Iranian Journal of Mathematical Sciences and Informatics
Vol. 20, No. 2 (2025), pp 31-39
DOI: 10.61186/ijmsi.20.2.31

# Fredholm Composition Operators on Harmonic Bloch-Type Spaces

Y. Estaremi[a*], A. Ebadian[b], S. Esmaeili[c]

[a]Department of Mathematics and Computer Sciences, Golestan University, Gorgan, Iran
[b]Department of Mathematics, Urmia University, Urmia, Iran
[c]Department of Mathematics, Payame Noor University, P. O. Box: 19395-3697, Tehran, Iran

E-mail: yestaremi@gu.ac.ir
E-mail: ebadian.ali@gmail.com
E-mail: dr.somaye.esmaili@gmail.com

Abstract. In this paper we characterize Fredholm and invertible composition operators on harmonic Bloch function spaces. Indeed, we provide some necessary and sufficient conditions for Fredholmness of composition operators. Also, we investigate the relation between the Fredholm and invertible composition operators on harmonic Bloch function spaces.

**Keywords:** Harmonic function, Fredholm operator, Composition operator, Univalent function.

**2010 Mathematics subject classification:** 47B33.

## 1. Introduction

Let $D$ be the open unit disk in the complex plane. For a continuously differentiable complex-valued function $f(z) = u(z) + iv(z)$, $z = x + iy$, we use the common notations for its formal derivatives:

*Corresponding Author

Received 02 August 2020; Accepted 16 December 2021

$$f_z = \frac{1}{2}(f_x - if_y),$$

$$f_{\bar{z}} = \frac{1}{2}(f_x + if_y).$$

A twice continuously differentiable complex-valued function $f = u + iv$ on $D$ is called a harmonic function if and only if the real-valued function $u$ and $v$ satisfy Laplace's equation, i,e., $\Delta u = \Delta v = 0$. A direct calculation shows that the Laplacian of $f$ is

$$\Delta f = 4f_{z\bar{z}}.$$

Thus for functions $f$ with continuous second partial derivatives, it is clear that $f$ is harmonic if and only if $\Delta f = 0$. We consider complex-valued harmonic function $f$ defined in a simply connected domain $D \subset C$. The function $f$ has a canonical decomposition $f = h + \bar{g}$, where $h$ and $g$ are analytic in $D$ (See,[5], p. 7). The aim of this paper is to characterize composition operators on harmonic Bloch type spaces. Analytic functions are preserved under composition, but harmonic functions are not. The composition of a harmonic function with an analytic function is harmonic, but the composition of an analytic function with a harmonic function need not be harmonic. A planar complex-valued harmonic function $f$ in $D$ is called a harmonic Bloch function if and only if

$$\beta_f = \sup_{z,w\in D, z\neq w} \frac{|f(z) - f(w)|}{\varrho(z,w)} < \infty.$$

Here $\beta_f$ is called the Lipschitz number of $f$ and

$$\begin{aligned}
\varrho(z,w) &= \frac{1}{2}\log\left(\frac{1 + |\frac{z-w}{1-\bar{z}w}|}{1 - |\frac{z-w}{1-\bar{z}w}|}\right) \\
&= \frac{1}{2}\log\left(\frac{1+\rho(z,w)}{1-\rho(z,w)}\right) \\
&= \arctan h\left|\frac{z-w}{1-\bar{z}w}\right|,
\end{aligned}$$

denotes the hyperbolic distance between $z$ and $w$ in $D$, in which $\rho(z,w)$ is the pseudo-hyperbolic distance on $D$. In this paper we denote the hyperbolic disk with center $a$ and radius $r > 0$, by $D(a,r) = \{z : \varrho(a,z) < r\}$.

