

Differentiation along Multivector Fields

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ABSTRACT. The Lie derivation of multivector fields along multivector fields has been introduced by Schouten (see [10, 11]), and studied for example in [5] and [12]. In the present paper we define the Lie derivation of differential forms along multivector fields, and we extend this concept to covariant derivation on tangent bundles and vector bundles, and find natural relations between them and other familiar concepts. Also in spinor bundles, we introduce a covariant derivation along multivector fields and call it the Clifford covariant derivation of that spinor bundle, which is related to its structure and has a natural relation to its Dirac operator.

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1. INTRODUCTION

The Schouten-Nijenhuis bracket was discovered by J. A. Schouten (see [10, 11]), and A. Nijenhuis established, in [7], its main properties. A strong renewal of interest in that bracket occurred when A. Lichnerowicz began to consider generalizations of symplectic or contact structures which involve contravariant tensor fields rather than differential forms.

The Properties of the Schouten-Nijenhuis bracket were very actively investigated in the last years [5, 8], as well as its very numerous applications, in particular to Poisson geometry and Poisson cohomology [3, 8, 13, 14, 17], bi-hamiltonian manifolds and integrable systems [4], Poisson-Lie groups [1].

In [10], Schouten introduced the differential invariant of two purely contravariant tensor fields. Then in [7], Nijenhuis showed that for skew symmetric contravariant tensor fields (also called skew multivector fields) this satisfies the Jacobi identity and gives at the same time a structure of a graded Lie algebra to the space of all multivector fields. The same is true for the symmetric multivector fields. In [12], Tulczyjew gave a coordinate free treatment of the bracket for skew multivector fields and clarified its relation to certain differential operators on the space of differential forms, which are similar to those of the better known and more important Frolicher-Nijenhuis bracket for tangent bundle valued differential forms.

In the paper [6], the authors have studied the algebra of derivation of scalar and vector-valued forms along the tangent bundle projection $\tau : TM \rightarrow M$. Spinor bundle and Dirac operator have important applications in various fields including geometry and theoretical physics [15, 16]. For this reason, in this paper, we introduce a covariant derivation along multivector fields in a Spinor bundle and find some relation between it and the corresponding Dirac operator.

2. PRELIMINARIES

First, we recall some notations from multilinear algebra. In this paper V is an arbitrary n dimensional real vector space. The dual of V is denoted by V^* and its l -outer product is denoted by $\Lambda^l V$. The elements of $\Lambda^l V$ and $\Lambda^l V^*$ are called respectively l -vectors and l -forms on V . $\Lambda^0 V = \mathbb{R}$, and set $\Lambda V = \bigoplus_{l=0}^n \Lambda^l V$. For a vector $v \in V$, the outer product of l -vectors by v is an operator which is denoted by μ_v , while the interior product with v of l -forms is an operator denoted by i_v .

$$\begin{array}{ccc} \mu_v : \Lambda^l V & \longrightarrow & \Lambda^{l+1} V \\ X & \longmapsto & v \wedge X \end{array} \quad , \quad \begin{array}{ccc} i_v : \Lambda^l V^* & \longrightarrow & \Lambda^{l-1} V^* \\ X & \longmapsto & i_v(X). \end{array}$$

For blade k -vectors $(1 \leq k)v_1 \wedge \cdots \wedge v_k$, the operators $\mu_{v_1 \wedge \cdots \wedge v_k}$ and $i_{v_1 \wedge \cdots \wedge v_k}$ are defined by

$$\mu_{v_1 \wedge \cdots \wedge v_k} = \mu_{v_1} \circ \cdots \circ \mu_{v_k} \quad , \quad i_{v_1 \wedge \cdots \wedge v_k} = i_{v_k} \circ \cdots \circ i_{v_1}.$$

These definitions are well defined and extend linearly to all k -vectors $X \in \Lambda^k V$

$$\mu_X : \Lambda^l V \longrightarrow \Lambda^{l+k} V \quad , \quad i_X : \Lambda^l V^* \longrightarrow \Lambda^{l-k} V^*$$

Then for scalars $\lambda \in \mathbb{R}$, the operators μ_λ and i_λ are both equal to the operator multiplication by λ .

The operators μ_X and i_X can be extended linearly to all $X \in \Lambda V$ and operate respectively on ΛV and ΛV^* . For the natural product between $\Lambda^l V$ and $\Lambda^l V^*$ ($l = 0, \dots, n$), μ_X and i_X are dual to each other (see [2]). For all $X \in \Lambda^k V, Y \in \Lambda^l V$ and $\omega \in \Lambda^{k+l} V^*$ we have

$$\langle \omega, \mu_X(Y) \rangle = \langle i_X(\omega), Y \rangle .$$

Similarly, for each k -form $\omega \in \Lambda^k V^*$ the operators $\mu_\omega : \Lambda^l V^* \longrightarrow \Lambda^{l+k} V^*$ and $i_\omega : \Lambda^l V \longrightarrow \Lambda^{l-k} V$ with the same properties can be defined.

To each nonzero n -form $\Omega \in \Lambda^n V^*$, one can associate a Hodge operator $H_\Omega : \Lambda^l V \longrightarrow \Lambda^{n-l} V^*$ which is defined by $H_\Omega(X) = i_X \Omega$. If $\Omega^* \in \Lambda^n V$ is the dual of Ω , then we also define $H'_\Omega : \Lambda^l V^* \longrightarrow \Lambda^{n-l} V$ by $H'_\Omega(\omega) = i_\omega \Omega^*$.

Let $\{e_i\}_{i=1}^n$ be a basis of V , $\{\alpha_i\}_{i=1}^n$ its dual basis and $\Omega = \alpha_1 \wedge \dots \wedge \alpha_n$. Then $\Omega^* = e_1 \wedge \dots \wedge e_n$ and for any transformation $\sigma \in S_n$ we have

$$\begin{aligned} H_\Omega(e_{\sigma(1)} \wedge \dots \wedge e_{\sigma(l)}) &= \epsilon_\sigma \alpha_{\sigma(l+1)} \wedge \dots \wedge \alpha_{\sigma(n)}, \\ H'_\Omega(\alpha_{\sigma(1)} \wedge \dots \wedge \alpha_{\sigma(l)}) &= \epsilon_\sigma e_{\sigma(l+1)} \wedge \dots \wedge e_{\sigma(n)}. \end{aligned}$$

The Hodge operators H_Ω and H'_Ω are nearly inverse to each other and the following relations hold.

$$\Lambda^l V \xleftarrow{\quad} \Lambda^{n-l} V^*, \quad H_\Omega \circ H'_\Omega = (-1)^{l(n-l)} \mathbf{1} \quad , \quad H'_\Omega \circ H_\Omega = (-1)^{l(n-l)} \mathbf{1}.$$

For all $X \in \Lambda^k V$, the operators μ_X and i_X make the following diagram commutative.

$$\begin{array}{ccc} \Lambda^l V & \xrightarrow{\mu_X} & \Lambda^{l+k} V \\ H_\Omega \downarrow & & \downarrow H_\Omega \\ \Lambda^{n-l} V^* & \xrightarrow{(-1)^{lk} i_X} & \Lambda^{n-l-k} V^* \end{array} \quad H_\Omega \circ \mu_X = (-1)^{lk} i_X \circ H_\Omega$$

In fact, for every $X \in \Lambda^k V$ and $Y \in \Lambda^l V$, we have

$$H_\Omega(X \wedge Y) = (-1)^{lk} i_X H_\Omega(Y) = i_Y H_\Omega(X).$$

We can also deduce that for every $\omega \in \Lambda^k V^*$ and $Y \in \Lambda^l V$

$$H_\Omega(i_\omega(Y)) = (-1)^{k(l+1)} \omega \wedge H_\Omega(Y).$$

In this paper, M is a n -dimensional manifold and $f, g \in C^\infty(M)$. The set of C^∞ sections of $\Lambda^k(TM)$ is denoted by $\mathfrak{X}^k(M)$ and they are called k -vector fields on M . Also the set of C^∞ sections of $\Lambda^k(TM^*)$ is denoted by $A^k(M)$ and these are k -differential forms on M .

