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On the Sets of Strongly f-Lacunary Summable Sequences

Ibrahim S. Ibrahim[a,b*], Rifat Çolak[a]

[a]Department of Mathematics, Faculty of Science, Firat University, Elazig, Türkiye
[b]Department of Mathematics, Faculty of Science, University of Zakho, Zakho, Iraq

E-mail: ibrahimmath95@gmail.com
E-mail: rftcolak@gmail.com

Abstract. The statistical convergence with respect to a modulus function has various applications in both mathematics and statistics. The main purpose of this research paper is to establish the relations between the sets of strongly f-lacunary summable and strongly g-lacunary summable sequences, strongly f-lacunary summable and g-lacunary statistically convergent sequences, where f and g are different modulus functions under certain conditions. Furthermore, for some special modulus functions, we establish the relations between the sets of strongly f-lacunary summable and strongly lacunary summable sequences.

Keywords: Modulus function, Statistical convergence, f-lacunary statistical convergence, Strong f-lacunary summability.

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1. Introduction

The notion of statistical convergence was first based on the first edition of the monograph of Zygmund [22], and its definition was introduced by Fast [10] in a short note and later reintroduced by Schoenberg [21] independently

*Corresponding Author

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with some of the basic properties of statistical convergence. In recent decades, statistical convergence has been confirmed in many several fields and under various names, such as measure theory, approximation theory, Banach spaces, hopfield neural network, locally convex spaces, trigonometric series, number theory, summability theory, ergodic theory, turnpike theory, Fourier analysis, and optimization. Subsequently, Šalát [20], Fridy [12], Connor [8], Rath and Tripathy [18], Et [9], and many others were further explored from the point of perspective of sequence spaces and related to summability theory.

The idea of a modulus function was structured by Nakano [17]. Some authors such as Ruckle [19] and Maddox [15] have introduced some sequence spaces by using a modulus function. Other than them, to establish a number of sequence spaces, Gosh and Srivastava [13], Bhardwaj and Singh [3], and some others have used a modulus function. More information on this principle, as well as its applications, can be found in [2, 5, 6, 7, 14].

We denote by $\mathbb{N}$ the set of natural numbers. The number $\delta(A)$ of a set $A \subset \mathbb{N}$ is called the natural density of A and is defined by

$$\delta(A) = \lim_{n\to\infty} \frac{1}{n} \left|\{k \leq n : k \in A\}\right|$$

where $|\{k \leq n : k \in A\}|$ denotes the number of elements of A which are less than or equal to n. One easily may see that $\delta(A) = 0$ if $A \subset \mathbb{N}$ is a finite set and $\delta(\mathbb{N}) = 1$ and also $\delta(\mathbb{N} - A) = \delta(\mathbb{N}) - \delta(A) = 1 - \delta(A)$.

In this paper, the spaces of bounded and convergent sequences are symbolized by ℓ_∞ and c, respectively, as well as the set of all complex numbers is symbolized by $\mathbb{C}$.

A sequence (x_k) in $\mathbb{C}$ is named statistically convergent to the number $l \in \mathbb{C}$ if $\delta\left(\{k \in \mathbb{N} : |x_k - l| \geq \varepsilon\}\right) = 0$ for each $\varepsilon > 0$.

We imply an increasing sequence $\theta = (k_r)$ of non-negative integer numbers with $k_0 = 0$ by a lacunary sequence such that $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals put by θ shall be represented by $I_r = (k_{r-1}\, , k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ can be shortened by q_r.

Fridy and Orhan [11] defined lacunary statistical convergence as the following expression.

Let $\theta = (k_r)$ be a lacunary sequence. A sequence (x_k) of numbers is named lacunary statistically convergent (or S_θ-convergent) to the number l, if

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\{k \in I_r : |x_k - l| \geq \varepsilon\}\right| = 0$$

for each $\varepsilon > 0$. In this case, we write $S_\theta - \lim x_k = l$. Throughout the paper, the class of S_θ-convergent sequences will be symbolized by S_θ.

