Iranian Journal of Mathematical Sciences and Informatics
Vol. 6, No. 1 (2011), pp 59-77
DOI: 10.7508/ijmsi.2011.01.006

On (Semi)Topological BL-algebras

R. A. Borzooei[a,*], G. R. Rezaei[b] and N. Kouhestani[b]

[a]Department of Mathematics, Shahid Beheshti University, Tehran, Iran
[b]Department of Mathematics, Sistan and Balouchestan University, Zahedan, Iran

E-mail: borzooei@sbu.ac.ir
E-mail: grezaei@hamoon.usb.ac.ir
E-mail: kouhestani@hamoon.usb.ac.ir

Abstract. In last ten years many mathematicians have studied properties of BL-algebras endowed with a topology. For example A. Di Nola and L. Leustean [5] studied compact representations of BL-algebras, L. C. Ciungu [4] investigated some concepts of convergence in the class of perfect BL-algebras, J. Mi Ko and Y. C. Kim [13] studied relationships between closure operators and BL-algebras, M.Haveshki, E. Eslami and A. Broumand Saeid [9] applied filters to construct a topology on BL-algebras.In this paper we define semitopological and topological BL-algebras and derive here conditions that imply a BL-algebra to be a semitopological or topological BL-algebra.

Keywords: BL-algebra, (semi)topological BL-algebra, filter, Hausdorff space, T_i-spaces.

2000 Mathematics subject classification: 06B10, 03G10.

*Corresponding Author

Received 28 May 2010; Accepted 22 December 2010

1. Introduction

Algebra and topology, the two fundamental domains of mathematics, play complementary roles. Topology studies continuity and convergence and provides a general framework to study the concept of a limit. Much of topology is devoted to handing infinite sets and infinity itself; the methods developed are qualitative and, in a certain sense, irrational. Algebra studies all kinds of operations and provides a basis for algorithms and calculations.
Because of this difference in nature, algebra and topology have a strong tendency to develop independency, not in direct contact with each other. However, in applications, in higher level domains of mathematics, such as functional analysis, dynamical systems, representation theory, and others, topology and algebra come in contact most naturally. Many of the most important objects of mathematics represent a blend of algebraic and of topological structures. Topological function spaces and linear topological spaces in general, topological groups and topological fields, transformation groups, topological lattices are objects of this kind. Very often an algebraic structure and a topology come naturally together; this is the case when they are both determined by the nature of the elements of the set considered. The rules that describe the relationship between a topology and algebraic operation are almost always transparent and natural-the operation has to be continuous, jointly continuous, jointly or separately. In the 20th century many topologists and algebraists have contributed to topological algebra. Some outstanding mathematicians were involved, among them J. Dieudonné, L. S. Pontryagin, A. Weyl.
BL-algebras have been introduced by Hájek [8] in order to investigate many-valued logic by algebraic means. His motivations for introducing BL-algebras were of two kinds. The first one was providing an algebraic counterpart of a propositional logic, called Basic Logic, which embodies a fragment common to some of the most important many-valued logics, namely Lukasiewicz Logic, Gödel Logic and Product Logic. This Basic Logic (BL for short) is proposed as "the most general" many-valude logic with truth values in [0,1] and BL-algebras are the corresponding Lindenbaum-tarski algebras. The second one was to provide an algebraic mean for the study of continuous t-norms (or triangular norms) on [0,1].
In section 3 of this note, we define semitopological and topological $BL-$algebras, and we state and prove some theorems that determine the relationships between them. It is quite clear that a topological BL-algebra is a semitopological BL-algebra, but the converse is not true. In this paper we find certain conditions under which the converse is true. In section 4 we deal with relations between T_i spaces and BL-algebras endowed with a topology. We bring a condition that T_1 spaces are equivalent to Hausdorff spaces on BL-algebras endowed with a topology.