Clearly, $\mathfrak{X}^0(M) = A^0(M) = C^\infty(M)$, and $\mathfrak{X}^1(M) = \mathfrak{X}(M)$. Set $A(M) = \bigoplus_{k=0}^n A^k(M)$ and $\mathfrak{X}^*(M) = \bigoplus_{k=0}^n \mathfrak{X}^k(M)$. For each k -vector field $U \in \mathfrak{X}^k(M)$,

$\mu_U : \mathfrak{X}^l(M) \longrightarrow \mathfrak{X}^{l+k}(M)$ and $i_U : A^l(M) \longrightarrow A^{l-k}(M)$ are defined pointwise. For a volume element $\Omega \in A^n(M) (\forall p \in M \quad \Omega_p \neq 0)$ Hodge operator $H_\Omega : \mathfrak{X}^l(M) \longrightarrow A^{n-l}(M)$ is defined pointwise. Clearly, the same relations also hold for these operators.

3. LIE DERIVATION ALONG MULTIVECTOR FIELDS

For a blade k -vector field $U = U_1 \wedge \cdots \wedge U_k (1 \leq k)$ on M and a l -vector field $V \in \mathfrak{X}^l(M)$, Lie derivation of V along U , is denoted by $L_U V$ and defined as follows

$$L_U V = \sum_{j=1}^k (-1)^{j+1} U_1 \wedge \cdots \wedge \hat{U}_j \wedge \cdots \wedge U_k \wedge L_{U_j} V.$$

This definition can be extended to all k -vector fields and this is Schouten bracket of multivector fields, which generalizes the notion introduced in [10, 11] and studied in [5]. For $U \in \mathfrak{X}^k(M)$, L_U is an operator $\mathfrak{X}^l(M) \xrightarrow{L_U} \mathfrak{X}^{l+k-1}(M)$, and for $k = 1$, L_U is the ordinary Lie derivation. For the function f , $L_U f \in \mathfrak{X}^{k-1}(M)$ and we denote it by $U(f)$. For $U \in \mathfrak{X}^k(M)$, $V \in \mathfrak{X}^l(M)$ and $W \in \mathfrak{X}^*(M)$, the following relations hold

$$\begin{aligned} (1) \quad & U(f) = i_{df}(U), \\ (2) \quad & L_{fU} V = f L_U V + (-1)^k U \wedge V(f), \\ (3) \quad & L_U V = (-1)^{kl} L_V U, \\ (4) \quad & L_U (V \wedge W) = L_U V \wedge W + (-1)^{l(k-1)} V \wedge L_U W, \\ (5) \quad & L_{U \wedge V} W = (-1)^k U \wedge L_V W + (-1)^{l(k-1)} V \wedge L_U W. \end{aligned}$$

These relations lead us to define $L_f g = 0$ and $L_f V = V(f)$, for $V \in \mathfrak{X}^*(M)$. To extend the definition to Lie derivation of differential forms along multivector

fields, it is useful to define the operator $A : \overbrace{\mathfrak{X}M \times \cdots \times \mathfrak{X}M}^{k \text{ times}} \longrightarrow \mathfrak{X}^{k-1}(M) (2 \leq k)$, as follows

$$\begin{aligned} A(U_1, \cdots, U_k) &= \sum_{1 \leq i < j \leq k} (-1)^{i+j} [U_i, U_j] \wedge U_1 \wedge \cdots \wedge \hat{U}_i \wedge \cdots \wedge \hat{U}_j \wedge \cdots \wedge U_k \\ &= \frac{1}{2} \sum_{j=1}^k (-1)^j L_{U_j} (U_1 \wedge \cdots \wedge \hat{U}_j \wedge \cdots \wedge U_k). \end{aligned}$$

The operator A is alternating but it can not be considered as an operator on $\mathfrak{X}^k(M)$, because the value of $A(U_1, \cdots, fU_{j_0}, \cdots, U_k)$ does depend on j_0 . In fact

$$\begin{aligned} A(U_1, \cdots, fU_{j_0}, \cdots, U_k) &= fA(U_1, \cdots, U_k) - (U_1 \wedge \cdots \wedge U_k)(f) \\ &\quad + (-1)^{j_0+1} U_{j_0}(f) U_1 \wedge \cdots \wedge \hat{U}_{j_0} \wedge \cdots \wedge U_k. \end{aligned}$$

For the case $k = 1$, set $A(U) = 0$. The following relation helps us in the computations in which A is involved.

$$A(U_1, \dots, U_k, V_1, \dots, V_l) = A(U_1, \dots, U_k) \wedge V_1 \wedge \dots \wedge V_l \\ + (-1)^k U_1 \wedge \dots \wedge U_k \wedge A(V_1, \dots, V_l) - L_{U_1 \wedge \dots \wedge U_k} (V_1 \wedge \dots \wedge V_l).$$

The operator A can be used to define exterior derivation as follows (see [9])

$$\langle d\omega, U_1 \wedge \dots \wedge U_k \rangle = \sum_{j=1}^k (-1)^{j+1} U_j \langle \omega, U_1 \wedge \dots \wedge \hat{U}_j \wedge \dots \wedge U_k \rangle \\ + \langle \omega, A(U_1, \dots, U_k) \rangle,$$

where and $U_1, \dots, U_k \in \mathfrak{X}^*(M)$.

Now, we can define the Lie derivation of differential forms along multivector fields,

$$L_U(\omega) = \sum_{j=1}^k (-1)^{j+1} i_{U_1 \wedge \dots \wedge \hat{U}_j \wedge \dots \wedge U_k} (L_{U_j} \omega) - i_{A(U_1, \dots, U_k)}(\omega),$$

where $\omega \in A(M)$ and a blade k -vector field $U = U_1 \wedge \dots \wedge U_k (1 \leq k)$. This Lie derivation is well defined and we can extend it along all the k -vector fields U by $L_U : A^l(M) \rightarrow A^{l-k+1}(M)$. It has the following properties

$$(6) \quad L_{fU}(\omega) = fL_U(\omega) + (-1)^{k+1} df \wedge i_U(\omega),$$

$$(7) \quad L_U(f\omega) = fL_U(\omega) + i_{U(f)}(\omega),$$

$$(8) \quad L_U(\omega) = i_U(d\omega) + (-1)^{k+1} d(i_U\omega),$$

$$(9) \quad d(L_U\omega) = (-1)^{k+1} L_U(d\omega),$$

$$(10) \quad L_{U \wedge V}(\omega) = i_V(L_U\omega) + (-1)^k L_V(i_U\omega),$$

$$(11) \quad L_U(i_V\omega) = (-1)^{l(k-1)} (i_V(L_U\omega) + i_{L_U V}(\omega)),$$

where $U \in \mathfrak{X}^k(M)$, $V \in \mathfrak{X}^l(M)$ and $\omega \in A(M)$. The above relations can be straightforwardly verified, although some of the computations are quite long.

Relation (6) suggests us to define $L_g(\omega) = -dg \wedge \omega$, for $\omega \in A(M)$. Clearly all above relations also hold for the case $U = g$. Note that for $U \in \mathfrak{X}^k(M)$ and $\omega \in A^{k-1}(M)$, we have $L_U(\omega) \in C^\infty(M)$ and in fact $L_U(\omega) = \langle d\omega, U \rangle$.

4. COVARIANT DERIVATIONS ALONG MULTIVECTOR FIELDS

Let ∇ be a connection on TM . For each vector field $U \in \mathfrak{X}M$, the covariant differentiation of l -vector fields along U is the operator $\nabla_U : \mathfrak{X}^l(M) \rightarrow \mathfrak{X}^l(M)$. Note that for the case $l = 0$, $\nabla_U f$ is defined as $U(f)$. Now, we extend this concept for blade k -vector fields $U = U_1 \wedge \dots \wedge U_k (1 \leq k)$ as follows

$$\nabla_U V = \sum_{j=1}^k (-1)^{j+1} U_1 \wedge \dots \wedge \hat{U}_j \wedge \dots \wedge U_k \wedge \nabla_{U_j} V, \quad \forall V \in \mathfrak{X}^*(M).$$

This operator is well defined and the definition can be extended to covariant differentiation along blade k -vector fields by $\nabla_U : \mathfrak{X}^l(M) \rightarrow \mathfrak{X}^{l+k-1}(M)$. In the case when $k = 0$ the properties of this operator lead us to define $\nabla_f V = 0$ and they can be extended to all $U \in \mathfrak{X}^*(M)$.