In [3] Colonna proved that

$$\beta_f = \sup_{z,w\in D, z\neq w} \frac{|f(z) - f(w)|}{\varrho(z,w)} = \sup_{z\in D}(1 - |z|^2)[|f_z(z)| + |f_{\bar{z}}(z)|].$$

Moreover, the set of all harmonic Bloch mappings, denoted by the symbol $HB(1)$ or $HB$, forms a complex Banach space with the norm $|||.|||$, given by

$$|||f|||_{HB(1)} = |f(0)| + \sup_{z\in D}(1-|z|^2)[|f_z(z)| + |f_{\bar{z}}(z)|].$$

For $\alpha \in (0,\infty)$, the harmonic $\alpha$-Bloch space $HB(\alpha)$ (also referred to as harmonic Bloch-type space) consists of complex-valued harmonic functions like $f$ defined on $D$ such that

$$\|f\|_{HB(\alpha)} = \sup_{z\in D}(1-|z|^2)^{\alpha}[|f_z(z)| + |f_{\bar{z}}(z)|] < \infty,$$

and the harmonic little $\alpha$-Bloch space $HB_0(\alpha)$ consists of all functions in $HB(\alpha)$ such that

$$\lim_{|z|\to 1}(1-|z|^2)^{\alpha}[|f_z(z)| + |f_{\bar{z}}(z)|] = 0.$$

Obviously, when $\alpha = 1$, $\|f\|_{HB(\alpha)} = \beta_f$. $HB(\alpha)$ is a Banach space with norm given by

$$|||f|||_{HB(\alpha)} = |f(0)| + \|f\|_{HB(\alpha)} = |f(0)| + \sup_{z\in D}(1-|z|^2)^{\alpha}[|f_z(z)| + |f_{\bar{z}}(z)|],$$

and $HB_0(\alpha)$ is a closed subspace of $HB(\alpha)$.
We denote the closed unit balls of $HB(\alpha)$ and the closed unit ball of $HB_0(\alpha)$ by $b(\alpha)$ and $b_0(\alpha)$ respectively.

Let $\varphi$ be an analytic self-map of $D$. The composition operator $C_\varphi$ induced by such a $\varphi$ is the linear map on the spaces of all harmonic functions on the unit disk defined by

$$C_\varphi f = fo\varphi.$$

It is easy to see that an operator defined in this manner is linear.

Composition operators can act on various types of function spaces. In each case the main goal is to discover the connection between the properties of the inducing function $\varphi$ and the operator theoretic properties of $C_\varphi$ such as, being bounded, compact, invertible, normal, subnormal, isometric, closed range, Fredholm, and many others. For instance, in [8], the author characterized Fredholm composition operators on the small Bloch-type spaces and gave some necessary and sufficient conditions for their semi-Fredholmness. Also, they extended some results about semi-Fredholmness of composition operators on the small Bloch-type spaces. In addition, some results about the properties of composition operators are obtained in [2]. For more results on composition operators acting on various other spaces of analytic functions see, for example, [4].

By the Closed Graph Theorem, $C_\varphi : X \to Y$, from Banach space $X$ to the other Banach space $Y$, is bounded if and only if $C_\varphi$ maps $X$ to $Y$. Clearly, $C_\varphi$

preserves the set of harmonic functions. To state the results obtained, we need the following definitions. Let $\rho(z,w) = |\varphi_z(w)|$ denote the pseudohyperbolic distance between $z$ and $w$ on $D$, in which $\varphi_z$ is a disk automorphism of $D$, that is,

$$\varphi_z(w) = \frac{z-w}{1-\bar{z}w}.$$

We say that subset $G \subset D$ is an $r$-net in $D$, for some $r \in (0,1)$ if for all $z \in D$, $\exists w \in G$, such that $\rho(z,w) < r$.