2. Preliminary

Recall that a set A with a family $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of its subsets is called a *topological space*, denoted by $(A,\mathcal{U})$, if $A, \emptyset \in \mathcal{U}$, the intersection of any finite numbers of members of $\mathcal{U}$ is in $\mathcal{U}$ and the arbitrary union of members of $\mathcal{U}$ is in $\mathcal{U}$. The members of $\mathcal{U}$ are called *open sets* of A and the complement of $A \in \mathcal{U}$, that is $A \setminus U$, is said to be a *closed set*. If B is a subset of A, the smallest closed set containing B is called the *closure* of B and denoted by $\overline{B}$ (or $cl_u B$). A subset P of A is said to be a *neighborhood* of $x \in A$, if there exists an open set U such that $x \in U \subseteq P$. A subfamily $\{U_\alpha\}$ of $\mathcal{U}$ is said to be a *base* of $\mathcal{U}$ if for each $x \in U \in \mathcal{U}$ there exists an $\alpha \in I$ such that $x \in U_\alpha \subseteq U$, or equivalently, each U in $\mathcal{U}$ is the union of members of $\{U_\alpha\}$. A subfamily $\{U_\beta\}$ of $\mathcal{U}$ is said to form a *subbase* for $\mathcal{U}$ if the family of finite intersections of members of $\{U_\beta\}$ forms a base of $\mathcal{U}$. Let $\mathcal{U}_x$ denote the totality of all neighborhoods of x in A. Then a subfamily $\mathcal{V}_x$ of $\mathcal{U}_x$ is said to form a *fundamental system* of neighborhoods of x, if for each U_x in $\mathcal{U}_x$, there exists a V_x in $\mathcal{V}_x$ such that $V_x \subseteq U_x$.

Definition 2.1. [11] Let $(A,\mathcal{U})$ be a topological space. We have the following separation axioms in $(A,\mathcal{U})$:
$\mathbf{T_0}$: For each $x, y \in A$ and $x \neq y$, there is at least one in an open neighborhood excluding the other.
$\mathbf{T_1}$: For each $x, y \in A$ and $x \neq y$, each has an open neighborhood not containing the other.
$\mathbf{T_2}$: For each $x, y \in A$ and $x \neq y$, both have disjoint open neighborhoods U, V such that $x \in U$ and $y \in V$.
$\mathbf{T_3}$: If C is any closed subset of $(A,\mathcal{U})$ and $x \in A$ such that $x \notin C$, then there exist disjoint open sets U, V such that $x \in U$ and $C \subseteq V$.
$\mathbf{T_4}$: If C and x are as in T_3, then there exists a real valued function $f : A \to [0, 1]$ such that $f(x) = 0$ and $f(C) = 1$.
$\mathbf{T_5}$: If C and D are two disjoint closed subsets of A, then there exist two disjoint open subsets U and V such that $C \subseteq U$ and $D \subseteq V$.

A topological space satisfying T_i is called a *T_i-space*. A T_2-space is also known as a *Hausdorff space*. A T_1-space satisfying T_i, $i = 3, 4, 5$, will be called *regular, completely regular* and *normal*, respectively. A topological space $(A,\mathcal{U})$ is said to be *compact*, if each open covering of A is reducible to a finite open covering, *locally compact*, if for each $x \in A$ there exist an open neighborhood U of x and a compact subset K such that $x \in U \subseteq K$.

Proposition 2.2. [11] (i) If $(A,\mathcal{U})$ is a Hausdorff space, then $(A,\mathcal{U})$ is locally compact if and only if for each $x \in A$ there exists an open neighborhood U of x such that $\overline{U}$ is compact.
(ii) A locally compact Hausdorff topological space A is normal if it is the union

of an increasing sequence $\{U_n\}$ of open sets such that $\overline{U_n}$ is compact for each $n \in \mathbb{N}$.