Theorem 4.1. For $U \in \mathfrak{X}^k(M)$, $V \in \mathfrak{X}^l(M)$, $W \in \mathfrak{X}^*(M)$, we have

$$(12) \quad \nabla_{fU}W = f\nabla_UW,$$

$$(13) \quad \nabla_U(fW) = f\nabla_UW + U(f) \wedge W,$$

$$(14) \quad \nabla_{U \wedge V}W = (-1)^k U \wedge \nabla_V W + (-1)^{l(k-1)} V \wedge \nabla_U W,$$

$$(15) \quad \nabla_U(V \wedge W) = \nabla_U V \wedge W + (-1)^{l(k-1)} V \wedge \nabla_U W.$$

If ∇ be torsion free, then

$$(16) \quad L_U V = \nabla_U V + (-1)^{lk} \nabla_V U.$$

Proof. All these relations can be checked by direct computation, so we prove only (16). We can assume $U = U_1 \wedge \cdots \wedge U_k$, $V = V_1 \wedge \cdots \wedge V_l$. Then we have

$$\begin{aligned} L_U V &= L_{U_1 \wedge \cdots \wedge U_k} V_1 \wedge \cdots \wedge V_l = \sum_{j=1}^k (-1)^{j+1} U_1 \wedge \cdots \wedge \hat{U}_j \wedge \cdots \wedge U_k \wedge L_{U_j} (V_1 \wedge \cdots \wedge V_l) \\ &= \sum_{j=1}^k \sum_{i=1}^l (-1)^{j+1} U_1 \wedge \cdots \wedge \hat{U}_j \wedge \cdots \wedge U_k \wedge V_1 \wedge \cdots \wedge L_{U_j} V_i \wedge \cdots \wedge V_l \\ &= \sum_{j=1}^k \sum_{i=1}^l (-1)^{j+1} U_1 \wedge \cdots \wedge \hat{U}_j \wedge \cdots \wedge U_k \wedge V_1 \wedge \cdots \wedge (\nabla_{U_j} V_i - \nabla_{V_i} U_j) \wedge \cdots \wedge V_l \\ &= \sum_{j=1}^k (-1)^{j+1} U_1 \wedge \cdots \wedge \hat{U}_j \wedge \cdots \wedge U_k \wedge (\sum_{i=1}^l V_1 \wedge \cdots \wedge \nabla_{U_j} V_i \wedge \cdots \wedge V_l) \\ &\quad - \sum_{i=1}^l (-1)^{j+1+j-1+l-i+l(k-1)} V_1 \wedge \cdots \wedge \hat{V}_i \wedge \cdots \wedge V_l \wedge (\sum_{j=1}^k U_1 \wedge \cdots \wedge \nabla_{V_i} U_j \wedge \cdots \wedge U_k) \\ &= \sum_{j=1}^k (-1)^{j+1} U_1 \wedge \cdots \wedge \hat{U}_j \wedge \cdots \wedge U_k \wedge \nabla_{U_j} (V_1 \wedge \cdots \wedge V_l) \\ &\quad + (-1)^{kl} \sum_{i=1}^l (-1)^{i+1} V_1 \wedge \cdots \wedge \hat{V}_i \wedge \cdots \wedge V_l \wedge \nabla_{V_i} (U_1 \wedge \cdots \wedge U_k) \\ &= \nabla_U V + (-1)^{kl} \nabla_V U. \quad \square \end{aligned}$$

For a connection ∇ , the covariant derivation of differential forms along vector fields $U \in \mathfrak{X}(M)$ is an operator $\nabla_U : A^l(M) \longrightarrow A^l(M)$. We can extend this concept for blade k -vector fields $U = U_1 \wedge \cdots \wedge U_k (1 \leq k)$ as follows

$$\nabla_U \omega = \sum (-1)^{j+1} i_{U_1 \wedge \cdots \wedge \hat{U}_j \wedge \cdots \wedge U_k} (\nabla_{U_j} \omega), \quad \forall \omega \in A(M).$$

This operator is well defined and it can be extended to all $U \in \mathfrak{X}^k(M)$, by $\nabla_U : A^l(M) \longrightarrow A^{l-k+1}(M)$. In the case $k = 0$, it is convenient to define $\nabla_f \omega = 0$, so this operator linearly extends to all $U \in \mathfrak{X}^*(M)$.

Theorem 4.2. For $U \in \mathfrak{X}^k(M)$ and $\omega \in A(M)$, the following relations hold

$$(17) \quad \nabla_{fU} \omega = f \nabla_U \omega,$$

$$(18) \quad \nabla_U (f\omega) = f \nabla_U \omega + i_{U(f)}(\omega),$$

$$(19) \quad \nabla_{U \wedge V} \omega = i_V (\nabla_U \omega) + (-1)^{kl} i_U (\nabla_V \omega),$$

$$(20) \quad \nabla_U (i_V \omega) = (-1)^{l(k-1)} (i_V (\nabla_U \omega) + i_{\nabla_U V}(\omega)).$$

If ∇ be torsion free, and $\omega \in A^{k-1}(M)$, then

$$(21) \quad \nabla_U \omega = L_U \omega = \langle d\omega, U \rangle .$$

Proof. Direct computations prove all the above relations. We prove only (19). We can assume $U = U_1 \wedge \cdots \wedge U_k$ and $V = V_1 \wedge \cdots \wedge V_l$. Then we have

$$\begin{aligned} \nabla_{U \wedge V} \omega &= \nabla_{U_1 \wedge \cdots \wedge U_k \wedge V_1 \wedge \cdots \wedge V_l} \omega = \sum_{j=1}^k (-1)^{j+1} i_{(U_1 \wedge \cdots \wedge \hat{U}_j \wedge \cdots \wedge U_k \wedge V_1 \wedge \cdots \wedge V_l)} (\nabla_{U_j} \omega) \\ &\quad + \sum_{i=1}^l (-1)^{k+i+1} i_{(U_1 \wedge \cdots \wedge U_k \wedge V_1 \wedge \cdots \wedge \hat{V}_i \wedge \cdots \wedge V_l)} (\nabla_{V_i} \omega) \\ &= i_{V_1 \wedge \cdots \wedge V_l} \left(\sum_{j=1}^k (-1)^{j+1} i_{U_1 \wedge \cdots \wedge \hat{U}_j \wedge \cdots \wedge U_k} (\nabla_{U_j} \omega) \right) \\ &\quad + (-1)^{k+k(l-1)} i_{U_1 \wedge \cdots \wedge U_k} \left(\sum_{i=1}^l (-1)^{i+1} i_{V_1 \wedge \cdots \wedge \hat{V}_i \wedge \cdots \wedge V_l} (\nabla_{V_i} \omega) \right) \\ &= i_V (\nabla_U \omega) + (-1)^{kl} i_U (\nabla_V \omega). \quad \square \end{aligned}$$

Now, it is natural that for every $\omega \in A^k(M)$ and $U \in \mathfrak{X}^l(M)$ to define $\nabla_\omega U$ as a $(k-l+1)$ -differential form by the following relation

$$\langle \nabla_\omega U, V \rangle = \langle \omega, \nabla_U V \rangle, \quad \forall V \in \mathfrak{X}^{k-l+1}(M).$$

So, for $\omega \in A^k(M)$, $\nabla_\omega : \mathfrak{X}^l(M) \longrightarrow A^{k-l+1}(M)$.