Let $\upsilon : [0,1] \to \Re$ be a continuous non-increasing function which is positive except $\upsilon(1) = 0$. We also denote by $\upsilon$ the function defined on the unit disk by $\upsilon(z) = \upsilon(|z|)$. To study Fredholm operators we need the associated weight function that is defined as

$$\tilde{\upsilon}(z) := 1/\sup\{|f(z)| : f \in HB(\alpha), \|f\|_{HB(\alpha)} \leq 1\} = 1/\|k_z\|_{(HB(\alpha))^*}.$$

Here $k_z : HB(\alpha) \to \mathbb{C}$ is the linear functional defined by evaluation at $z$, that is, $k_z f = f(z)$. There are many cases in which $\tilde{\upsilon}(z) = \upsilon(z)$. For instance, if $g$ is any harmonic function on $D$ and $\upsilon(z) = 1/\max_{|w|=|z|} |g(w)|$ is a weight, then $|f(z)| \leq \max_{|w|=|z|} |g(w)|$ whenever $\|f\| \leq 1$. Since $\|g\| \leq 1$, we find that $\tilde{\upsilon}(z) = \upsilon(z)$. This is the case of the harmonic Bloch-type spaces in which $\upsilon(z) = (1-|z|^2)^\alpha$ where $\alpha > 0$. In this case $g(z) = (1-z^2)^{-\alpha}$. Recall that the composition operator $C_\varphi : HB(\alpha) \to HB(\alpha)$ is bounded if and only if $\sup_{z\in D} \frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^\alpha}|\varphi^{'}(z)| < \infty$ and $C_\varphi : HB_0(\alpha) \to HB_0(\alpha)$ is bounded if and only if $\varphi \in B_0(\alpha)$ and $\sup_{z\in D} \frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^\alpha}|\varphi^{'}(z)|$ is finite.

In this paper we investigate the Fredholm composition operators on the harmonic Bloch-type spaces. Indeed, we provide some necessary and sufficient conditions for Fredholmness of composition operators. Also, we prove that the composition operator $C_\varphi$ on harmonic Bloch function spaces is Fredholm if and only if it is invertible if and only if $\varphi$ is disc automorphism.

## 2. Main results

In this section we are going to investigate composition operators on harmonic Bloch function spaces. First we prove that the point evaluation functionals are continuous on $HB(\alpha)$.

**Lemma 2.1.** *For every $z \in D$, the point evaluation $k_z : HB(\alpha) \to \mathbb{C}, f \to k_z(f) = f(z)$, is continuous and $\|k_z\|_{(HB(\alpha))^*} = \|k_z\|_{(HB_0(\alpha))^*}$.*

*Proof.* Let $\alpha = 1$, $z, w \in D$ and $z \neq w$. Then there exists a positive constant $C$ such that

$$|f(z)| - |f(w)| \leq C\pi \|f\|_{HB(\alpha)} \max\{\frac{|z-w|}{(1-|z|)^{\frac{1}{2}}}, \frac{|z-w|}{(1-|w|)^{\frac{1}{2}}}\},$$

for all $f \in HB(\alpha)$. Hence

$$\|k_z - k_w\|_{(HB(\alpha))^*} \leq C\pi \max\{\frac{|z-w|}{(1-|z|)^{\frac{1}{2}}}, \frac{|z-w|}{(1-|w|)^{\frac{1}{2}}}\},$$

for all $z, w \in D$ with $z \neq w$. This implies that $z \mapsto k_z$ is continuous. Moreover, for $\alpha \geq \lambda > 0, \alpha > 1$,

$$\|k_z - k_w\|_{(HB(\alpha))^*} \leq C\pi |z-w| \beta(1-\lambda, 1+\lambda-\alpha),$$

in which $\beta(.,.)$ is the Beta function. Since $b(\alpha) \supset b_0(\alpha)$, then

$$\sup\{|f(z)| : f \in b(\alpha)\} \geq \sup\{|f(z)| : f \in b_0(\alpha)\},$$

for each $z \in D$. This means that $\|x\|_{(HB_0(\alpha))^*} \leq \|x\|_{(HB(\alpha))^*}$, for all $x \in (HB(\alpha))^*$. On the other hand,

$$\|k_z\|_{(HB(\alpha))^*} = \sup\{|f(z)| : \|f\|_{HB(\alpha)} \leq 1, f \in HB(\alpha)\}.$$