A function $f : [0, \infty) \to [0, \infty)$ is called a modulus function (or a modulus) if

(1) $f(t) = 0$ if and only if $t = 0$,

(2) $f(t_1+t_2) \le f(t_1)+f(t_2)$ for every $t_1, t_2 \in [0, \infty)$,
(3) f is increasing,
(4) f is continuous from the right at 0.

From these properties, it is clear that a modulus function must be continuous everywhere on $[0, \infty)$. A modulus function may be either unbounded or bounded. For instance, $f(t) = \log(t+1)$ is unbounded, but $f(t) = \frac{t}{t+1}$ is bounded.

Lemma 1.1. [16] *For any modulus f, $\lim_{t\to\infty} \frac{f(t)}{t}$ exists and $\lim_{t\to\infty} \frac{f(t)}{t} = \inf_{t\in(0,\infty)} \frac{f(t)}{t}$.*

The following definition was presented by Aizpuru et al. [1].

The number $\delta_f(A)$ of a set $A \subset \mathbb{N}$ is defined by

$$\delta_f(A) = \lim_{n\to\infty} \frac{f\left(|k \le n : k \in A|\right)}{f\left(n\right)}$$

and is called the f-density of A, where f is an unbounded modulus function.

A sequence (x_k) in $\mathbb{C}$ is named f-statistically convergent to $l \in \mathbb{C}$ if for each $\varepsilon > 0$,

$$\delta_f(A) = (\{k \in \mathbb{N} : |x_k - l| \ge \varepsilon\}) = 0.$$

Throughout the paper, S^f symbolizes the class of all f-statistically convergent sequences.

Definition 1.2. Suppose $\theta = (k_r)$ is a lacunary sequence and f is an unbounded modulus. Then, the sequence (x_k) in $\mathbb{C}$ is named f-lacunary statistically convergent (or S_θ^f-convergent) to $l \in \mathbb{C}$, if

$$\lim_{r\to\infty} \frac{1}{f\left(h_r\right)} f\left(|\{k \in I_r : |x_k - l| \ge \varepsilon\}|\right) = 0$$

for every $\varepsilon > 0$. We write $S_\theta^f - \lim x_k = l$ if (x_k) is S_θ^f-convergent to l. Throughout the paper, S_θ^f symbolizes the class of all sequences which are S_θ^f-convergent, that is,

$$S_\theta^f = \left\{(x_k) : \lim_{r\to\infty} \frac{1}{f\left(h_r\right)} f\left(|\{k \in I_r : |x_k - l| \ge \varepsilon\}|\right) = 0, \text{ for every } \varepsilon > 0\right\}.$$

We write S_θ instead of S_θ^f in the case $f(t) = t$ and for $\theta = (2^r)$, we write S^f instead of S_θ^f and also in the particular case $f(t) = t$ and $\theta = (2^r)$, we write S instead of S_θ^f.

Lemma 1.3. *The S_θ^f-limit of every S_θ^f-convergent sequence is unique.*

2. Main results

In this section, we first give the definition of N_θ^f and then establish the relations between N_θ^f and N_θ^g, N_θ and N_θ^f, N_θ^f and S_θ^g, $\ell_\infty \cap S_\theta^f$ and N_θ^g, for different modulus functions f and g under some conditions on the considered modulus functions. However, the relations between the sets N_θ and S_θ, S_θ and S are known already in [11].

Definition 2.1. Suppose $\theta = (k_r)$ is a lacunary sequence and f is a modulus function. Then, the sequence (x_k) in $\mathbb{C}$ is named strongly $f-$lacunary summable (or strongly N_θ^f-summable) to $l \in \mathbb{C}$, if

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} f\left(|x_k - l|\right) = 0.$$

If the sequence (x_k) is strongly N_θ^f-summable to l, we write $N_\theta^f - \lim x_k = l$. The class of strongly N_θ^f-summable sequences will be symbolized by N_θ^f, that is,

$$N_\theta^f = \left\{ (x_k) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} f\left(|x_k - l|\right) = 0 \text{ for some number } l \right\}.$$

Note that the modulus function f need not be unbounded in this definition.