Theorem 4.3. For $U \in \mathfrak{X}^k(M)$, $V \in \mathfrak{X}^l(M)$ and $\omega \in A^m(M)$, the following relations hold

$$(22) \quad \nabla_{f\omega} U = f \nabla_\omega U,$$

$$(23) \quad \nabla_\omega (fU) = f \nabla_\omega U + (-1)^{k(m+1)} df \wedge i_U \omega,$$

$$(24) \quad \nabla_\omega (U \wedge V) = (-1)^{lm} i_V (\nabla_\omega U) + (-1)^{k(l+m)} i_U (\nabla_\omega V) + (-1)^{m(k+l)+1} i_{L_U V} (\omega).$$

If ∇ be torsion free, then

$$(25) \quad L_U \omega = \nabla_U \omega - (-1)^{km} \nabla_\omega U.$$

Proof. Direct computations prove all the above relations. For example we prove (25). For $W \in \mathfrak{X}^{m-k+1}(M)$, we have

$$\begin{aligned} \langle L_U \omega, W \rangle &= i_W (L_U \omega) = L_{U \wedge W} \omega - (-1)^k L_W (i_U \omega) \\ &= \nabla_{U \wedge W} \omega - (-1)^k \nabla_W (i_U \omega) \\ &= i_W (\nabla_U \omega) + (-1)^{k(m-k+1)} i_U (\nabla_W \omega) - (-1)^{k+k(m-k)} (i_U (\nabla_W \omega) + \\ &\quad i_{\nabla_W U} (\omega)) \\ &= \langle \nabla_U \omega, W \rangle - (-1)^{km} \langle \omega, \nabla_W U \rangle \\ &= \langle \nabla_U \omega - (-1)^{km} \nabla_\omega U, W \rangle. \quad \square \end{aligned}$$

5. DIVERGENCE

Consider a fixed volume element Ω on the manifold M . Associated to Ω , there exists a dual operator to the exterior derivation $d : A^k(M) \longrightarrow A^{k+1}(M)$ which is called divergence operator. This is a homogenous differential operator on multivector fields of degree -1 , $\delta : \mathfrak{X}^k(M) \longrightarrow \mathfrak{X}^{k-1}(M)$, and defined as follows (see [9])

$$\begin{array}{ccc} A^{n-k}(M) & \xrightarrow{d} & A^{n-k+1}(M) \\ H_\Omega \uparrow & & \downarrow H'_\Omega \\ \mathfrak{X}^k(M) & \xrightarrow{(-1)^{n(k+1)+1}\delta} & \mathfrak{X}^{k-1}(M) \end{array} \quad \delta = (-1)^{n(k+1)+1} H'_\Omega \circ d \circ H_\Omega$$

δ depends on Ω , but for any nonzero scalar λ , changing Ω to $\lambda\Omega$ does not change δ .

Theorem 5.1. *For $U \in \mathfrak{X}^k(M)$, $V \in \mathfrak{X}^*(M)$, and $\omega \in A^m(M)$, the following relations hold for the divergence operator*

$$(26) \quad \delta^2 = 0,$$

$$(27) \quad H_\Omega(\delta U) = -L_U \Omega,$$

$$(28) \quad \delta U = (-1)^{(n+1)(k+1)+1} H'_\Omega(L_U \Omega),$$

$$(29) \quad \delta(fU) = f\delta U - U(f),$$

$$(30) \quad \delta(U \wedge V) = \delta U \wedge V + (-1)^k U \wedge \delta V - L_U V,$$

$$(31) \quad \delta(L_U V) = -L_{\delta U} V + (-1)^{k+1} L_U \delta V,$$

$$(32) \quad \delta(i_\omega U) = (-1)^m (i_\omega \delta U - i_{d\omega} U).$$

Proof. We prove only (30). If $V \in \mathfrak{X}^l(M)$, then we have

$$\begin{aligned} \delta(U \wedge V) &= (-1)^{(n+1)(k+l+1)+1} H'_\Omega(L_{U \wedge V} \Omega) \\ &= (-1)^{(n+1)(k+l+1)+1} H'_\Omega(i_V(L_U \Omega) + (-1)^k L_V(i_U \Omega)) \\ &= (-1)^{(n+1)(k+l+1)+1} H'_\Omega(i_V(L_U \Omega) + (-1)^{k+k(l+1)}(i_U(L_V \Omega) + i_{L_V U} \Omega)) \\ &= (-1)^{(n+1)(k+l+1)+1} ((-1)^{l(n-k)} V \wedge H'_\Omega(L_U \Omega) + (-1)^{kl+k(n-l)} U \wedge H'_\Omega(L_V \Omega) \\ &\quad + (-1)^{kl} H'_\Omega H_\Omega(L_V U)) \\ &= (-1)^{l(k+1)} V \wedge \delta U + (-1)^k U \wedge \delta V + (-1)^{kl+1} L_V U \\ &= \delta U \wedge V + (-1)^k U \wedge \delta V - L_U V. \quad \square \end{aligned}$$

Let ∇ be a torsion free connection on M , and $\{E_i\}_{i=1}^n$ a local basis of vector fields and $\{\alpha_i\}_{i=1}^n$ its dual basis of local 1-differential forms. Then for any $\omega \in A(M)$, $d\omega$ can be expressed by [9]

$$d\omega = \sum_{i=1}^n \alpha_i \wedge \nabla_{E_i} \omega.$$

Also, if $\nabla \Omega = 0$, then for any $U \in \mathfrak{X}^*(M)$, δU can be expressed by

$$\delta U = - \sum_{i=1}^n i_{\alpha_i} (\nabla_{E_i} U).$$

6. DIFFERENTIATION IN CLIFFORD BUNDLES

In this section, first we recall some preliminary notions. Let V be a n dimensional real inner product vector space. we can identify V and V^* via the following isomorphisms

$$\begin{aligned} \# : V &\longrightarrow V^* & , & & b : V^* &\longrightarrow V \\ v^\#(u) &= \langle v, u \rangle & , & & \langle \alpha^b, v \rangle &= \alpha(v), \end{aligned}$$

Where $u, v \in V$ and $\alpha \in V^*$. These isomorphisms are inverse to each other and they can be extended to isomorphism between $\Lambda^k V$ and $\Lambda^k V^*$ with the same properties. If V is oriented, then V has a canonical volume element $\Omega \in \Lambda^n V^*$. If $\{e_i\}_{i=1}^n$ is a positive oriented orthonormal basis, ($\langle e_i, e_j \rangle = \pm \delta_{ij}$), and $\{\alpha_i\}_{i=1}^n$ is its dual basis, then $\Omega = \alpha_1 \wedge \cdots \wedge \alpha_n$ and Ω do not depend on the choice of the basis. Then, we have the canonical Hodge operators H_Ω and H'_Ω . By identification of $\Lambda^k V$ and $\Lambda^k V^*$, H_Ω and H'_Ω give rise to the operators $H, H' : \Lambda^k V \longrightarrow \Lambda^{n-k} V$ defined by the following relations

$$H(U) = H_\Omega(U)^b, H'(U) = H'_\Omega(U^\#).$$

If inner product of V is of the type (r, s) , then $H' = (-1)^s H$. For any $U \in \Lambda^k V$, the interior product operator i_U can be considered as an operator from $\Lambda^l V$ to $\Lambda^{l-k} V$ as follows

$$i_U W = (i_U W^\#)^b = i_{U^\#} W, \quad \forall W \in \Lambda^l V.$$

The pervious properties of Hodge operator hold in this case. For example, if $U \in \Lambda^k V$ and $W \in \Lambda^l V$, then

$$\begin{aligned} H(U \wedge W) &= (-1)^{ki} i_U H(W), \\ H(i_U(W)) &= (-1)^{k(l+1)} U \wedge H(W). \end{aligned}$$

Now, if M is a semi-Riemannian oriented n -dimensional manifold, then all above notions hold pointwise in the fibers of TM , and we can identify $A^k(M)$ and $\mathfrak{X}^k(M)$ by the following isomorphisms

$$\mathfrak{X}^k(M) \xrightarrow{\#} A^k(M), \quad A^k(M) \xrightarrow{b} \mathfrak{X}^k(M).$$

There exists a canonical volume element $\Omega \in A^n(M)$, and we get the canonical Hodge operator $H : \mathfrak{X}^k(M) \longrightarrow \mathfrak{X}^{n-k}(M)$. Now, we can reduce all operators with domain and range $\mathfrak{X}^*(M)$ or $A(M)$ to operators whose domains and ranges are $\mathfrak{X}^*(M)$. For example, for $U \in \mathfrak{X}^k(M)$, we define $i_U : \mathfrak{X}^l(M) \longrightarrow \mathfrak{X}^{l-k}(M)$ as follows

$$i_U V = i_{U^\#}(V) = (i_U V^\#)^b.$$

Lie derivation $L_U : A^l(M) \longrightarrow A^{l-k+1}(M)$, defines a derivation

$$\begin{aligned} L'_U : \mathfrak{X}^l(M) &\longrightarrow \mathfrak{X}^{l-k+1}(M), \\ V &\longrightarrow (L_U V^\#)^b. \end{aligned}$$