So by Proposition 1.2.1 of [7] we find $f_0 \in b(\alpha)$ such that $|f_0(z)| = \sup\{|f(z)| : \|f\|_{HB(\alpha)} \leq 1\}$. We know that the closd unit ball of $HB_0(\alpha)$ is dense in the closed unit ball of $HB(\alpha)$ under the compact open topology ($co$). Hence $f_0 \in \overline{b(\alpha)}^{co}$. Therefore, for each $f \in HB(\alpha), \|f\|_{HB(\alpha)} \leq 1$, there exists $\{g_i\}, g_i \in HB_0(\alpha), \|g_i\|_{HB(\alpha)} \leq 1$ such that $g_i \to f$ in the compact open topology. Hence $g_i(z) \to f(z)$ and so $k_z(g_i) \to k_z(f)$. Since $|k_z(g_i)| \leq \|k_z\|_{(HB_0(\alpha))^*}$,

$$|k_z(f)| = \lim_{z \in D} |k_z(g_i)| \leq \|k_z\|_{(HB_0(\alpha))^*}.$$

Consequently,

$$\|k_z\|_{(HB(\alpha))^*} \leq \|k_z\|_{(HB_0(\alpha))^*}.$$

□

Here we obtain another technical Lemma for point evaluation functionals.

**Lemma 2.2.** *The following hold for point evaluation functionals:*
*a)* $\lim_{|z|\to 1} \|k_z\|_{(HB(\alpha))^*} = \lim_{|z|\to 1} \|k_z\|_{(HB_0(\alpha))^*} = \infty$.
*b)* $\lim_{|z|\to 1}(k_z(f)/\|k_z\|_{(HB(\alpha))^*}) = 0$ *for all* $f \in HB_0(\alpha)$.

*Proof.* a) By Proposition 1.1 of [1] and Lemma 2.2 we get the result.

b) Let $P$ be a polynomial. Then

$$\lim_{|z|\to 1} \frac{|k_z(P)|}{\|k_z\|_{(HB(\alpha))^*})} \leq \sup\{|P(z)| : z \in \bar{D}\} \lim_{|z|\to 1} \frac{1}{\|k_z\|_{(HB(\alpha))^*})} = 0.$$

Since $\{k_z/\|k_z\|_{(HB(\alpha))^*}) : z \in D\}$ is equi-continuous in $(HB_0(\alpha))^*$ and polynomials are dense in $(HB_0(\alpha))^*$, we have

$$\lim_{|z|\to 1}(k_z(f)/\|k_z\|_{(HB(\alpha))^*}) = 0, \quad \text{for all} \quad f \in HB_0(\alpha).$$

$\square$

In the sequel we begin to investigate the Fredholm composition operators.

**Lemma 2.3.** *Let analytic self-map $\varphi : D \to D$ belongs to $B_0(\alpha)$. The composition operator $C_\varphi : HB(\alpha) \to HB(\alpha)$ is Fredholm if and only if $C_\varphi : HB_0(\alpha) \to HB_0(\alpha)$ is Fredholm.*

*Proof.* It is a direct consequence of the fact that $HB(\alpha)$ is the bi-dual of $HB_0(\alpha)$ and $C_\varphi : HB(\alpha) \to HB(\alpha)$ coincides with the bi-adjoint map of $C_\varphi : HB_0(\alpha) \to HB_0(\alpha)$, whenever both operators are well defined. $\square$

We recall that a harmonic function $\varphi : D \to D$ is called univalent if it is injective. In the next theorem we prove that the Fredholmeness of $C_\varphi$ implies that $\varphi$ is univalent.