The strong N_θ^f-summability will reduce to the strong N_θ-summability if we take $f(t) = t$.

Theorem 2.2. *Suppose f and g are modulus functions and $\theta = (k_r)$ is a lacunary sequence. If $\sup\limits_{t\in(0,\infty)} \frac{f(t)}{g(t)} < \infty$, then $N_\theta^g \subset N_\theta^f$.*

Proof. Assume that $\alpha = \sup\limits_{t\in(0,\infty)} \frac{f(t)}{g(t)} < \infty$, then we have $\frac{f(t)}{g(t)} \le \alpha$, and so that $f(t) \le \alpha g(t)$ for every $t \in [0, \infty)$. Now, it is clear that $\alpha > 0$ and if (x_k) is strongly N_θ^g-summable to some $l \in \mathbb{C}$, we may write

$$\frac{1}{h_r} \sum_{k\in I_r} f\left(|x_k - l|\right) \le \alpha \frac{1}{h_r} \sum_{k\in I_r} g\left(|x_k - l|\right).$$

Taking the limits on both sides as $r \to \infty$, we obtain that $(x_k) \in N_\theta^g$ implies $(x_k) \in N_\theta^f$. □

Remark 2.3. The following example shows that the inclusion $N_\theta^g \subset N_\theta^f$ is strict at least for some special modulus functions f and g.

Example 2.4. Let the lacunary sequence $\theta = (k_r)$ be given and consider the sequence (x_k) such that x_k to be $\left[\sqrt{h_r}\right]$ at the first $\left[\sqrt{h_r}\right]$ integers in I_r, and $x_k = 0$ otherwise, where $[t]$ denotes the integer part of a real number t. Now, if we take modulus functions $f(t) = \frac{t}{t+1}$ and $g(t) = t$, then $\sup\limits_{t\in(0,\infty)} \frac{f(t)}{g(t)} = 1 < \infty$ and so that $N_\theta^g \subset N_\theta^f$. By using the equality $f(0) = 0$, we have

$$\frac{1}{h_r} \sum_{k\in I_r} f\left(|x_k|\right) = \frac{1}{h_r} \left[\sqrt{h_r}\right] f\left(\left[\sqrt{h_r}\right]\right) = \frac{\left[\sqrt{h_r}\right]\left[\sqrt{h_r}\right]}{h_r\left(\left[\sqrt{h_r}\right]+1\right)}.$$

By taking the limits as $r \to \infty$, we get that $N_\theta^f - \lim x_k = 0$ and so $(x_k) \in N_\theta^f$. But since

$$\frac{1}{h_r} \sum_{k \in I_r} g(|x_k|) = \frac{1}{h_r} \left[\sqrt{h_r}\right] g\left(\left[\sqrt{h_r}\right]\right) = \frac{\left[\sqrt{h_r}\right]\left[\sqrt{h_r}\right]}{h_r}$$

and $\frac{[\sqrt{h_r}][\sqrt{h_r}]}{h_r} \to 1$ as $r \to \infty$, we get that $(x_k) \notin N_\theta^g$. Hence, $(x_k) \in N_\theta^f - N_\theta^g$ and the inclusion $N_\theta^g \subset N_\theta^f$ is strict.

