is not cyclic.

In the following theorem, J. Cain, L. Mathewson and A. Wilkens proved that $\Gamma'(R)$ preserves connectedness over a direct product.

Theorem 2.6. [21, Theorem 3.2] *Let $R \cong R_1 \times R_2 \times \cdots \times R_n$, where R_i is a commutative ring for all $i \in \{1, 2, \ldots, n\}$. Let $x, y \in R$ with $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$. If $x_i - y_i$ in $\Gamma'(R_i)$ for some $i \in \{1, 2, \ldots, n\}$, then $x - y$ in $\Gamma'(R)$.*

The converse does not hold. If $x - y$ in $\Gamma'(R_1 \times R_2 \times \cdots \times R_n)$, it is possible that x_i is not adjacent to y_i for all $i \in \{1, 2, \ldots, n\}$. For example, in $\Gamma'(\mathbb{Z}_{16} \times \mathbb{Z}_{16}), (2, 4) - (4, 2)$, but 2 is not adjacent to 4 in $\Gamma'(\mathbb{Z}_{16})$. In fact, $\Gamma'(\mathbb{Z}_{16})$ is empty.

Theorem 2.7. [3, Theorem 2.7] *Let $(R, \mathfrak{m})$ be a Noetherian local ring such that $\mathfrak{m}$ is a principal ideal. Then there is no adjacency between vertices in $\Gamma'(R)$.*

Theorem 2.8. [3, Theorem 2.8] *Let R be a non-local ring. Then $gr(\Gamma'(R) - J(R)) \leq 5$ or $gr(\Gamma'(R) - J(R)) = \infty$.*

Theorem 2.9. [3, Theorem 2.9] *Suppose that $|Max(R)| \geq 3$. Then $gr(\Gamma'(R)) = 3$.*

The following theorem shows that a ring R is finite whenever the cozero-divisor graph $\Gamma'(R)$ is finite.

Theorem 2.10. [3, Theorem 2.10] *Suppose that $Z(R) \neq W(R)$. Then $\Gamma'(R)$ is finite if and only if R is finite.*

Corollary 2.11. [3, Corollary 2.11] *Let R be a commutative domain and $\Gamma'(R)$ be finite. Then R is a field.*

J. Cain, L. Mathewson and A. Wilkens provide proof for a more generalized statement that $\Gamma'(R)$ is finite if and only if R is finite in [21]. The following theorem gives the special case that $\Gamma'(R) - J(R)$ is a complete bipartite graph.

Proposition 2.12. [3, Proposition 2.12] *Let R be a ring with $Max(R) = \{\mathfrak{m}_1, \mathfrak{m}_2\}$. Then $\Gamma'(R) - J(R)$ is a complete bipartite graph with parts $\mathfrak{m}_i - J(R)$ for $i = 1, 2$ if and only if, every cyclic ideals $\mathfrak{a}, \mathfrak{b} \subseteq \mathfrak{m}_i - J(R)$, for some $i = 1, 2$ are totally ordered(i.e.,either $\mathfrak{a} \subseteq \mathfrak{b}$ or $\mathfrak{b} \subseteq \mathfrak{a}$).*

Theorem 2.13. [3, Theorem 2.14] *Let R be a commutative ring with identity.*

- *(i) Let R not be a field. If R has an infinite number of maximal ideals, then $\omega(\Gamma'(R))$ is also infinite; otherwise $\omega(\Gamma'(R)) \geq |Max(R)|$.*
- *(ii) If $\chi(\Gamma'(R)) < \infty$, then $|Max(R)| < \infty$.*

We refer to the reader the papers [2, 5, 6] for some properties of cozero-divisor graphs.

3. Relation between $\Gamma(R)$ and $\Gamma'(R)$

In this section, we present results concerning the relation between the zero divisor graph and cozero divisor graph. Assume that $Z(R) = W(R)$ in the following results.

Theorem 3.1. [3, Theorem 3.1] *Suppose that* $\Gamma'(R)$ *is complete. Then* $\Gamma(R)$ *is also complete.*

The Converse need not be true. For example in the ring $\mathbb{Z}_9$, $\Gamma(\mathbb{Z}_9)$ is complete but $\Gamma'(\mathbb{Z}_9)$ is not complete. The following theorem shows that over a reduced ring, the converse of Theorem 3.1 is true.

Theorem 3.2. [3, Theorem 3.3] *Let* R *be a reduced ring and* $\Gamma(R)$ *be complete. Then* $\Gamma'(R)$ *is also complete.*

Corollary 3.3. [3, Corollary 3.4] *In a reduced ring* R, $\Gamma(R)$ *is complete if and only if* $\Gamma'(R)$ *is complete.*

The following result gives the relation between adjacency in the graphs $\Gamma(R)$ and $\Gamma'(R)$.

Lemma 3.4. [3, Lemma 3.5] *Assume that* $a, b \in W^*(R)$ *are distinct vertices in* $\Gamma'(R)$ *and* a *is not adjacent to* b*. Then* $d_{\Gamma(R)}(a, b) \leq 2$.

Recall that a simple undirected graph is called a planar graph if it can be drawn on the plane in such a way that no edges cross each other.

Theorem 3.5. [3, Theorem 3.9] *Assume that* $|Max(R)| \geq 5$*. Then* $\Gamma'(R)$ *is not planar.*

Lemma 3.6. [3, Lemma 3.10] *Either the complement of* $\Gamma'(R)$ *or its subdivision is a subgraph of* $\Gamma(R)$.

Theorem 3.7. [3, Theorem 3.11] *Assume that the complement of* $\Gamma'(R)$ *is not planar. Then* $\Gamma(R)$ *is not planar.*

For any vertex x of a connected graph G, the eccentricity of x, denoted by $e(x)$, is the maximum of the distances from x to the other vertices of G, and the minimum value of the eccentricity is the radius of G, which is denoted by $r(G)$. The radius of cozero-divisor graphs is given by the following Theorems.