**Theorem 2.4.** *Let $\varphi : D \to D$ be an analytic function. If $C_\varphi : HB(\alpha) \to HB(\alpha)$ is Fredholm, then $\varphi$ is univalent.*

*Proof.* We know that point evaluations $k_z$ are bounded linear functionals on the harmonic Bloch-type spaces, and for adjoint composition operator we have $C^*_\varphi(k_z) = k_{\varphi(z)}$, for $z \in D$. We claim that for every point in $D$ there is a neighborhood where $\varphi$ is univalent. Otherwise, there exists $z \in D$ and there are disjoint infinite sequences $\{u_n\}$ and $\{\upsilon_n\}$ in $D$ such that $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \upsilon_n = z$ and $\varphi(u_n) = \varphi(\upsilon_n)$. For $n \in N$, we may define $\ell_n := k_{u_n} - k_{\upsilon_n} \in (HB(\alpha))^*$. So for $f \in HB(\alpha)$,

$$\begin{aligned} C^*_\varphi(\ell_n)(f) &= \langle k_{u_n} - k_{\upsilon_n}, C_\varphi(f)\rangle \\ &= f(\varphi(u_n)) - f(\varphi(\upsilon_n)) = 0. \end{aligned}$$

Hence, $\{\ell_n\} \subseteq KerC^*_\varphi$. Direct computations show that $\{\ell_n\}$ is an infinite linearly independent sequence in $(HB(\alpha))^*$. Moreover, since $C_\varphi$ is Fredholm, then $C^*_\varphi$ is also Fredholm and $dim(KerC^*_\varphi) < \infty$. This is a contradiction.

Note that $\varphi$ cannot be constant. Now we show that $\varphi$ is univalent. Suppose on the contrary. Then there are $z, w \in D$, $z \neq w$ such that $\varphi(z) = \varphi(w) := t$. So there are $B_z = B(z, r_1)$ and $B_w = B(w, r_2)$, $B_z \cap B_w = \emptyset$, such that $\varphi|B_z$ and $\varphi|B_w$ are univalent. By Open mapping theorem we get that $\varphi(B_z)$ and $\varphi(B_w)$ are open in $D$. Hence $\varphi(B_z) \cap \varphi(B_w)$ is open in $D$ and it is an open neighborhood of $t$. Therefore

$$\exists\{u_n\} \subseteq B_z, \qquad \exists\{\upsilon_n\} \subseteq B_w$$

such that $\varphi(u_n) = \varphi(\upsilon_n)$. We define $\{\ell_n\}$ as above. It is an infinite linearly independent sequence in $KerC_\varphi^*$. This is a contradiction, since $C_\varphi^*$ is Fredholm. $\square$

In the next theorem we prove that if $C_\varphi$ is Fredholm, then $\varphi$ is an automorphism.

**Theorem 2.5.** *Let $\varphi : D \to D$ be analytic function and $\varphi \in B_0(\alpha)$. For $\alpha = 1$, if $C_\varphi : HB(\alpha) \to HB(\alpha)$ or $C_\varphi : HB_0(\alpha) \to HB_0(\alpha)$ is Fredholm, then $\varphi$ is an automorphism.*

*Proof.* By Lemma 2.3 we know that the Fredholmness of $C_\varphi : HB_0(\alpha) \to HB_0(\alpha)$ is equivalent to the Fredholmness of $C_\varphi : HB(\alpha) \to HB(\alpha)$. So it is enough to consider only the cace $C_\varphi : HB_0(\alpha) \to HB_0(\alpha)$. If $C_\varphi$ is Fredholm, then $C_\varphi^*$ is also a Fredholm operator. From ([6], Chapter III, Theorem 13), there are bounded operators $S$ and $K$ on $(HB_0(\alpha))^*$, with $K$ compact, such that $SC_\varphi^* = I + K$. Theorem 2.4 implies that $\varphi$ is univalent. Hence it suffices to prove that $\varphi$ is an inner function to show that $\varphi$ is an automorphism. For $z \in D$, we set $\nu_z = \frac{k_z}{\|k_z\|_{(HB(\alpha))^*}}$. By Lemma 2.2, $\nu_z$ weakly converges to 0 in $(HB_0(\alpha))^*$ as $|z|$ goes to 1 and $\nu_z$ has norm one. Since $K$ is compact operator, $\|K\nu_z\|_{(HB(\alpha))^*} \to 0$ as $|z| \to 1$. Hence there exists $r$, $0 < r < 1$, such that $\|K\nu_z\|_{(HB(\alpha))^*} < \frac{1}{2}$ for all $z \in D$ with $r < |z| < 1$. Since $C_\varphi^* k_z = k_{\varphi(z)}$,