The exterior derivation $d : A^k(M) \longrightarrow A^{k+1}(M)$ induces an operator

$$\begin{aligned} \bar{d} : \mathfrak{X}^k(M) &\longrightarrow \mathfrak{X}^{k+1}(M), \\ U &\longrightarrow (dU^\#)^b. \end{aligned}$$

For a local orthonormal basis $\{E_i\}_{i=1}^n$ ($\langle E_i, E_j \rangle = \pm \delta_{ij}$) of M , set $\hat{j} = \langle E_j, E_j \rangle$. If ∇ is the Levi-civita connection of M , then for every $U \in \mathfrak{X}^*(M)$, we have

$$\bar{d}(U) = \sum_{j=1}^n \hat{j} E_j \wedge \nabla_{E_j} U,$$

$$\delta U = - \sum_{j=1}^n \hat{j} i_{E_j}(\nabla_{E_j} U).$$

For $U \in \mathfrak{X}^k(M)$ and $w \in A^k(M)$, the operators

$$\nabla_U : A^l(M) \longrightarrow A^{l-k+1}(M), \quad \nabla_\omega : \mathfrak{X}^l(M) \longrightarrow A^{k-l+1}(M),$$

induce the following operators

$$\begin{aligned} \nabla'_U : \mathfrak{X}^l(M) &\longrightarrow \mathfrak{X}^{l-k+1}(M) & , & \quad \nabla''_U : \mathfrak{X}^l(M) \longrightarrow \mathfrak{X}^{k-l+1}(M) \\ \nabla'_U V &= (\nabla_U V^\#)^b & , & \quad \nabla''_U V = (\nabla_{U^\#} V)^b. \end{aligned}$$

By (17) – (25), we can find similar relations for these operators. For example from relation (25) , for $U \in \mathfrak{X}^k(M)$ and $V \in \mathfrak{X}^l(M)$ we can infer

$$L'_U V = \nabla'_U V - (-1)^{kl} \nabla''_U U.$$

Let us denote the Clifford algebra of each $T_p M$ by $Cl(T_p M)$. Then we can construct the Clifford bundle of M , $Cl(TM) = \cup_{p \in M} Cl(T_p M)$. $Cl(TM)$, as a vector bundle, is isomorphic to $\wedge TM$, and its sections are multivector fields (see [9]). For $U \in \mathfrak{X}^*(M)$, we can consider all the operators $L_U, L'_U, \bar{d}, \delta, \nabla_U, \nabla'_U, \nabla''_U$, as operators which act on the sections of $Cl(TM)$ and yield sections of $Cl(TM)$. But the Clifford multiplication of $Cl(TM)$ gives it a new structure and this new structure produce a new covariant derivation. For a blade k - vector field $U = U_1 \wedge \cdots \wedge U_k$ ($1 \leq k$) and a multivector field V , we define

$$\hat{\nabla}_U V = \sum_{j=1}^k (-1)^{j+1} (U_1 \wedge \cdots \wedge \hat{U}_j \wedge \cdots \wedge U_k) \nabla_{U_j} V.$$

The above multiplication between $U_1 \wedge \cdots \wedge \hat{U}_j \wedge \cdots \wedge U_k$ and $\nabla_{U_j} V$ is Clifford multiplication. This operator is well defined and the definition can be extended along all $U \in \mathfrak{X}^*(M)$. For the case $k = 0$, we define $\hat{\nabla}_f V = 0$, so this operator can be extended linearly to all $U \in \mathfrak{X}^*(M)$. We call this operator Clifford covariant derivation of $Cl(TM)$. Clearly, if $U \in \mathfrak{X}^1(M)$, then for all $V \in \Gamma Cl(TM)$

$$\hat{\nabla}_U V = \nabla_U V = \nabla'_U V,$$

And, if $U \in \mathfrak{X}^2(M)$, then

$$\hat{\nabla}_U V = \nabla_U V - \nabla'_U V,$$

Theorem 6.1. For vector fields U_1, \dots, U_k and $U, V \in \mathfrak{X}^*(M)$ the following relations hold

$$(33) \quad \hat{\nabla}_{fU} V = f \hat{\nabla}_U V,$$

$$(34) \quad \hat{\nabla}_U fV = f \hat{\nabla}_U V + U(f)V,$$

$$(35) \quad \hat{\nabla}_{U_1 \dots U_k} V = \sum_{j=1}^k (-1)^{j+1} U_1 \cdots \hat{U}_j \cdots U_k \nabla_{U_j} V.$$

Proof. We prove only (35). Suppose $\{E_i\}_{i=1}^n$ is a local orthonormal basis of vector fields on M . Due to the $C^\infty(M)$ -linearity of both sides of (35) with respect to U_1, \dots, U_k , it is sufficient to prove (35) in the case when U_1, \dots, U_k are between E_1, \dots, E_n . First assume that $U_1 = E_{i_1}, \dots, U_k = E_{i_k}$ and i_1, \dots, i_k are mutually distinct. Then we have

$$\begin{aligned} \hat{\nabla}_{E_{i_1} \cdots E_{i_k}} V &= \hat{\nabla}_{E_{i_1} \wedge \cdots \wedge E_{i_k}} V = \sum_{j=1}^k (-1)^{j+1} (E_{i_1} \wedge \cdots \wedge \hat{E}_{i_j} \wedge \cdots \wedge E_{i_k}) \nabla_{E_{i_j}} V \\ &= \sum_{j=1}^k (-1)^{j+1} E_{i_1} \cdots \hat{E}_{i_j} \cdots E_{i_k} \nabla_{E_{i_j}} V. \end{aligned}$$

If two of E_{i_1}, \dots, E_{i_k} are equal, then a simple computation shows that the equality remains valid. By induction we can infer that the equality is valid in general case. \square

The Clifford covariant derivation has a natural relation with the Dirac operator of $Cl(TM)$. If $E_{i=1}^n$ is a local orthonormal positive oriented basis for TM , then the Dirac operator $D: \mathfrak{X}^*(M) \rightarrow \mathfrak{X}^*(M)$ is defined by

$$D(U) = \sum_{j=1}^n \hat{j} E_j \nabla_{E_j} U.$$

D does not depend on the choice of the local basis, and we can find a better formula for D . From the properties of Clifford multiplication, we know that

$$E_j \nabla_{E_j} U = E_j \wedge \nabla_{E_j} U + i_{E_j}(\nabla_{E_j} U).$$

Then, we have

$$\begin{aligned} D(U) &= \sum_{j=1}^n \hat{j} E_j \nabla_{E_j} U = \sum_{j=1}^n \hat{j} (E_j \wedge \nabla_{E_j} U + i_{E_j}(\nabla_{E_j} U)) \\ &= \sum_{j=1}^n \hat{j} E_j \wedge \nabla_{E_j} U + \sum_{j=1}^n \hat{j} i_{E_j}(\nabla_{E_j} U) = \bar{d}(U) - \delta(U). \end{aligned}$$

This implies that $D = \bar{d} - \delta$. Writing this equation for fU , we have

$$\begin{aligned} D(fU) &= \bar{d}(fU) - \delta(fU) = \bar{d}f \wedge U + f\bar{d}U - (f\delta U - U(f)) \\ &= \bar{d}f \wedge U + f(\bar{d}U - \delta U) + i_{\bar{d}f}(U) \\ &= fD(U) + (\bar{d}f)U. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} D(U \wedge V) &= \bar{d}(U \wedge V) - \delta(U \wedge V) \\ &= \bar{d}U \wedge V + (-1)^k U \wedge \bar{d}V - \delta U \wedge V - (-1)^k U \wedge \delta V + L_U V \\ &= (\bar{d}U - \delta U) \wedge V + (-1)^k U \wedge (\bar{d}V - \delta V) + L_U V \\ &= (DU) \wedge V + (-1)^k U \wedge (DV) + L_U V. \end{aligned}$$