Theorem 3.8. [3, Theorem 3.12] *Suppose that* R *is a non-local ring with* $J(R) = 0$*. Then* $r(\Gamma'(R)) = 0$ *if and only if* $r(\Gamma(R)) = 1$.

Theorem 3.9. [3, Theorem 3.13] *Assume that* R *is a non-local ring with* $J(R) = 0$ *and* $r(\Gamma'(R)) = diam(\Gamma'(R)) = 1$*. Then* $r(\Gamma(R)) = 1$.

Theorem 3.10. [3, Theorem 3.15] *Let* R *be a non-local Noetherian ring with* $J(R) = 0$*. Assume that the complement of* $\Gamma'(R)$ *is a subgraph of the complement of* $\Gamma(R)$*. If* $r(\Gamma'(R)) = 2$*, then* $r(\Gamma(R)) = 2$.

4. Some properties of $\Gamma'(R)$

In this section we mention some properties of $\Gamma'(R)$. In particular, we list the characterization of star and unicyclic graph.

Theorem 4.1. [9, Lemma 7] *Let R be a commutative ring. If $\Gamma'(R)$ is a forest with at least one edge, then the following conditions hold:*
(i) R is not local.
(ii) $|Max(R)| = 2$.
(iii) If $Max(R) = \mathfrak{m}_1, \mathfrak{m}_2$, then $|\mathfrak{m}_2 \setminus \mathfrak{m}_1| = 1$ or $|\mathfrak{m}_1 \setminus \mathfrak{m}_2| = 1$.
(iv) $J(R) = 0$.

Theorem 4.2. [5, Theorem 2.1] *Let R be a non-local finite ring.*
(i) If $\Gamma'(R)$ is a forest(contains no cycles), then $R \cong \mathbb{Z}_2 \times \mathbb{F}$, where $\mathbb{F}$ is a field.
(ii) If $R \cong \mathbb{Z}_2 \times \mathbb{F}$, where $\mathbb{F}$ is a field, then $\Gamma'(R)$ is a star graph.

Theorem 4.3. [5, Theorem 2.2] *Let R be a finite ring.*
(i) If R is non-local, then $\Gamma'(R)$ is a double-star graph if and only if $R \cong \mathbb{Z}_2 \times \mathbb{F}$, where $\mathbb{F}$ is a field.
(ii) If R is local with principal maximal ideal $\mathfrak{m}$, then $\Gamma'(R)$ is a double-star graph if and only if R is either $\mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$ or $\mathbb{F}$, where $\mathbb{F}$ is a field.
(iii) If R is local with non-principal maximal ideal $\mathfrak{m}$ and $\Gamma'(R)$ is a double-star graph, then the minimal generating set of $\mathfrak{m}$ has two elements.

According to [5], if $\Gamma'(R)$ is a unicyclic graph, then $|Max(R)| \leq 3$. In 2014, S. Akbari and S. Khojasteh proved the following improvement in this result for unicyclic graph.

Theorem 4.4. [8, Theorem 1] *Let R be a commutative ring. If $\Gamma'(R)$ is a unicyclic graph, then $|Max(R)| \leq 2$.*

Theorem 4.5. [8, Theorem 2] *Let R be a commutative non-local ring. If $\Gamma'(R)$ is a unicyclic graph, then R is a finite ring.*

The following results give the characterization of unicyclic cozero-divisor graph.

Theorem 4.6. [5, Theorem 2.3] *Let $R \cong R_1 \times \cdots \times R_n$ be a finite commutative ring with identity, where each R_i is local ring and $n \geq 2$. Then $\Gamma'(R)$ is unicyclic if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_4$, $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle} \times \mathbb{Z}_2$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$.*

Theorem 4.7. [8, Theorem 4] *Let R be a commutative local ring. Then $\Gamma'(R)$ is a unicyclic graph if and only if $R \cong \frac{\mathbb{Z}_2[x,y]}{\langle x,y \rangle^2}, \frac{\mathbb{Z}_4[x]}{\langle 2x,x^2 \rangle}$.*

A refinement of a graph H is a graph G such that the vertex sets of G and H are the same and every edge in H is an edge in G.

Proposition 4.8. [5, Proposition 2.1] $\Gamma'(R)$ *is the refinement of a star graph if and only if there exists an element a in $W^*(R)$ such that $|aR| = 2$ and, for all $b \in W^*(R)$ with $a \neq b$, $a \notin bR$.*

In particular, if there exists a maximal ideal $\mathfrak{m}$ of R such that $|\mathfrak{m}| = 2$, then $\Gamma'(R)$ is the refinement of a star graph.

Theorem 4.9. [5, Theorem 2.4] *Let R be a non-local finite ring. Then $\Gamma'(R)$ is a union of cycle graphs if and only if $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.*

An Eulerian graph is a graph that has a path that visits each edge exactly once which starts and ends on the same vertex. Clearly, a nontrivial connected graph G is Eulerian if and only if, every vertex of G has an even degree.

Theorem 4.10. [5, Theorem 2.6] *Suppose that R contains a principal maximal ideal $\mathfrak{m}$ such that $|W(R) - \mathfrak{m}|$ is an odd number. Then $\Gamma'(R) - J(R)$ is not Eulerian.*

Theorem 4.11. [5, Theorem 2.7] *Assume that R is a non-local ring. Then the following conditions are equivalent:*
(i) $\Gamma'(R) - J(R)$ is complete bipartite.
(ii) $\Gamma'(R) - J(R)$ is bipartite.
(iii) $\Gamma'(R) - J(R)$ contains no triangles.