$$\begin{aligned}
1 - \|K\nu_z\| &= \|\nu_z\| - \|K\nu_z\| \\
&\leq \|\nu_z + K\nu_z\| \\
&= \|SC_\varphi^*\nu_z\| \\
&\leq \|S\| \frac{k_{\varphi(z)}}{\|k_z\|}.
\end{aligned}$$

For $z \in D$, with $r < |z| < 1$, we have

$$\begin{aligned}
\|S\| \frac{k_{\varphi(z)}}{\|k_z\|} &\geq 1 - \|K\nu_z\| \\
&\geq 1 - \frac{1}{2} = \frac{1}{2}.
\end{aligned}$$

Hence for $z \in D$ with $r < |z| < 1$,

$$\frac{1}{2}\|k_{\varphi(z)}\| \leq \frac{\|S\|}{\|k_z\|}.$$

From Lemma 2.2, we know that $\lim_{|z|\to 1} \|k_z\| = \infty$. Hence,

$$\lim_{|z|\to 1} \frac{1}{\|k_{\varphi(z)}\|} = 0.$$

It follows that $\lim_{|z|\to 1} |\varphi(z)| = 1$ and $\varphi$ is an inner function. $\square$

Here we obtain that if $\varphi$ is an automorphism, then the inverse of $C_\varphi$ is again a composition operator.

**Theorem 2.6.** *If $\varphi$ is an automorphism on $D$ and $C_\varphi : HB_0(\alpha) \to HB_0(\alpha)$ is bounded, then $C_{\varphi^{-1}} : HB_0(\alpha) \to HB_0(\alpha)$ is bounded and $C_\varphi^{-1} = C_{\varphi^{-1}}$.*

*Proof.* For the authomorphism $\varphi$, there are $\theta \in \mathbb{R}$ and $a \in D$ such with $\varphi(z) = e^{i\theta}\varphi_a(z)$, where $\varphi_a(z) = \frac{(a-z)}{1-\bar{a}z}$. Hence $\varphi^{-1}(z) = \varphi_a(e^{-i\theta}z)$. Since $\tilde{\nu}$ is radial,

$$\begin{aligned}\sup_{w\in D} \frac{(1-|w|^2)^\alpha}{(1-|\varphi^{-1}(w)|^2)^\alpha} &= \sup_{z\in D} \frac{(1-|\varphi(z)|^2)^\alpha}{(1-|z|^2)^\alpha} \sup_{z\in D} \frac{(1-|\varphi_a(z)|^2)^\alpha}{(1-|z|^2)^\alpha} \\ &= \sup_{z\in D} \frac{(1-|z|^2)^\alpha}{(1-|\varphi_a(z)|^2)^\alpha} \\ &= \sup_{z\in D} \frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^\alpha}.\end{aligned}$$

Moreover, boundedness of $C_\varphi : HB_0(\alpha) \to HB_0(\alpha)$ implies that $C_{\varphi^{-1}} : HB_0(\alpha) \to HB_0(\alpha)$ is also bounded and easiliy we get that $C_\varphi^{-1} = C_{\varphi^{-1}}$. □

Our results up to here give us tha if we replace $HB_0(\alpha)$ by $HB(\alpha)$ in Theorem 2.6, the results come true. In the next theorem we find that $C_\varphi$ is Fredholm if and only if it is invertible.