Now, we note that if $E \in \mathfrak{X}^1(M)$, then

$$\begin{aligned} EU &= E \wedge U + i_E(U), \\ UE &= U \wedge E + (-1)^{k+1} i_E(U). \end{aligned}$$

So, $EU = (-1)^k UE + 2i_E(U)$. Therefore we get

$$\begin{aligned}
D(UV) &= \sum_{j=1}^n \hat{j} E_j \nabla_{E_j} (UV) \\
&= \sum_{j=1}^n \hat{j} E_j (\nabla_{E_j} U) V + \sum_{j=1}^n \hat{j} E_j U \nabla_{E_j} V \\
&= (DU)V + \sum_{j=1}^n \hat{j} ((-1)^k U E_j + 2i_{E_j}(U)) \nabla_{E_j} V \\
&= (DU)V + (-1)^k U (DV) + 2 \sum_{j=1}^n \hat{j} i_{E_j}(U) \nabla_{E_j} V.
\end{aligned}$$

To compute the last term, we assume that $U = U_1 \wedge \cdots \wedge U_k (1 \leq k)$, so

$$\begin{aligned}
&\sum_{j=1}^n \hat{j} i_{E_j}(U) \nabla_{E_j} V \sum_{j=1}^n \hat{j} \sum_{i=1}^k (-1)^{i+1} \langle E_j, U_i \rangle (U_1 \wedge \cdots \wedge \hat{U}_i \wedge \cdots \wedge U_k) \nabla_{E_j} V \\
&= \sum_{i=1}^k (-1)^{i+1} (U_1 \wedge \cdots \wedge \hat{U}_i \wedge \cdots \wedge U_k) \nabla_{\sum_{j=1}^n \hat{j} \langle E_j, U_i \rangle E_j} V \\
&= \sum_{i=1}^k (-1)^{i+1} (U_1 \wedge \cdots \wedge \hat{U}_i \wedge \cdots \wedge U_k) \nabla_{U_i} V \\
&= \hat{\nabla}_U V.
\end{aligned}$$

Due to the linearity with respect to U of both sides of this equality, this relation holds for all $U \in \mathfrak{X}^k(M)$. Then, we have

$$D(UV) = (DU)V + (-1)^k U(DV) + 2\hat{\nabla}_U V.$$

Therefore, we prove the following

Theorem 6.2. *For $U \in \mathfrak{X}^k(M)$ and $V \in \mathfrak{X}^*(M)$ the following relations hold*

$$(36) \quad D(fU) = fD(U) + (\bar{d}f)U,$$

$$(37) \quad D(U \wedge V) = (DU) \wedge V + (-1)^k U \wedge (DV) + L_U V,$$

$$(38) \quad D(UV) = (DU)V + (-1)^k U(DV) + 2\hat{\nabla}_U V.$$

7. EXTENSION TO VECTOR VALUED MULTIVECTOR FIELDS

To extend the results of the preceding sections to vector valued multivector fields, consider a vector bundle $E \rightarrow M$. E -valued k -differential forms on M are sections of $\Lambda^k TM^* \otimes E$, and E -valued k -multivector fields on M are sections of $\Lambda^k TM \otimes E$. Typical E -valued k -differential forms can be written as $\omega \otimes X$ in which $\omega \in A^k(M)$ and $X \in \Gamma E$, while typical E -valued k -multivector fields can be written as $U \otimes X$ in which $U \in \mathfrak{X}^k(M)$ and $X \in \Gamma E$. The set of all E -valued k -differential forms and multivector fields are denoted respectively by $A^k(M, E)$ and $\mathfrak{X}^k(M, E)$ and we set $A(M, E) = \bigoplus_{k=0}^n A^k(M, E)$, $\mathfrak{X}^*(M, E) = \bigoplus_{k=0}^n \mathfrak{X}^k(M, E)$.

For three vector bundles E, F, G on M and a linear bundle morphism $F \otimes E \rightarrow G$, can

extend the exterior and interior product operators between F -valued and E -valued differential forms and multivector fields as follows

$$\begin{aligned}(\omega \otimes X) \wedge (\theta \otimes Y) &= (\omega \wedge \theta) \otimes (XY) \in A(M, G), \\(U \otimes X) \wedge (V \otimes Y) &= (U \wedge V) \otimes (XY) \in \mathfrak{X}^*(M, G), \\i_{U \otimes X}(\theta \otimes Y) &= i_U(\theta) \otimes (XY) \in A(M, G), \\i_{\theta \otimes X}(U \otimes Y) &= i_\theta U \otimes (XY) \in \mathfrak{X}^*(M, G),\end{aligned}$$

for every $\omega, \theta \in A(M), U, V \in \mathfrak{X}^*(M)$ and $X \in \Gamma F, Y \in \Gamma E$, where we assume that $(XY)_p = X_p \otimes Y_p$. These operators are well defined and their actions can be extended to all vector valued differential forms and multivector fields. For example, if $F = M \times \mathbb{R}, G = E$ and

$$\begin{aligned}(M \times \mathbb{R}) \otimes E &\longrightarrow E, \\(p, \lambda) \otimes \xi &\longmapsto \lambda \xi.\end{aligned}$$

then we have the exterior and interior product operators between ordinary differential forms and multivector fields on M , E -valued differential forms and multivector fields.

Consider a fixed connection ∇ on a vector bundle E . We can define covariant derivation of sections of E along multivector fields on M as follows. For a blade k -vector field $U = U_1 \wedge \cdots \wedge U_k (1 \leq k)$ and $X \in \Gamma E$, we define

$$\nabla_U X = \sum_{j=1}^k (-1)^{j+1} (U_1 \wedge \cdots \wedge \hat{U}_j \wedge \cdots \wedge U_k) \otimes \nabla_{U_j} X.$$

This operator is well defined and it extends to all $U \in \mathfrak{X}^k(M)$, by

$$\nabla_U : \mathfrak{X}^0(M, E) = \Gamma E \longrightarrow \mathfrak{X}^{k-1}(M, E).$$

To define ∇_U as an operator on $\mathfrak{X}^l(M, E)$, we need a connection on M . If ∇ is another connection on M , then these connections induce a connection on $\Lambda(TM) \otimes E$ and we denote all of them by ∇ . Note that, for $U \in \mathfrak{X}^*(M)$, $X \in \Gamma E$ and a vector fields V , we have

$$\nabla_V(U \otimes X) = (\nabla_V U) \otimes X + U \otimes \nabla_V X.$$

Now, for $U = U_1 \wedge \cdots \wedge U_k (1 \leq k)$, we can define $\nabla_U : \mathfrak{X}^l(M, E) \longrightarrow \mathfrak{X}^{l+k-1}(M, E)$ as follows. For $\mathbf{Y} \in \mathfrak{X}^*(M, E)$, we put

$$\nabla_U \mathbf{Y} = \sum_{j=1}^k (-1)^{j+1} (U_1 \wedge \cdots \wedge \hat{U}_j \wedge \cdots \wedge U_k) \wedge \nabla_{U_j} \mathbf{Y}.$$

This operator is well defined and it can be extended along all $U \in \mathfrak{X}^k(M)$. It can easily be shown that for each typical E -valued l -vector fields $V \otimes X$, we have

$$\nabla_U(V \otimes X) = (\nabla_U V) \otimes X + (-1)^{l(k-1)} V \wedge \nabla_U X.$$

With direct computation, we have

Theorem 7.1. *For $U \in \mathfrak{X}(M)$, $V \in \mathfrak{X}^l(M)$ and $\mathbf{Y} \in \mathfrak{X}^*(M, E)$, the following relations hold*

$$(39) \quad \nabla_{fU} \mathbf{Y} = f \nabla_U \mathbf{Y},$$

$$(40) \quad \nabla_U (f \mathbf{Y}) = f \nabla_U \mathbf{Y} + U(f) \wedge \mathbf{Y},$$

$$(41) \quad \nabla_{U \wedge V} \mathbf{Y} = (-1)^k U \wedge \nabla_V \mathbf{Y} + (-1)^{l(k-1)} V \wedge \nabla_U \mathbf{Y},$$

$$(42) \quad \nabla_U (V \wedge \mathbf{Y}) = (\nabla_U V) \wedge \mathbf{Y} + (-1)^{l(k-1)} V \wedge \nabla_U \mathbf{Y}.$$

To extend the definition of operator ∇_U for all $U \in \mathfrak{X}^*(M)$, we need to define $\nabla_f \mathbf{Y}$.