The next theorem, which is due to Afkhami and Khashyarmanesh [5], characterizes Hamiltonian $\Gamma'(R) - J(R)$ when R is a finite ring with two maximal ideals.

Theorem 4.12. [5, Theorem 2.8] *Let R be a finite ring with two maximal ideals $\mathfrak{m}_1$ and $\mathfrak{m}_2$ such that $|\mathfrak{m}_1| = |\mathfrak{m}_2|$. Then $\Gamma'(R) - J(R)$ is Hamiltonian.*

Theorem 4.13. [5, Theorem 2.9] *Assume that R is non-local. Then $\chi(\Gamma'(R) - J(R)) = 2$ if and only if $\omega(\Gamma'(R) - J(R)) = 2$.*

Proposition 4.14. [6, Proposition 4.5] *In $\Gamma'(R_1 \times R_2)$, we have the following inequalities:*
clique $(\Gamma'(R_1 \times R_2)) \geq$ Max $\{clique(\Gamma'(R_1)),\ clique\ (\Gamma'(R_2))\}$;
$\chi(\Gamma'(R_1 \times R_2)) \geq Max\{\chi(\Gamma'(R_1)), \chi(\Gamma'(R_2))\}$.

Theorem 4.15. [6, Theorem 4.7] *Assume that R_1 and R_2 are fields. Then $\Gamma'(R_1 \times R_2)$ is a complete bipartite graph.*

Theorem 4.16. [6, Theorem 4.10] *The cozero-divisor graph $\Gamma'(R_1 \times R_2)$ is connected and $diam(\Gamma'(R_1 \times R_2)) \leq 3$.*

Theorem 4.17. [6, Theorem 4.13] *(i) If at least one of the cozero-divisor graph $\Gamma'(R_1)$ or $\Gamma'(R_2)$ is not totally disconnected, the $gr(\Gamma'(R_1 \times R_2) = 3$.*
(ii) If $R_1 \neq \mathbb{Z}_2$ and $R_2 \neq \mathbb{Z}_2$, then $gr(\Gamma'(R_1 \times R_2) \leq 4$.
(iii) If $R_1 = \mathbb{Z}_2$ and R_2 is a field, then $gr(\Gamma'(R_1 \times R_2) = \infty$.
(iv) $R_1 = \mathbb{Z}_2$ and R_2 is not a field, then $gr(\Gamma'(R_1 \times R_2) = 3, 4$ or ∞.

5. Rings whose cozero-divisor graphs are of bounded degree

In this section, we see that for every positive integer Δ, the set of commutative nonlocal rings with maximum degree Δ is finite. S. Akbari et. al classified all rings whose cozero-divisor graph has a maximum degree at most 3.

Theorem 5.1. [8, Theorem 5] *Let R be a commutative nonlocal ring and Δ be the maximum degree of $\Gamma'(R)$. If Δ is finite, then $|Max(R)| \leq \Delta + 1$, and R is a finite ring.*

Theorem 5.2. [8, Theorem 6] *Let Δ be a positive integer and $\mathscr{A}$ be the set of all commutative nonlocal rings R with maximum degree at most Δ. Then $|R| \leq (\Delta + 1)^{\Delta+1}$, and $\mathscr{A}$ is finite.*

Theorem 5.3. [8, Theorem 7] *Let R be a commutative nonlocal ring and Δ be the maximum degree of $\Gamma'(R)$. Then $\Delta \leq 3$ if and only if R is isomorphic to one of the rings: $\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{F}_4, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2), \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_3 \times \mathbb{F}_4, \mathbb{F}_4 \times \mathbb{F}_4$, and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Lemma 5.4. [8, Lemma 7] *Let $(R, \mathfrak{m})$ be a commutative local ring, $\Delta > 0$ be the maximum degree of $\Gamma'(R)$. If Δ is finite, then R is finite, $\mathfrak{m}$ is principal, or $\mathfrak{m} = \mathfrak{m}^2$.*

Theorem 5.5. [8, Theorem 8] *Let $(R, \mathfrak{m})$ be a commutative local ring and Δ be the maximum degree of $\Gamma'(R)$. If $0 < \Delta \leq 3$, then $\Gamma'(R)$ is a 3-cycle.*

Corollary 5.6. [8, Corollary 7] *Let R be a commutative local ring and Δ be the maximum degree of $\Gamma'(R)$. Then $0 < \Delta \leq 3$ if and only if $R \cong \mathbb{Z}_2[x, y]/(x, y)^2$, $\mathbb{Z}_4[x]/(2x, x^2)$.*

Corollary 5.7. [8, Corollary 9] *Let R be a commutative nonlocal ring and $\Gamma'(R)$ be a disjoint union of cycles. Then $\Gamma'(R)$ is a 4-cycle and $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.*

S. Akbari and S. Khojasteh have improved Lemma 2.2 of [5].

Corollary 5.8. [8, Corollary 10,11] *Let R be a commutative ring. If $\Gamma'(R)$ is a disjoint union of cycles, then the following hold:*
(i)$|Max(R)| \leq 2$.
(ii)$\Gamma'(R) \in \{C_3, C_4\}$;
(iii)$\Gamma'(R) = C_3$ *if and only if* $R \cong \mathbb{Z}_2[x, y]/(x, y)^2, \mathbb{Z}_4[x]/(2x, x^2)$;
(iv)$\Gamma'(R) = C_4$ *if and only if* $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.

In particular, $\Gamma'(R)$ is a disjoint union of cycles if and only if R is isomorphic to one of the following rings: $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_2[x, y]/(x, y)^2, \mathbb{Z}_4[x]/(2x, x^2)$.