Definition 2.3. [11] Let $(A, *)$ be an algebra of type 2 and $\mathcal{U}$ be a topology on A. Then $\mathcal{A} = (A, *, \mathcal{U})$ is called a
(i) *Right (left) topological algebra*, if for all $a \in A$ the map $* : A \to A$ is defined by $x \to a * x$ ($x \to x * a$) is continuous, or equivalently, for any x in A and any open set U of $a * x$ $(x * a)$, there exists an open set V of x such that $a * V \subseteq U$ $(V * a \subseteq U)$.
(ii) *Semitopological algebra*, if $\mathcal{A}$ is a right and left topological algebra.
(iii) *Topological algebra*, if the operation $*$ is continuous, or equivalently, if for any x, y in A and any open set (neighborhood) W of $x * y$ there exist two open sets (neighborhoods) U and V of x and y, respectively, such that $U * V \in W$.

Proposition 2.4. Let $(A, *)$ be a algebra of type 2 and $\mathcal{U}$ be a topology on A.
(i) If $(A, *, \mathcal{U})$ is a finite semitopological algebra, then it is a topological algebra.
(ii) If $(A, *)$ is commutative algebra, then right and left topological algebras are equivalent. Moreover $(A, *, \mathcal{U})$ is a semitopological algebra iff it is right or left topological algebra.

Definition 2.5. Let A be a nonempty set and $\{*_i\}_{i \in I}$ be a family of operations of type 2 on A and $\mathcal{U}$ be a topology on A. Then
(i) $(A, \{*_i\}_{i\in I}, \mathcal{U})$ is a right(left) topological algebra, if for any $i \in I$, $(A, *_i, \mathcal{U})$ is a right (left) topological algebra.
(ii) $(A, \{*_i\}_{i\in I}, \mathcal{U})$ is a semitopological (topological) algebra if for all $i \in I$, $(A, *_i, \mathcal{U})$ is a semitopological (topological) algebra.

Definition 2.6. [8] A *BL-algebra* is an algebra $\mathcal{A} = (A, \wedge, \vee, \odot, \to, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ such that $(A, \wedge, \vee, 0, 1)$ is a bounded lattice, $(A, \odot, 1)$ is a commutative monoid and for any $a, b, c \in A$

$$c \leq a \to b \Leftrightarrow a \odot c \leq b, \quad a \wedge b = a \odot (a \to b), \quad (a \to b) \vee (b \to a) = 1$$

Let A be a BL-algebra. We define $a' = a \to 0$ and denote $(a')'$ by a''. The map $c : A \to A$ by $c(a) = a'$, for any $a \in A$, is called the *negation map*. Also, we define $a^0 = 1$ and $a^n = a^{n-1} \odot a$ for all natural numbers n.

Notation: Let A be a BL-algebra and $B \subseteq A$. For any $a \in A$, we define $B \odot a, a \odot B, B \to a$ and $a \to B$ as follows:

$$B \odot a = \{x \odot a : x \in B\}, a \odot B = \{a \odot x : x \in B\}, B \to a = \{x \to a : x \in B\},$$

$$a \to B = \{a \to x : x \in B\}$$

Definition 2.7. [8] A filter of A is a nonempty set $F \subseteq A$ such that $x, y \in F$ implies $x \odot y \in F$ and if $x \in F$ and $x \leq y$ imply $y \in F$, for any $x, y \in A$.

Proposition 2.8. [3]Let A be a BL-algebra. Then
(i) If $1 \in F \subseteq A$, then F is a filter if and only if $x \in F$ and $x \to y \in F$ imply $y \in F$.
(ii) If F is a filter in A, then for each $x, y \in F$, $x \wedge y$, $x \vee y$ and $x \to y$ are in F.