Theorem 2.5. *Suppose f and g are modulus functions and $\theta = (k_r)$ is a lacunary sequence. If $\inf\limits_{t \in (0,\infty)} \frac{f(t)}{g(t)} > 0$, then $N_\theta^f \subset N_\theta^g$ and the inclusion is strict.*

Proof. Assume that $\beta = \inf\limits_{t \in (0,\infty)} \frac{f(t)}{g(t)} > 0$. Then, we have $\frac{f(t)}{g(t)} \geq \beta$ and so that $\beta g(t) \leq f(t)$ for every $t \in [0, \infty)$. Now, if (x_k) is strongly N_θ^f-summable to some $l \in \mathbb{C}$, we may write

$$\frac{1}{h_r} \sum_{k \in I_r} g(|x_k - l|) \leq \frac{1}{h_r} \sum_{k \in I_r} \frac{1}{\beta} f(|x_k - l|).$$

Taking the limits on both sides as $r \to \infty$, we obtain that $(x_k) \in N_\theta^f$ implies $(x_k) \in N_\theta^g$.

For the strict inclusion, if we take the sequence of Example 2.4 with modulus functions $f(t) = t$ and $g(t) = \frac{t}{t+1}$, then the strict inclusion will happen. □

The result below is obtained from Theorem 2.2 and Theorem 2.5.

Corollary 2.6. *Suppose f and g are modulus functions and $\theta = (k_r)$ is a lacunary sequence. If*

$$0 < \inf_{t \in (0,\infty)} \frac{f(t)}{g(t)} \leq \sup_{t \in (0,\infty)} \frac{f(t)}{g(t)} < \infty,$$

then $N_\theta^f = N_\theta^g$.

Theorem 2.7. *For any modulus function f and lacunary sequence $\theta = (k_r)$, we have $N_\theta \subset N_\theta^f$.*

Proof. Assume that $(x_k) \in N_\theta$. So that

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} |x_k - l| = 0$$

for some $l \in \mathbb{C}$. Given $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f(t) < \varepsilon$ for $t \in (0, \delta]$. Now, consider

$$\sum_{k \in I_r} f(|x_k - l|) = \sum_{\substack{k \in I_r \\ |x_k - l| \leq \delta}} f(|x_k - l|) + \sum_{\substack{k \in I_r \\ |x_k - l| > \delta}} f(|x_k - l|).$$

Since $f(|x_k - l|) < \varepsilon$ for $|x_k - l| \leq \delta$, then

$$\sum_{\substack{k\in I_r \\ |x_k - l| \leq \delta}} f(|x_k - l|) < \varepsilon h_r,$$

and also for $|x_k - l| > \delta$, we get

$$|x_k - l| < \frac{|x_k - l|}{\delta} < 1 + \left[\frac{|x_k - l|}{\delta}\right],$$

Since f is a modulus, we have

$$f(|x_k - l|) \leq f\left(1 + \left[\frac{|x_k - l|}{\delta}\right]\right) \leq f(1)\left(1 + \left[\frac{|x_k - l|}{\delta}\right]\right) \leq 2f(1)\frac{|x_k - l|}{\delta}.$$

So, we get

$$\sum_{\substack{k\in I_r \\ |x_k - l| > \delta}} f(|x_k - l|) \leq \frac{2f(1)}{\delta} \sum_{\substack{k\in I_r \\ |x_k - l| > \delta}} |x_k - l|.$$

Thus,

$$\begin{aligned} \tfrac{1}{h_r}\sum_{k\in I_r} f(|x_k - l|) &\leq \varepsilon + \tfrac{2f(1)}{\delta}\tfrac{1}{h_r} \sum_{\substack{k\in I_r \\ |x_k - l| > \delta}} |x_k - l| \\ &\leq \varepsilon + \tfrac{2f(1)}{\delta}\tfrac{1}{h_r} \sum_{k\in I_r} |x_k - l|. \end{aligned}$$

Since $(x_k) \in N_\theta$, we obtain that $(x_k) \in N_\theta^f$. □

Corollary 2.8. *Suppose f is any modulus function and $\theta = (k_r)$ is a lacunary sequence. If $\inf\limits_{t\in(0,\infty)} \frac{f(t)}{t} > 0$, then $N_\theta^f = N_\theta$.*