In the remaining part of this section, we see the connection between two graphs $\Gamma(R)$ and $\overline{\Gamma'(R)}$. The complement of the cozero-divisor graph $\Gamma'(R)$, is the Cayley graph $Cay(W^*(R), R \setminus \{1\})$.

Proposition 5.9. [5, Proposition 3.1] *Let R be a finite ring such that $\Gamma(R)$ is not a refinement of a complete r-partite graph, where r is a positive integer. Then $\overline{\Gamma'(R)}$ is connected.*

Proposition 5.10. [5, Proposition 3.2] *$\overline{\Gamma'(R)}$ is complete if and only if the set of all principal ideals of R is totally ordered by inclusion.*

Proposition 5.11. [5, Proposition 3.3] *Let R be a Noetherian ring. If $\overline{\Gamma'(R)}$ has an infinite clique,then R has a principal ideal with infinite order which contains all vertices of the clique.*

Note that $\overline{\Gamma'(R_1 \times R_2)}$ is not connected, in general. For example, $\overline{\Gamma'(\mathbb{Z}_2 \times \mathbb{Z}_2)}$ is disconnected. In the following theorem, we study the girth of $\overline{\Gamma'(R_1 \times R_2)}$.

Theorem 5.12. [5, Theorem 3.1] *$gr(\overline{\Gamma'(R_1 \times R_2)}) = 3, 6$ or ∞.*

Lemma 5.13. [5, Lemma 3.1] *Suppose that R_1 and R_2 are non-trivial commutative rings with identities. Then $\overline{\Gamma'(R_1 \times R_2)}$ contains a subgraph isomorphic to K_t, where t is the number of unit elements of R_i , for some $i = 1, 2$. Moreover, if $|W(R_1)| > 1$ (or $|W(R_2)| > 1$), then K_{t+1} is isomorphic to a subgraph of $\overline{\Gamma'(R_1 \times R_2)}$.*

Proposition 5.14. [5, Proposition 3.5] *$\overline{\Gamma'(R_1 \times R_2)}$ is not planar if one of the following conditions holds:*
(i) $|U(R_1)| \geq 5$.
(ii) $|U(R_1) \geq 4$ and $|W(R_1)| > 1$.

6. Reduced cozero-divisor graphs of commutative rings

Amanda Wilkens et al. continued investigating the algebraic implications of the graph by developing the reduced cozero-divisor graph, which is a simpler analog. It is clear from the definition of the cozero-divisor graph that any two elements which generate the same ideal will play similar roles within the structure of the graph, as this next theorem demonstrates.

Theorem 6.1. [27, Theorem 2.1] *Let R be a ring, and let $x, y \in W(R)^*$. If $(x) = (y)$, then x is not adjacent to y, and for all $z \in W(R)^*, x - z$ if and only if $y - z$.*

This tells us that any two points that generate the same ideal will have exactly the same set of neighbors. In this way, $\Gamma'(R)$ is somewhat redundant in its portrayal of relationships between principal ideals. The inclusion of multiple generators of the same ideal serves only to complicate the graph as the rings get larger.

The reduced cozero-divisor graph of R, denoted by $\Gamma_r(R)$, and defined as follows. Let $\Omega(R)$ designate the set of principal ideals of R, and let $\Omega(R)^* = \Omega(R) \setminus \{(0), R\}$ (i.e. $\Omega(R)^*$ is the set of nontrivial principal ideals of R). Then

$V(\Gamma_r(R)) = \Omega(R)^*$, and $(a) - (b)$ if and only if $(a) \nsubseteq (b)$ and $(b) \nsubseteq (a)$. Since the vertex set of $\Gamma_r(R)$ is the principal ideals themselves and not elements of R, the redundancies of $\Gamma'(R)$ are eliminated.

Theorem 6.2. [27, Theorem 2.4] *Let R be a ring. Then $\Gamma_r(R)$ is complete if and only if every principal ideal of R is a maximal principal ideal.*

The following theorem investigates connections in $\Gamma_r(R)$ over the decomposition of rings.

Theorem 6.3. [27, Theorem 2.5] *Let $R \cong R_1 \times \cdots \times R_n$ where R_i is a commutative ring with identity for $1 \leq i \leq n$. Let (x), $(y) \in \Omega(R)^*$. If $(x_i) - (y_i)$ in $\Gamma_r(R_i)$ for some $i \in \{1, 2, \ldots, n\}$, then $(x) - (y)$ in $\Gamma_r(R)$.*

Theorem 6.4. [27, Theorem 3.2] *Let $R_1 \cong F_1 \times F_2 \times \cdots \times F_n$ and $R_2 \cong G_1 \times G_2 \times \cdots \times G_m$, where F_i and G_j are fields for all $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$. Then $n = m$ if and only if $\Gamma_r(R_1) \cong \Gamma_r(R_2)$.*

A ring R is a special principal ideal ring, or SPIR, if it is a local principal ideal ring (PIR) whose maximal ideal is nilpotent.

Theorem 6.5. [27, Theorem 3.4] *Let R_1 and R_2 be rings with $R_1 \cong S_1 \times S_2 \times \cdots \times S_n$ and $R_2 \cong T_1 \times T_2 \times \cdots \times T_m$, where $(S_i, (p_i))$ is an SPIR with nilpotency degree a_i for all $i \in \{1, 2, \ldots, n\}$ and $(T_j, (q_j))$ is an SPIR with nilpotency index b_j for all $j \in \{1, 2, \ldots, m\}$. If $n = m$ and $a_1, a_2, \ldots, a_n$ is a permutation of $b_1, b_2, \ldots, b_n$, then $\Gamma_r(R_1) \cong \Gamma_r(R_2)$.*

Theorem 6.6. [27, Theorem 4.3] *Let R be a non-local Artinian ring. Then $\Gamma_r(R)$ is connected with $diam(\Gamma_r(R)) \leq 3$.*

Theorem 6.7. [27, Theorem 4.5] *Let R be a non-local Artinian ring. Then $gr(\Gamma_r(R))$
≤ 4 or $gr(\Gamma_r(R)) = \infty$.*

Proposition 6.8. [27, Proposition 4.8] *Let $(R, \mathfrak{m})$ be an Artinian SPIR. Then $\Gamma_r(R)$ is complete if and only if $\mathfrak{m}^2 = 0$.*

Theorem 6.9. [27, Theorem 4.9] *Let R be an Artinian PIR. Then the following are equivalent:*
(i) $\Gamma_r(R)$ is complete,
(ii) $R \cong F_1 \times F_2$, and
(iii) $\Gamma_r(R) \cong K^2$.