Example 2.9. (i) *Let "$\odot$" and "$\to$" on the real unit interval $I = [0, 1]$ be defined as follows:*

$$x \odot y = \min\{x, y\}$$

$$x \to y = \begin{cases} 1 & , x \leq y, \\ y & , \textit{otherwise}. \end{cases}$$

Then $\mathcal{I} = (I, \min, \max, \odot, \to, 0, 1)$ is a BL-algebra.
(ii) *Let $\odot$ be the usual multiplication of real numbers on the unit interval $I = [0, 1]$ and $x \to y = 1$ iff $x \leq y$ and y/x otherwise. Then $\mathcal{I} = (I, \min, \max, \odot, \to, 0, 1)$ is a BL-algebra.*

Proposition 2.10. [8] Let A be a BL-algebra. The following properties hold.
(B_1) $x \odot y \leq x, y$ and $x \odot 0 = 0$,
(B_2) $x \leq y$ implies $x \odot z \leq y \odot z$,
(B_3) $x \leq y$ iff $x \to y = 1$,
(B_4) $1 \to x = x$, $x \to x = 1$, $x \to 1 = 1$ and $1 \odot x = x$,
(B_5) $y \leq x \to y$,
(B_6) $x \to (y \to z) = (x \odot y) \to z = y \to (x \to z)$,
(B_7) $1' = 0$ and $0' = 1$,
(B_8) $x' = 1 \Leftrightarrow x = 0$,
(B_9) $(x \wedge y)' = x' \vee y'$ and $(x \vee y)' = x' \wedge y'$,
(B_{10}) $(x \wedge y)'' = x'' \vee y''$ and $(x \vee y)'' = x'' \wedge y''$,
(B_{11}) $(x \odot y)'' = x'' \odot y''$ and $(x \to y)'' = x'' \to y''$,
(B_{12}) $x \to y' = y \to x' = (x \odot y)' = x'' \to y'$,
(B_{13}) $x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$,
(B_{14}) $x''' = x'$,
(B_{15}) $((x \to y) \to y) \to y = x \to y$.

Note. From now on, in this paper we let A be a BL-algebra and $\mathcal{U}$ be a topology on A, unless otherwise state.

3. (Semi)Topological BL-algebras

In this section we state and prove some theorems about semitopological and topological BL-algebras and we investigate some conditions that imply a semitopological BL-algebra is a topological BL-algebra.

Definition 3.1. (i) Let $(A, \{*_i\}, \mathcal{U})$, where $\{*_i\} \subseteq \{\wedge, \vee, \odot, \to\}$, be a semitopological (topological) algebra. Then $(A, \{*_i\}, \mathcal{U})$ is called a semitopological (topological) BL-algebra.
(ii) Let F be a filter in A and $\mathcal{U}_F$ be a subspace topology of $(A, \mathcal{U})$. If $(F, \mathcal{U}_F)$ is

semitopological (topological) algebra, then $(F,\mathcal{U}_F)$ is called a semitopological (topological) filter.

Remark 3.2. If $\{*_i\} = \{\wedge, \vee, \odot, \rightarrow\}$, we consider $\mathcal{A} = (A,\mathcal{U})$ instate of $(A, \{\wedge, \vee, \odot, \rightarrow\}, \mathcal{U})$, for simplicity.

Proposition 3.3. Let $(A, \{\odot, \rightarrow\}, \mathcal{U})$ be a semitopological BL-algebra. Then $(A, \wedge, \mathcal{U})$ is a semitopological BL-algebra. Moreover the negation map c is continuous and if c is one-to-one, then c is homeomorphism (i.e, c and c^{-1} are continuous).

Proof. Let $x, y \in A$, $U \in \mathcal{U}$ and $x \wedge y \in U$. Since $(A, \odot, \mathcal{U})$ is a right topological BL-algebra and $x \odot (x \rightarrow y) = x \wedge y \in U$, there is a $W \in \mathcal{U}$ such that $x \rightarrow y \in W$ and $x \odot W \subseteq U$. Since $(A, \rightarrow, \mathcal{U})$ is a right topological BL-algebra, there exists $V \in \mathcal{U}$ such that $y \in V$ and $x \rightarrow V \subseteq W$. Hence

$$x \wedge y \in x \wedge V = x \odot (x \rightarrow V) \subseteq x \odot W \subseteq U$$