Simply, define $\nabla_f \mathbf{Y} = 0$, then the definition can be extend linearly ∇_U to all $U \in \mathfrak{X}^*(M)$. We can easily check that all relations (39)-(42) also hold for the case $k = 0$ or $l = 0$.

Now, we define covariant derivation of E -valued differential forms along multivector fields. For a blade k -vector field $U = U_1 \wedge \cdots \wedge U_k$ ($k \geq l$), and $\Phi \in A(M, E)$, we define

$$\nabla_U \Phi = \sum_{j=1}^k (-1)^{j+1} i_{U_1 \wedge \cdots \wedge \hat{U}_j \wedge \cdots \wedge U_k} (\nabla_{U_j} \Phi).$$

This definition is well defined and extends to all $U \in \mathfrak{X}^k(M)$, and

$$\nabla_U : A^1(M, E) \longrightarrow A^{1-k+1}(M, E).$$

Note that the operator ∇_U depends on a connection on E and a connection on M . For a typical F -valued differential form on X we have

$$\nabla_U (\omega \otimes X) = (\nabla_U \omega) \otimes X + i_{\nabla_U X} \omega.$$

In the case $k = 0$, this relation leads us to define $\nabla_f \Phi = 0$. Similarly to the proof of Theorem 4.2, we can state the following result

Theorem 7.2. *For $U \in \mathfrak{X}^k(M)$, $V \in \mathfrak{X}^l(M)$ and $\Phi \in A(M, E)$, the following relations hold*

$$(43) \quad \nabla_{fU} \Phi = f \nabla_U \Phi,$$

$$(44) \quad \nabla_U (f \Phi) = f \nabla_U \Phi + i_{U(f)} (\Phi),$$

$$(45) \quad \nabla_{U \wedge V} \Phi = i_V (\nabla_U \Phi) + (-1)^{kl} i_U (\nabla_V \Phi),$$

$$(46) \quad \nabla_U (i_V \Phi) = (-1)^{l(k-1)} (i_{\nabla_U V} (\Phi) + i_V (\nabla_U \Phi)).$$

For each $\omega \in A^k(M)$, we can also define $\nabla_\omega : \mathfrak{X}^l(M, E) \longrightarrow \mathfrak{X}^{k-l+1}(M, E)$ as follows

$$(\nabla_\omega \mathbf{Y})(V_1, \cdots, V_{k-l+1}) = i_\omega (\nabla_{V_1 \wedge \cdots \wedge V_{k-l+1}} \mathbf{Y})$$

for $\mathbf{Y} \in \mathfrak{X}^l(M, E)$ and every vector fields $V_1, \cdots, V_{k-l+1} \in \mathfrak{X}(M)$. For $1 \leq l$, ∇_ω depends on the both connections on E and M , but in the case $l = 0$, $\nabla_\omega : \mathfrak{X}^0(M, E) = \Gamma E \longrightarrow \mathfrak{X}^{k+1}(M, E)$ depends only on the connection on E . Then for $X \in \Gamma E$, we have

$$(\nabla_\omega X)(V_1, \cdots, V_{k-l+1}) = \sum_{j=1}^{k+1} (-1)^{j+1} \omega(V_1, \cdots, \hat{V}_j, \cdots, V_{k+1}) \nabla_{V_j} X.$$

Therefore, if we consider ∇X as an E -valued 1- differential form, then $\nabla_w X = \nabla X \wedge w$. For a typical E -valued l - vector field $U \otimes X$, we have

$$(47) \quad \nabla_w(U \otimes X) = (\nabla_w U) \otimes X + (-1)^{l(k-1)} \nabla_{i_U(w)} X$$

To extend the concept of Lie derivation of E - valued multivector fields and differential forms along multivector fields, we need a connection on E . First, for any vector field $U \in \mathfrak{X}(M)$, and a typical E -valued multivector field $V \otimes X$, it is natural to define the operator L_U^∇ which is a combination of Lie derivation and covariant derivation as follows

$$L_U^\nabla(V \otimes X) = L_U V \otimes X + V \otimes \nabla_U X.$$

This operator is well defined and it can be extended to all E -valued multivector fields. Now, for a blade k - vector field $U = U_1 \wedge \cdots \wedge U_k (1 \leq k)$ and $\mathbf{Y} \in \mathfrak{X}^*(M, E)$, we can define $L_U^\nabla \mathbf{Y}$ as follows

$$L_U^\nabla \mathbf{Y} = \sum_{j=1}^k (-1)^{j+1} U_1 \wedge \cdots \wedge \hat{U}_j \wedge \cdots \wedge U_k \wedge (L_{U_j}^\nabla \mathbf{Y}).$$

This definition is well defined and extends to all $U \in \mathfrak{X}^k(M)$. Then for a typical E -valued l - vector field $V \otimes X$ we have

$$(48) \quad L_U^\nabla(V \otimes X) = (L_U V) \otimes X + (-1)^{l(k-1)} V \wedge \nabla_U X.$$

In the case $k = 0$, the above relation suggests us to define $L_f^\nabla(V \otimes X) = V(f) \otimes X$. Clearly for $U \in \mathfrak{X}^k(M)$, we have $L_U^\nabla : \mathfrak{X}^l(M, E) \longrightarrow \mathfrak{X}^{l+k-1}(M, E)$.

Using the relation (1)-(5),(39)-(42) and (48) we can prove the following results for typical E -valued multivector fields.

Theorem 7.3. For $U \in \mathfrak{X}^k(M)$, $V \in \mathfrak{X}^l(M)$, $\mathbf{Y} \in \mathfrak{X}^*(M, E)$, the following relations hold.

$$(49) \quad L_U^\nabla(f\mathbf{Y}) = fL_U^\nabla \mathbf{Y} + U(f) \wedge \mathbf{Y},$$

$$(50) \quad L_{fU}^\nabla \mathbf{Y} = fL_U^\nabla \mathbf{Y} + (-1)^k U \wedge i_{df}(\mathbf{Y}),$$

$$(51) \quad L_U^\nabla(V \wedge \mathbf{Y}) = (L_U V) \wedge \mathbf{Y} + (-1)^{l(k-1)} V \wedge L_U^\nabla \mathbf{Y},$$

$$(52) \quad L_{U \wedge V}^\nabla \mathbf{Y} = (-1)^K U \wedge L_V^\nabla \mathbf{Y} + (-1)^{l(K-1)} V \wedge (L_U^\nabla \mathbf{Y}).$$

In the same manner we can define Lie derivation of E -valued differential forms. First, for any vector field $U \in \mathfrak{X}(M)$, and any typical element $\omega \otimes X$ of $A(M, E)$ we define

$$L_U^\nabla(\omega \otimes X) = L_U \omega \otimes X + \omega \otimes \nabla_U X.$$

This operator is well defined and it can be extended to the Lie derivation of all the elements of $A(M, E)$. Now, for a blade k - vector field $U = U_1 \wedge \cdots \wedge U_k (1 \leq k)$ and $\Phi \in A(M, E)$ we define

$$L_U^\nabla \Phi = \sum_{j=1}^k (-1)^{j+1} i_{U_1 \wedge \cdots \wedge \hat{U}_j \wedge \cdots \wedge U_k} (L_{U_j}^\nabla \Phi) - i_{A(U_1, \dots, U_k)}(\Phi).$$

It is obvious that, the above operator is well defined and we can extend the definition to all $U \in \mathfrak{X}^k(M)$. For typical elements $\omega \otimes X$ of $A(M, E)$ we have

$$(53) \quad L_U^\nabla(\omega \otimes X) = (L_U \omega) \otimes X + i_{\nabla_U X}(\omega).$$