There are significant advantages to studying reduced cozero-divisor graphs as opposed to the cozero-divisor graphs themselves; first and foremost, $\Gamma_r(R)$ is typically a much simpler graph that provides the same information regarding principal ideal relationships.

The next result comments on the relationship between the connectedness of $\Gamma'(R)$ and $\Gamma_r(R)$.

Theorem 6.10. [21, Theorem 4.2] *(1) If* $|\Omega(R)^*| > 1$*, then* $\Gamma_r R)$ *is connected if and only if* $\Gamma'(R)$ *is connected.*
(2) If $|\Omega(R)^*| = 1$*, then* $\Gamma'(R)$ *is connected if and only if* $|W(R)^*| = 1$*.*

Theorem 6.11. [21, Theorem 5.6] *Let* R_1 *and* R_2 *be rings with* $R_1 \cong \mathbb{Z}_{p_1^{a_1}} \times \mathbb{Z}_{p_2^{a_2}} \times \cdots \mathbb{Z}_{p_n^{a_n}}$ *and* $R_2 \cong \mathbb{Z}_{q_1^{b_1}} \times \mathbb{Z}_{q_2^{b_2}} \times \cdots \mathbb{Z}_{q_m^{a_m}}$*, where* p_i *and* q_i *are prime integers and* $a_i, b_j > 1$ *for* $i \in \{1, 2, \ldots, n\}$ *and* $j \in \{1, 2, \ldots, m\}$*. If* $n = m$ *and* $a_1, a_2, \ldots, a_n$ *is a permutation of* $b_1, b_2, \ldots b_n$*, then* $\Gamma^*(R_1) \cong \Gamma^*(R_2)$*.*

7. Cozero-divisor graph of $R[x]$ and $R[[x]]$

In [6], M. Afkhami and K. Khashyarmanesh investigated some combinatorial properties of the cozero-divisor graphs $\Gamma'(R[x])$ and $\Gamma'(R\,[[x]])$ such as connectivity, diameter, girth, clique numbers and planarity. They also study the cozero-divisor graphs of the direct products of two arbitrary commutative rings. They stated that $\Gamma'(R[x])$ is always connected and its diameter is not exceeding three. Recently, in [9], Akbari et. al made an improvement and determined the diameter of cozero-divisor graph of $R[x]$ is two.

Theorem 7.1. [9, Theorem 3] *Let* R *be a commutative ring. Then*

$$diam(\Gamma'(R[x])) = 2.$$

Theorem 7.2. [6, Theorem 2.7] $gr(\Gamma'(R[x])) = 3$*.*

Theorem 7.3. [6, Theorem 2.8] *In the graph* $\Gamma'(R[x])$*, clique* $(\Gamma'(R[x]))$ *is infinity and hence the chromatic number* $\chi(\Gamma'(R[x]))$ *is infinity.*

Proposition 7.4. [6, Proposition 2.10] *If there exists a maximal ideal* m *of* R *with* $|m| = 2$*, then there is a refinement of a star graph in the structure of* $\Gamma'(R[x])$*.*

Theorem 7.5. [9, Theorem 4] *If* R *is a field, then* $\Gamma'(R[[x]])$ *is totally disconnected.*

Theorem 7.6. [9, Theorem 8] *Let* R *be a commutative Artinian non-local ring. If* $J(R) = 0$*, then* $diam(\Gamma'(R[[x]])) = 3$*.*

Theorem 7.7. [9, Theorem 5] *Let* $(R, \mathfrak{m})$ *be a commutative local ring and* $\mathfrak{m} \neq 0$*. Then* $diam(\Gamma'(R[[x]])) \leq 3$*.*

Theorem 7.8. [9, Theorem 6] *Let* $(R, \mathfrak{m})$ *be a local ring,* $\mathfrak{m} \neq 0$ *and* $\mathfrak{m}^2 = 0$*. Then* $diam(\Gamma'(R[[x]])) = 2$*.*

Lemma 7.9. [6, Lemma 3.5] *Let* $a \in W^*(R)$ *and let* i *and* j *be positive integers such that* $i < j < 2i$*. Then the vertices* $a + x^i$ *and* $a + x^j$ *are adjacent in* $\Gamma'(R\,[[x]])$*.*

8. Genus of cozero-divisor graphs

The authors in [6] investigated the planarity of $\Gamma'(R)$. Also, they characterized all finite non-local commutative rings such that $\Gamma'(R)$ is planar. Assume that R is isomorphic to the ring $R_1 \times R_2 \times \cdots \times R_n$, where R_i is local for every $i = 1, \ldots, n$.

Proposition 8.1. [4, Proposition 2.1, 2.2] *(i) If $n \geq 4$, then $\Gamma'(R)$ is not planar.*
(ii) Assume that $n = 3$. If there exists $1 \leq i \leq 3$, such that R_i has at least three elements, then $\Gamma'(R)$ is not planar.