which implies that $(A, \wedge, \mathcal{U})$ is a right topological BL-algebra. Since $\wedge$ is commutative, by Proposition 2.4(ii), $(A, \wedge, \mathcal{U})$ is a semitopological BL-algebra. Now we prove negation map $c : A \rightarrow A$ is continuous. For this let $x' \in U \in \mathcal{U}$, where $x \in A$. Since $x' = x \rightarrow 0$ and $(A, \rightarrow, \mathcal{U})$ is a semitopological BL-algebra, there is a $V \in \mathcal{U}$ such that x is in V and $V' = V \rightarrow 0 \subseteq U$.
Now let c is one-to-one, we show that c^{-1} is continuous. By B_{14}, for each $x \in A$, $c(x'') = x''' = x' = c(x)$, so $x'' = x$ and c is onto. But c=c^{-1} because for each $x \in A$, $cc^{-1}(x) = x = x'' = cc(x)$, which implies that $c^{-1}(x) = c(x)$. □

Proposition 3.4. Let $(A, \{\wedge, \rightarrow\}, \mathcal{U})$ be a topological BL-algebra. Then $(A, \vee, \mathcal{U})$ is a topological BL-algebra.

Proof. Let $x \vee y \in U \in \mathcal{U}$. Then by B_{13}, $((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x) = x \vee y \in U$. Since $\wedge$ is continuous, there exist $W_1, W_2 \in \mathcal{U}$ such that $(x \rightarrow y) \rightarrow y) \in W_1$, $(y \rightarrow x) \rightarrow x) \in W_2$ and $W_1 \wedge W_2 \subseteq U$. Since $(A, \rightarrow, \mathcal{U})$ is a topological BL-algebra, there are $V_1, V_2 \in \mathcal{U}$ such that $x \rightarrow y \in V_1$, $y \in V_2$ and $V_1 \rightarrow V_2 \subseteq W_1$. Also there are $V_3, V_4 \in \mathcal{U}$ such that $x \in V_3, y \in V_4$ and $V_3 \rightarrow V_4 \subseteq V_1$. Now, let $V = V_4 \cap V_2$. Then $x \in V_3$, $y \in V$ and $(V_3 \rightarrow V) \rightarrow V \subseteq (V_3 \rightarrow V_4) \rightarrow V_2 \subseteq V_1 \rightarrow V_2 \subseteq W_1$. Similarly, there are two open sets W_3 and W such that $y \in W_3$, $x \in W$ and $(W_3 \rightarrow W) \rightarrow W \subseteq W_2$. If $P = V_3 \cap W$ and $Q = V \cap W_3$, then $P, Q \in \mathcal{U}$ and $x \in P$, $y \in Q$ and

$$\begin{aligned} P \vee Q &\subseteq ((P \rightarrow Q) \rightarrow Q) \wedge ((Q \rightarrow P) \rightarrow P) \\ &\subseteq ((V_3 \rightarrow V) \rightarrow V) \wedge (W_3 \rightarrow W) \rightarrow W) \\ &\subseteq W_1 \wedge W_2 \subseteq U. \end{aligned}$$

Hence $(A, \vee, \mathcal{U})$ is a topological BL-algebra. □

Theorem 3.5. *Let $(A, \{\odot, \rightarrow\}, \mathcal{U})$ be a topological BL-algebra. Then $\mathcal{A} = (A, \mathcal{U})$ is a topological BL-algebra.*

Proof. By Proposition 3.4, it is enough to verify that $(A, \wedge, \mathcal{U})$ is a topological algebra. Since $\rightarrow$ is continuous, the mapping $f : (x, y) \mapsto (x, x \rightarrow y)$ from $A \times A$ into $A \times A$ is continuous. Now, since $\wedge = \odot \circ f$, $\wedge$ is continuous. Therefore, $\mathcal{A} = (A, \mathcal{U})$ is a topological BL-algebra. □

In the following Example we show that the converse of Proposition 3.3 and 3.4 is not correct.