Proof. Since $N_\theta \subset N_\theta^f$ for any modulus function f by Theorem 2.7, taking $g(t) = t$ in Theorem 2.5, we get $N_\theta^f \subset N_\theta$ if $\inf\limits_{t\in(0,\infty)} \frac{f(t)}{t} > 0$. Therefore, $N_\theta^f = N_\theta$ if $\inf\limits_{t\in(0,\infty)} \frac{f(t)}{t} > 0$. □

Theorem 2.9. *Assume that f and g are unbounded modulus functions and $\theta = (k_r)$ is a lacunary sequence. If $\inf\limits_{t\in(0,\infty)} \frac{f(t)}{g(t)} > 0$ and $\lim\limits_{t\to\infty} \frac{g(t)}{t} > 0$, then every strongly N_θ^f-summable sequence is S_θ^g-statistically convergent.*

Proof. Suppose that $\beta = \inf\limits_{t\in(0,\infty)} \frac{f(t)}{g(t)} > 0$. Then, we have $\frac{f(t)}{g(t)} \geq \beta$ and so that $\beta g(t) \leq f(t)$ for every $t \in [0,\infty)$. Now, if (x_k) is strongly N_θ^f-summable to some $l \in \mathbb{C}$, then we may write

$$\begin{aligned}
\frac{1}{h_r}\sum_{k\in I_r} f\left(|x_k-l|\right) &\geq \beta\frac{1}{h_r}\sum_{k\in I_r} g\left(|x_k-l|\right)\\
&= \beta\frac{1}{h_r}\sum_{\substack{k\in I_r\\|x_k-l|\geq\varepsilon}} g\left(|x_k-l|\right)+\beta\frac{1}{h_r}\sum_{\substack{k\in I_r\\|x_k-l|<\varepsilon}} g\left(|x_k-l|\right)\\
&\geq \beta\frac{1}{h_r}\sum_{\substack{k\in I_r\\|x_k-l|\geq\varepsilon}} g\left(|x_k-l|\right)\\
&\geq \beta\frac{1}{h_r}\left|\{k\in I_r:|x_k-l|\geq\varepsilon\}\right| g\left(\varepsilon\right).
\end{aligned}$$

Since $|\{k\in I_r:|x_k-l|\geq\varepsilon\}|$ is a positive integer and g is a modulus, we have

$$\begin{aligned}
\frac{1}{h_r}\sum_{k\in I_r} f\left(|x_k-l|\right) &\geq \beta\frac{1}{h_r}g\left(\left|\{k\in I_r:|x_k-l|\geq\varepsilon\}\right|\right)\frac{g(\varepsilon)}{g(1)}\\
&= \frac{g(|\{k\in I_r:|x_k-l|\geq\varepsilon\}|)}{g(h_r)}\frac{g(h_r)}{h_r}\frac{g(\varepsilon)}{g(1)}\beta.
\end{aligned}$$

Taking the limits on both sides as $r\to\infty$, we obtain that $(x_k)\in N_\theta^f$ implies $(x_k)\in S_\theta^g$ since $\lim_{t\to\infty}\frac{g(t)}{t}>0$. This fulfills the proof. $\square$

Remark 2.10. The converse of the above theorem is not true, in general. It can be shown in the following example.

Example 2.11. Let θ be given and select the sequence (x_k) as in Example 2.4 and also consider the modulus function $g(t)=f(t)=t$. Then, we have $\inf_{t\in(0,\infty)}\frac{f(t)}{g(t)}>0$ and $\lim_{t\to\infty}\frac{g(t)}{t}>0$. Given any $\varepsilon>0$, we have

$$\lim_{r\to\infty}\frac{1}{g\left(h_r\right)}g\left(\left|\{k\in I_r:|x_k|\geq\varepsilon\}\right|\right)=\lim_{r\to\infty}\frac{g\left(\left[\sqrt{h_r}\right]\right)}{h_r}=0.$$

So that $S_\theta^g-\lim x_k=0$ and thus $(x_k)\in S_\theta^g$. On the other hand,

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r} f\left(|x_k|\right)=\lim_{r\to\infty}\frac{\left[\sqrt{h_r}\right]\left[\sqrt{h_r}\right]}{h_r}=1.$$

Hence, $N_\theta^f-\lim x_k\neq 0$ and thus $(x_k)\notin N_\theta^f$.