Theorem 8.2. [6, Theorem 2.9] *The cozero-divisor graph $\Gamma'(R[x])$ is not planar.*

Theorem 8.3. [4, Theorem 2.5] *Let R be a non-local ring. Then $\Gamma'(R)$ is planar if and only if R is one of the following rings: $\cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\cong \mathbb{Z}_2 \times \mathbb{F}$, $\cong \mathbb{Z}_2 \times \mathbb{Z}_4$, $\cong \mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$, $\cong \mathbb{Z}_3 \times \mathbb{F}$, $\cong \mathbb{Z}_3 \times \mathbb{Z}_4$, $\cong \mathbb{Z}_3 \times \frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$, where $\mathbb{F}$ is a finite field.*

Theorem 8.4. [4, Theorem 2.6] *Suppose that $(R, \mathfrak{m})$ is a local ring such that $\mathfrak{m}$ is a principal ideal. Then $\Gamma'(R)$ is planar.*

In [24], S. Kavitha and R. Kala determined all isomorphism classes of finite commutative rings R with identity whose $\Gamma'(R)$ has genus one. Also they characterized all non-local rings for which the reduced cozero-divisor graph $\Gamma_r(R)$ is planar.

Theorem 8.5. [24, Theorem 2.1] *Let $R = F_1 \times \cdots \times F_n$ be a finite commutative ring with identity, where each F_j is a field and $n \geq 2$. Then $\gamma(\Gamma'(R)) = 1$ if and only if R is isomorphic to one of the following rings: $\mathbb{F}_4 \times \mathbb{F}_4, \mathbb{F}_4 \times \mathbb{Z}_5, \mathbb{Z}_5 \times \mathbb{Z}_5, \mathbb{F}_4 \times \mathbb{F}_7$ or $\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Theorem 8.6. [24, Theorem 2.3] *Let $R = R_1 \times \cdots \times R_n \times F_1 \times \cdots \times F_m$ be a commutative ring with identity, where each (R_i, m_i) is a local ring with $m_i \neq \{0\}, F_j$ is a field and $n, m \geq 1$. Then $\gamma(\Gamma'(R)) = 1$ if and only if R is isomorphic to one of the following rings: $\mathbb{Z}_4 \times \mathbb{F}_4$, $\frac{Z_2[x]}{\langle x^2 \rangle} \times \mathbb{F}_4, \mathbb{Z}_4 \times \mathbb{F}_5$, $\frac{Z_2[x]}{\langle x^2 \rangle} \times \mathbb{F}_5, \mathbb{Z}_4 \times \mathbb{F}_7$ or $\frac{Z_2[x]}{\langle x^2 \rangle} \times \mathbb{F}_7, \mathbb{Z}_9 \times \mathbb{Z}_2$, $\frac{Z_3[x]}{\langle x^2 \rangle} \times \mathbb{Z}_2, \mathbb{Z}_8 \times \mathbb{Z}_2$, $\frac{Z_2[x]}{\langle x^3 \rangle} \times \mathbb{Z}_2$, $\frac{Z_4[x]}{\langle 2x, x^2-2 \rangle} \times \mathbb{Z}_2$, $\frac{Z_4[x]}{\langle 2, x \rangle^2} \times \mathbb{Z}_2$ or $\frac{Z_2[x,y]}{\langle x, y \rangle^2} \times \mathbb{Z}_2$.*

Theorem 8.7. [24, Theorem 3.1] *Let $R = F_1 \times \cdots \times F_n$ be a finite commutative ring with identity, where each F_j is a field and $n \geq 2$. Then $\Gamma_r(R)$ is planar if and only if R is isomorphic to one of the following rings: $F_1 \times F_2 \times F_3$ or $F_1 \times F_2$.*

Theorem 8.8. [24, Theorem 3.2] *Let $R = R_1 \times \cdots \times R_n$ be a commutative ring with identity 1, where each $(R_i, \mathfrak{m}_i)$ is a local ring with $\mathfrak{m}_i \neq \{0\}$ and $n \geq 2$.*

Then $\Gamma_r(R)$ is planar if and only if $R \cong R_1 \times R_2$ such that $\mathfrak{m}_i$ is the only non-zero principal ideal in R_i.

Theorem 8.9. [24, Theorem 3.3] *Let $R = R_1 \times \cdots \times R_n \times F_1 \times \cdots \times F_m$ be a finite commutative ring with identity, where each $(R_i, \mathfrak{m}_i)$ is a local ring and each F_j is a field, $m, n \geq 1$. Then $\Gamma_r(R)$ is planar if and only if R satisfies the following conditions:*
(i) $n + m = 2$.
(ii) There exists only two non-zero principal ideals $\langle a_1 \rangle, \langle a_2 \rangle$ in R_1 such that $\langle a_1 \rangle \nsubseteq \langle a_2 \rangle$ and $\langle a_2 \rangle \nsubseteq \langle a_1 \rangle$.
(iii) $\mathfrak{m}_1 = \langle a \rangle$ is a principal ideal with nilpotency at most $k = 4$ and
if $k = 2$, then $\langle a \rangle$ is the only ideal in R_1
if $k = 3$, then $\langle a \rangle, \langle a^2 \rangle$ are the only ideals in R_1
if $k = 3$, then $\langle a \rangle, \langle a^2 \rangle, \langle a^3 \rangle$ are the only ideals in R_1.