**Theorem 2.7.** *Let the analytic function $\varphi : D \to D$ belonges to $\varphi \in B_0(\alpha)$. Then for $\alpha = 1$ and bounded operator $C_\varphi$ on $HB_0(\alpha)$ or $HB(\alpha)$, the following assertions are equivalent:*

(1)$C_\varphi : HB_0(\alpha) \to HB_0(\alpha)$ *is Fredholm.*
(2)$C_\varphi : HB(\alpha) \to HB(\alpha)$ *is Fredholm.*
(3)$\varphi$ *is an automorphism.*
(4)$C_\varphi : HB_0(\alpha) \to HB_0(\alpha)$ *is invertible.*
(5)$C_\varphi : HB(\alpha) \to HB(\alpha)$ *is invertible.*

*Proof.* Since $(HB_0(\alpha))^{**} = HB(\alpha)$ and $C_\varphi : HB(\alpha) \to HB(\alpha)$ is the bi-adjoint map of $C_\varphi : HB_0(\alpha) \to HB_0(\alpha)$, then $C_\varphi : HB(\alpha) \to HB(\alpha)$ is Fredholm if and only if $C_\varphi : HB_0(\alpha) \to HB_0(\alpha)$ is as well. Hence we have the implication (1) $\Leftrightarrow$ (2). The implications (1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (3) are direct sequence of Theorem 2.5. Also, the implications (3) $\Rightarrow$ (1) and (3) $\Rightarrow$ (2) are direct sequence of the fact that $C_\varphi^{-1} = C_{\varphi^{-1}}$ is bounded. Moreover, the implications (3) $\Rightarrow$ (4) and (3) $\Rightarrow$ (5) follow by the Theorem 2.6 . The implications (4) $\Rightarrow$ (1) and (5) $\Rightarrow$ (2) are clear.

□

In the next theorem we get that if $\varphi$ is univalent, then $C_\varphi$ has a closed range if and only if $\varphi$ is a disc automorphism.

**Theorem 2.8.** *Let$\alpha > 0, \alpha \neq 1$, and let $\varphi$ is univalent self-map of $D$. Then $C_\varphi$ has a closed range on $HB(\alpha)$ if and if $\varphi$ be a disk-automorphism.*

*Proof.* The proof is similar [8]. □

**Corollary 2.9.** *Let$\alpha > 0, \alpha \neq 1$. The composition operator $C_\varphi : HB(\alpha) \to HB(\alpha)$ is Fredholm if and only if $\varphi$ is a disk automorphism, i.e. if and only if $C_\varphi$ is invertible.*

## Acknowledgments

The authors would like to thank the reviewers for their thoughtful comments and efforts towards improving our manuscript.

## References

1. J. Bonet, P. Domanski, M. Lindstrom, J. Taskinen, Composition Operators Between Weithed Banach Spaces of Analytic Functions, *J. Austral. Math. Soc*, **64**, (1998), 101-118.
2. M. Carmen Gomes-Collado, D. Jornet, Fredholm Weighted Composition Operators on Weighted Banach Spaces of Analytic Functions of Type $H^\infty$, *J. Function Spaces*, (2015), 1-6.
3. F. Colonna, The Bloch Constant of Bounded Harmonic Mappings, *Indiana Univ. Math. J*, **38**, (1989), 829-840.
4. C. Cowen, B. MacCluer, *Composition Operators on Spaces of Analytic Functions*, CRC Press, Boca Raton, 1995.
5. P. Duren, *Harmonic Mapping in the Plane*, Cambridge Univ. Press, 2004.
6. V. Muller, *Spectral Theory of Linear Operators and Spectral Systems in Banach Algebras*, Birkhaasser, Basel, Switzerland, 2007.
7. A. M. Zarco Garcia, *Weighted Banach Spaces of Harmonic Function*, thesis, 2015.
8. N. Zorboska, Fredholm and Semi-Fredholm Composition Operators on the Small Bloch-type Spaces, *Integr. Equ. Oper. Theory*, **75**, (2013), 559–571.