The following result is obtained by taking $g(t)=f(t)$ in Theorem 2.9.

Corollary 2.12. *Assume that f is an unbounded modulus function and $\theta=(k_r)$ is a lacunary sequence. If $\lim_{t\to\infty}\frac{f(t)}{t}>0$, then every strongly N_θ^f-summable sequence is S_θ^f-statistically convergent.*

The following result is obtained by taking $g(t)=t$ in Theorem 2.9.

Corollary 2.13. *Assume that f is an unbounded modulus function and $\theta=(k_r)$ is a lacunary sequence. If $\inf_{t\in(0,\infty)}\frac{f(t)}{t}>0$, then every strongly N_θ^f-summable sequence is S_θ-statistically convergent.*

The following result is obtained by taking $f(t)=t$ in Corollary 2.13, which is also the first part of Theorem 1 of [11].

Corollary 2.14. *A strongly N_θ-summable sequence is S_θ-statistically convergent.*

Theorem 2.15. *Suppose $\theta = (k_r)$ is a lacunary sequence. Then, for any unbounded modulus functions f and g, we have $\ell_\infty \cap S_\theta^f \subset N_\theta^g$.*

Proof. Let f and g be unbounded modulus functions and $\theta = (k_r)$ be given. Since $S_\theta^f \subset S_\theta$ for every modulus f by Theorem 11 of [4], and since $\ell_\infty \cap S_\theta \subset N_\theta$ by the second part of Theorem 1 of [11], then we have $\ell_\infty \cap S_\theta^f \subset \ell_\infty \cap S_\theta \subset N_\theta$, that is, $\ell_\infty \cap S_\theta^f \subset N_\theta$. On the other hand, $N_\theta \subset N_\theta^g$ for any modulus g by Theorem 2.7. Therefore, $\ell_\infty \cap S_\theta^f \subset N_\theta^g$. □

Remark 2.16. The following example shows that the inclusion $\ell_\infty \cap S_\theta^f \subset N_\theta^g$ is strict at least for some special modulus functions f and g.

Example 2.17. As an example, let the lacunary sequence $\theta = (k_r)$ be provided and consider the sequence (x_k) such that x_k to be $\sqrt[3]{h_r}$ at the first $\left[\sqrt{h_r}\right]$ integers in I_r, and $x_k = 0$ otherwise. Now, if we take modulus functions $f(t) = \frac{t}{t+1}$ and $g(t) = t$, then for every $\varepsilon > 0$, we have

$$\lim_{r\to\infty} \frac{1}{f(h_r)} f(|\{k \in I_r : |x_k| \geq \varepsilon\}|) = \lim_{r\to\infty} \frac{f\left(\left[\sqrt{h_r}\right]\right)}{f(h_r)} = 1.$$

So that $(x_k) \notin S_\theta^f$, but since

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} g(|x_k|) = \lim_{r\to\infty} \frac{\left[\sqrt{h_r}\right] g\left(\sqrt[3]{h_r}\right)}{h_r} = 0,$$

we get $(x_k) \in N_\theta^g$.

The outcome below is a result of Theorem 2.15.

Corollary 2.18. *Suppose $\theta = (k_r)$ is a lacunary sequence and f is any unbounded modulus function. Then,*

(1) $\ell_\infty \cap S_\theta^f \subset N_\theta^f$.
(2) $\ell_\infty \cap S_\theta^f \subset N$.
(3) $\ell_\infty \cap S \subset N_\theta^f$.

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