Lemma 8.10. [6, Lemma 4.1] *Suppose that $R = R_1 \times \cdots \times R_n$ is a direct product of finite commutative rings. If a_i is adjacent to b_i in $\Gamma'(R_i)$, for some $1 \leq i \leq n$, then every element in R with i-th component a_i is adjacent to all elements in R with i-th component b_i.*

Proposition 8.11. [6, Proposition 4.4] *Assume that either $\Gamma'(R_1)$ or $\Gamma'(R_2)$ is not planar. Then $\Gamma'(R_1 \times R_2)$ is not planar.*

The embedding of graphs in a non-orientable surface is not an easy one. In 2017, A. Mallika and R. Kala aimed at the embedding of $\Gamma'(R)$ in non-orientable compact surfaces [25]. In particular, they classified all finite non-local rings R (upto isomorphism) with $0 < \overline{\gamma}(\Gamma'(R)) \leq 2$.

Theorem 8.12. [25, Corollary 3.4] *Let $R \cong F_1 \times \cdots \times F_n$ be a finite commutative ring with identity, where F_i's are finite fields for $1 \leq i \leq n$ and $|F_1| \leq |F_2| \leq \cdots \leq |F_n|$. Then*
(i) $\overline{\gamma}(\Gamma'(R)) = 1$ if and only if R is isomorphic to one of the following rings: $\mathbb{F}_4 \times \mathbb{F}_4, \mathbb{F}_4 \times \mathbb{Z}_5$.
(ii)$\overline{\gamma}(\Gamma'(R)) = 2$ if and only if R is isomorphic to one of the following rings: $\mathbb{F}_4 \times \mathbb{Z}_7, \mathbb{Z}_5 \times \mathbb{Z}_5, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$.

Theorem 8.13. [25, Corollary 3.7] *Let $R = R_1 \times \cdots \times R_n \times F_1 \times \cdots \times F_m$ be a finite commutative ring with identity, where R_i's are finite local rings which is not a field with $|R_i| \leq |R_{i+1}|$ for $1 \leq i \leq n$ and F_j's are finite fields with $|F_j| \leq |F_{j+1}|$ for $1 \leq j \leq m$. Then*
(i) $\overline{\gamma}(\Gamma'(R)) = 1$ if and only if R is isomorphic to one of the following rings: $\mathbb{Z}_4 \times \mathbb{F}_4$, $\mathbb{Z}_4 \times \mathbb{Z}_5$, $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle} \times \mathbb{F}_4$, $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle} \times \mathbb{Z}_5$.
(ii)$\overline{\gamma}(\Gamma'(R)) = 2$ if and only if R is isomorphic to one of the following rings: $\mathbb{Z}_8 \times \mathbb{Z}_2, \frac{\mathbb{Z}_4[x]}{\langle x^2-2, x^3 \rangle} \times \mathbb{Z}_2, \frac{\mathbb{Z}_2[x]}{\langle x^3 \rangle} \times \mathbb{Z}_2$, $\mathbb{Z}_9 \times \mathbb{Z}_2$, $\frac{\mathbb{Z}_3[x]}{\langle x^2 \rangle} \times \mathbb{Z}_2$, $\mathbb{Z}_4 \times \mathbb{Z}_7$, $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle} \times \mathbb{Z}_7$.

9. The cozero-divisor graph of a module

The concept of the zero-divisor graph of commutative rings has been generalized to modules over commutative rings; see for instance, [19] and [26]. In 2018, A. Alibemani et. all studied some connections between the graph-theoretic properties of $\Gamma'_R(M)$ and algebraic-theoretic properties of M and R [12]. The cozero-divisor graph of a non-zero unital left R-module M, denoted by $\Gamma'_R(M)$, is a graph with vertices $W_R(M)^* = \{m \in M | Rm \neq M\} \setminus \{0\}$ and two distinct vertices m and n are adjacent if $m \notin Rn$ and $n \notin Rm$.

Proposition 9.1. [12, Proposition 2.1] *Let R be a ring and M a left R-module. Then (1) $\Gamma'_R(M)$ is empty if and only if M is a simple R-module.*
(2) $\Gamma'_R(M)$ is empty if and only if R is a division ring.

Proposition 9.2. [12, Proposition 2.3] *Let R be a ring and M a left R-module such that $\Gamma'_R(M)$ contains a cycle. Then $gr(\Gamma'_R(M) \in \{3, 4\}$.*

Corollary 9.3. [12, Corollary 2.4] *Let R be a ring and M a left R-module. Then $gr(\Gamma'_R(M) \in \{3, 4, \infty\}$.*

Example 9.4. [12, Example 2.5] Now, we show that all three cases of Corollary 9.3 may happen.
(1) $gr(\Gamma'_{\mathbb{Z}}(\mathbb{Z}_4)) = \infty$.
(2) $gr(\Gamma'_{\mathbb{Z}_3\times\mathbb{Z}_3}(\mathbb{Z}_3 \times \mathbb{Z}_3)) = 4$.
(3) $gr(\Gamma'_{\mathbb{Z}}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)) = 3$.

Proposition 9.5. [12, Proposition 2.9] *Let R be a ring and M a Noetherian left R-module. Then $\Gamma'_R(M)$ is a complete bipartite graph with non-empty edge set if and only if $M = N_1 \oplus N_2$, where N_1 and N_2 are all simple submodules of M.*

Proposition 9.6. [12, Proposition 2.12] *Let R be a reduced ring and $\omega(\Gamma'_R(R)) < \infty$. Then*
(1) The number of minimal prime ideals of R is at most $\omega(\Gamma'_R(R))$.
(2) The number of annihilator ideals of R is at most $2^{\omega(\Gamma'_R(R))}$.