Using (6), (7) and (53), we can prove for $U \in \mathfrak{X}^k(M)$, $\Phi \in A(M, E)$, the following relations

$$(54) \quad L_{fU}^\nabla \Phi = fL_U^\nabla \Phi + (-1)^{k+1} df \wedge i_U(\Phi),$$

$$(55) \quad L_U^\nabla(f\Phi) = fL_U^\nabla(\Phi) + i_{U(f)}(\Phi).$$

For a connection ∇ on a vector bundle E , its associated exterior derivation $d^\nabla : A^k(M, E) \rightarrow A^{k+1}(M, E)$ was defined in [9], as follows

$$(d^\nabla \Phi)(U_1, \dots, U_{k+1}) = \sum_{j=1}^{k+1} (-1)^{j+1} \nabla_{U_j} \Phi(U_1, \dots, \hat{U}_j, \dots, U_{k+1}) + i_{A(U_1, \dots, U_{k+1})}(\Phi).$$

Where $\Phi \in A^k(M, E)$ and $U_1, \dots, U_{k+1} \in \mathfrak{X}(M)$. For a typical element $\omega \otimes X$ of $A^k(M, E)$, a simple computation shows that

$$(56) \quad d^\nabla(\omega \otimes X) = d\omega \otimes X + \nabla_\omega X = d\omega \otimes X + (-1)^k \omega \wedge \nabla X.$$

For a volume element $\Omega \in A^n(M)$, the Hodge operators H_Ω and H'_Ω can be extended to E -valued differential forms and multivector fields as follows

$$\begin{aligned} H_\Omega : \mathfrak{X}^k(M, E) &\rightarrow A^{n-k}(M, E) & , & \quad H'_\Omega : A^k(M, E) \rightarrow \mathfrak{X}^{n-k}(M, E) \\ U \otimes X &\mapsto H_\Omega(U) \otimes X & , & \quad \omega \otimes X \mapsto H'_\Omega(\omega) \otimes X. \end{aligned}$$

These operators are well defined and by them we can define a divergence operator on E -valued multivector fields, which depends on ∇ and which is denoted by δ^∇ .

$$\delta^\nabla : \mathfrak{X}^k(M, E) \rightarrow \mathfrak{X}^{k-1}(M, E) \quad \delta^\nabla = (-1)^{n(k+1)+1} H'_\Omega \circ d^\nabla \circ H_\Omega.$$

By definition, we can see that δ^∇ also depends on Ω .

Theorem 7.4. *If $U \in \mathfrak{X}^*(M)$ and $X \in \Gamma E$, then*

$$(57) \quad \delta^\nabla(U \otimes X) = (\delta U) \otimes X - \nabla_U X.$$

Proof. Let $U \in \mathfrak{X}^k(M)$. In the first we prove

$$\nabla_{H_\Omega(U)} X = (-1)^{k+1} H_\Omega(\nabla_U X).$$

For an arbitrary $W \in \mathfrak{X}^{n-k+1}(M)$, we compute $\langle W, \nabla_{H_\Omega(U)} X \rangle$ and $\langle W, H_\Omega(\nabla_U X) \rangle$.

$$\begin{aligned} \langle W, \nabla_{H_\Omega(U)} X \rangle &= \langle H_\Omega(U), \nabla_W X \rangle = \langle i_U(\Omega), \nabla_W X \rangle = \langle \Omega, U \wedge \nabla_W X \rangle, \\ \langle W, H_\Omega(\nabla_U X) \rangle &= \langle W, i_{(\nabla_U X)} \Omega \rangle = \langle \nabla_U X \wedge W, \Omega \rangle. \end{aligned}$$

From (41) we have

$$\nabla_{U \wedge W} X = (-1)^k U \wedge \nabla_W X + (-1)^{(n-k+1)(k-1)} W \wedge \nabla_U X = (-1)^k U \wedge \nabla_W X + \nabla_U X \wedge W.$$

Since the order of $U \wedge W$ is $n+1$, we have that $U \wedge W = 0$ and $\nabla_U X \wedge W = (-1)^{k+1} U \wedge \nabla_W X$ which yield the result. Now, we compute $\delta^\nabla(U \otimes X)$.

$$\begin{aligned} \delta^\nabla(U \otimes X) &= (-1)^{n(k+1)+1} H'_\Omega(d^\nabla(H_\Omega(U \otimes X))) = (-1)^{n(k+1)+1} H'_\Omega(d^\nabla(H_\Omega(U) \otimes X)) \\ &= (-1)^{n(k+1)+1} H'_\Omega(dH_\Omega(U) \otimes X + \nabla_{H_\Omega(U)} X) \\ &= (-1)^{n(k+1)+1} H'_\Omega \circ d \circ H_\Omega(U) \otimes X + (-1)^{n(k+1)+1} H'_\Omega((-1)^{k+1} H_\Omega(\nabla_U X)) \\ &= \delta U \otimes X - \nabla_U X. \quad \square \end{aligned}$$

Corollary 7.5. For $U \in \mathfrak{X}^k(M)$ and $\mathbf{Y} \in \mathfrak{X}^*(M, E)$ the following relations hold

$$(58) \quad \delta^\nabla(f\mathbf{Y}) = f\delta^\nabla\mathbf{Y} - i_{df}(\mathbf{Y}),$$

$$(59) \quad \delta^\nabla(U \wedge \mathbf{Y}) = (\delta U) \wedge \mathbf{Y} + (-1)^k U \wedge \delta^\nabla\mathbf{Y} - L_U^\nabla\mathbf{Y}.$$

Proof. It is sufficient to prove the above relations for typical elements of $\mathfrak{X}^*(M, E)$. The relations (29), (30) and (57) make the computations easy.

8. DIFFERENTIATION IN SPINOR BUNDLES

Let M be a semi-Riemannian manifold, $Cl(TM)$ its clifford bundle and $E \rightarrow M$ a spinor bundle on M . Then for a multiplication $Cl(TM) \otimes E \rightarrow E$ and a connection ∇ on E , together with Levi-Civita connection of M , this multiplication is parallel. If $\{E_i\}_{i=1}^n$ is an orthonormal local basis for M , then the Dirac operator of this spinor bundle is defined as follows

$$D : \Gamma E \rightarrow \Gamma E, \\ D(X) = \sum_{i=1}^n \hat{i}E_i \nabla_{E_i} X.$$

This definition does not depend on the choice of the local basis. Now we can define a covariant derivation of the sections of E along the sections of $Cl(T, M)$, which is related to the spinor structure of E . We call it Clifford covariant derivation of E and we denote it by $\hat{\nabla}$. For a blade multivector field $U = U_1 \wedge \cdots \wedge U_k (1 \leq k)$ and $X \in \Gamma E$ we define

$$\hat{\nabla}_U X = \sum_{j=1}^k (U_1 \wedge \cdots \wedge \hat{U}_j \wedge \cdots \wedge U_k) \nabla_{U_j} X.$$

This operator is well defined and it can be extended for all $U \in \mathfrak{X}^k(M)$. For the case $k = 0$, we define $\hat{\nabla}_f X = 0$, and then we extend the linear operator $\hat{\nabla}_U$ to all $U \in \mathfrak{X}^*(M) = \Gamma Cl(TM)$. By the same method mentioned in the proof of Theorem 6.1, we can prove the following.

Theorem 8.1. For $U \in \mathfrak{X}^*(M)$, $X \in \Gamma E$, $U_1, \dots, U_k \in \mathfrak{X}(M)$, the following relations hold

$$(60) \quad \hat{\nabla}_{fU} X = f\hat{\nabla}_U X,$$

$$(61) \quad \hat{\nabla}_U fX = f\hat{\nabla}_U X + U(f)X,$$

$$(62) \quad \hat{\nabla}_{U_1 \cdots U_k} X = \sum_{j=1}^k (-1)^{j+1} U_1 \cdots \hat{U}_j \cdots U_k \nabla_{U_j} X.$$

Note that $\hat{\nabla}$ and the Dirac operator are related by same formula as in Theorem 6.2. Then by a simple calculation, as we explore in the proof of theorem 6.2, we have the following.

Theorem 8.2. For $X \in \Gamma E$ and $U \in \mathfrak{X}^k(M)$ we have

$$(63) \quad D(fX) = fD(X) + (\bar{d}f)X,$$

$$(64) \quad D(UX) = (DU)X + (-1)^k U(DX) + 2\hat{\nabla}_U X.$$

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