Lemma 9.7. [12, Lemma 2.18] *Let R be a reduced ring. If $\Gamma'_R(R)$ is planar, then $|Min(R)| \leq 4$.*

Proposition 9.8. [12, Proposition 2.20] *Assume that R is a reduced ring with $Z(R) \neq 0$. If $\Gamma'_R(R)$ is planar, then there exists a ring isomorphism between R and one of the following rings: $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times S$, $\mathbb{Z}_3 \times S$, where S is a division ring.*

10. The cozero-divisor graph relative to finitely generated modules

In [16], H. Ansari-Toroghy et al. introduced a certain subgraph $\acute{\Gamma}_R(M)$ of the cozero-divisor graph $\acute{\Gamma}(R)$, called the cozero-divisor graph relative to M

and obtained some results similar to those of [3] and [5]. This graph, with a different point of view, can be regarded as a reduction of $\acute{\Gamma}(R)$, namely, we have $\acute{\Gamma}_R(R) = \acute{\Gamma}(R)$.

From now on, M denote a finitely generated R-module.

Definition 10.1. [16, Definition 1.] Define the cozero-divisor graph relative to M, denoted by $\acute{\Gamma}_R(M)$ as a graph with vertices $W_R^*(M) = W_R(M) \setminus \{0\}$ and two distinct vertices r and s are adjacent if and only if $r \notin (sM :_R M)$ and $s \notin (rM :_R M)$.

Definition 10.2. [16, Definition 2.] Define the strongly cozero-divisor graph relative to M , denoted by $\widetilde{\Gamma}_R(M)$ as a graph with vertices $W_R^*(M) = W_R(M)\setminus \{0\}$ and two distinct vertices r and s are adjacent if and only if $r \notin \sqrt{sM :_R M}$ and $s \notin \sqrt{rM :_R M}$.

The following example shows that $\acute{\Gamma}(R)$, $\acute{\Gamma}_R(M)$ and $\widetilde{\Gamma}_R(M)$ are different.

Set $R = \mathbb{Z}$ (here $\mathbb{Z}$ denotes the ring of integers) and $M = \mathbb{Z}_{12}$. Then $W_R^*(R) = \mathbb{Z} \setminus \{-1, 1, 0\}$ and $W_R^*(M) = \mathbb{Z} \setminus (\{m : (m, 12) = 1\} \cup \{0\})$ where $(m, 12)$ denotes the greatest common divisor of m and 12. The elements 8 and 12 are adjacent in $\acute{\Gamma}(R)$ but they are not adjacent in $\acute{\Gamma}_R(M)$. Also, 6 and 8 are adjacent in $\acute{\Gamma}_R(M)$ but they are not adjacent in $\widetilde{\Gamma}_R(M)$. Moreover, 6 and 10 are adjacent in $\widetilde{\Gamma}_R(R)$ but they are not adjacent in $\widetilde{\Gamma}_R(M)$.

An R-module L is said to be a multiplication module if for every submodule N of L there exists an ideal I of R such that $N = IL$.

Theorem 10.3. [16, Theorem 1] *$(a)\acute{\Gamma}_R(M)$ is a subgraph of $\acute{\Gamma}(R)$.*
$(b)\widetilde{\Gamma}_R(R)$ is a subgraph of $\acute{\Gamma}(R)$.
(c) If M is a faithful R-module, then $W_R^*(M) = W^*(R)$.
(d) If M is a faithful R-module, then $\widetilde{\Gamma}_R(M) = \widetilde{\Gamma}_R(R)$.
(e) If M is a faithful multiplication R-module, then $\acute{\Gamma}_R(M) = \acute{\Gamma}(R)$.

Theorem 10.4. [16, Theorem 2] *$\acute{\Gamma}_R(M)$ is complete if and only if $\widetilde{\Gamma}_R(M)$ is complete.*

Theorem 10.5. [16, Theorem 4] *Let M be a non-local module such that for every element $r \in J(M)$, there exist $P \in Max(M)$ and $s \in P \setminus J(M)$ with $r \notin (sM :_R M)$. Then $\acute{\Gamma}_R(M)$ is connected and $diam(\acute{\Gamma}_R(M)) \leq 3$.*

Theorem 10.6. [16, Theorem 6] *Let $|Max(M)| \geq 3$. Then $gr(\acute{\Gamma}_R(M)) = 3$.*

Theorem 10.7. [16, Theorem 8] *Assume that $|Max(M)| \geq 5$. Then $\acute{\Gamma}_R(M)$ is not planar.*

Theorem 10.8. [16, Theorem 10] *Let R be a Noetherian ring. If $\acute{\Gamma}_R(M)$ is totally disconnected, then M is a local module with maximal ideal of the form $(xM :_R M)$ for some $x \in W_R^*(M)$.*

Proposition 10.9. [16, Proposition 2] *If the graph* $\acute{\Gamma}_R(M) \setminus J(M)$ *is* n*-partite for some positive integer* n*, then* $|Max(M)| \leq n$*.*

Theorem 10.10. [16, Theorem 12] *Let* M *be an* R*-module with* $Max(M) = \{m_1, m_2\}$*. Then* $\acute{\Gamma}_R(M) \setminus J(M)$ *is a complete bipartite graph with parts* $m_i \setminus J(M), i = 1, 2$*, if and only if every pair of ideals* $(rM :_R M), (sM :_R M)$ *contained in* $(m_1 \setminus J(M))$ *or* $(m_2 \setminus J(M))$*, where* $r, s \in R$*, are totally ordered.*

Theorem 10.11. [16, Theorem 13] *Let* M *be a faithful* R*-module and* $Z_R(M) \neq W_R(M)$*. Then* $\acute{\Gamma}_R(M)$ *is finite if and only if* R *is finite.*

Future Scope

There are many possibilities for future research into this topic. This could include proving the converse of Theorem 6.11, and investigating more general decompositions as well. The connections may be able to be made between $\Gamma(R)$, $\Gamma^*(R)$ and $\acute{\Gamma}(R)$ along the lines of research in [3]. In [27], the authors discussed the connectedness only for Artinian rings. There is an open problem to classify all rings whose $\Gamma_r(R)$ is connected.

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