Example 3.6. *Let $\mathcal{I}$ be the BL-algebra in Example 2.9 (i) and $\mathcal{U}$ be a topology on $\mathcal{I}$ with the base $S = \{[a, b) \cap \mathcal{I} : a, b \in \mathbb{R}\}$. Then $(\mathcal{I}, \{\wedge, \vee\}, \mathcal{U})$ is a topological BL-algebra, but $(\mathcal{I}, \rightarrow, \mathcal{U})$ is not a semitopological BL-algebra.*

Proof. Let $x \wedge y \in U \in \mathcal{U}$, where $x, y \in I$. W.L.O.G, let $x \leq y$. Then $x \wedge y = x \in U$. If $x \neq y$, then $U \cap [x, y)$ and $[y, 1]$ are two open set of x and y, respectively, and $(U \cap [x, y)) \wedge [y, 1] \subseteq U$. If $x = y$, then $U \cap [x, 1]$ is a open set of x and $((U \cap [x, 1]) \wedge (U \cap [x, 1])) \subseteq U$. Hence $(\mathcal{I}, \wedge, \mathcal{U})$ is a topological BL-algebra. By the similar argument we can shows that $(\mathcal{I}, \vee, \mathcal{U})$ is a topological BL-algebra. Therefore, $(\mathcal{I}, \vee, \wedge, \mathcal{U})$ is a topological BL-algebra. Now we prove that $(\mathcal{I}, \rightarrow, \mathcal{U})$ is not a left topological BL-algebra. For this, let $0 \rightarrow 0 = 1 \in [1/2, 1]$ and let U be an arbitrary open set of 0. Then $U \rightarrow 0 = \{0, 1\} \nsubseteq [1/2, 1]$. □

By Proposition 2.4 (i), each finite semitopological BL-algebra is a topological BL-algebra. In the Examples 3.7, we introduce an infinite topological BL-algebra and in the Examples 3.8, we introduce a semitoplogical BL-algebra which is not a topological BL-algebra.

Example 3.7. *Let $\mathcal{I}$ be the BL-algebra in Example 2.9 (i) and $\mathcal{U}$ be a topology on $\mathcal{I}$ with the base $S = \{[a, b] \cap \mathcal{I} : a, b \in \mathbb{R}\}$. Then $(\mathcal{I}, \mathcal{U})$ is a topological BL-algebra.*

Proof. By Theorem 3.5, it is enough to prove that $(\mathcal{I}, \{\odot, \rightarrow\}, \mathcal{U})$ is a topological BL-algebra. Let $x \odot y \in U \in \mathcal{U}$, where $x, y \in I$. W.O.L.G, let $x \leq y$. Then $x \odot y = x \in U$. Since $x \in [0, x] \cap U \in \mathcal{U}$, $y \in [x, 1] \in \mathcal{U}$ and $([0, x] \cap U) \odot [x, 1] \subseteq U$, then $(\mathcal{I}, \odot, \mathcal{U})$ is a topological BL-algebra. Now, let $x \rightarrow y \in U \in \mathcal{U}$, where $x, y \in I$. If $x \leq y$, then $x \rightarrow y = 1 \in U$. So, $x \in [0, y] \in \mathcal{U}$ and $y \in [y, 1] \in \mathcal{U}$ and $[0, y] \rightarrow [y, 1] \subseteq U$. If $x > y$, then $x \rightarrow y = y \in U$. Thus $x \in [y, x] \in \mathcal{U}$, $y \in [0, y] \cap U \in \mathcal{U}$ and $[y, x] \rightarrow ([0, y] \cap U) \subseteq U$. Hence $(\mathcal{I}, \rightarrow, \mathcal{U})$ is a topological BL-algebra. Therefore, $(\mathcal{I}, \mathcal{U})$ is a topological BL-algebra.

□

Example 3.8. *Let $\mathcal{I}$ be the BL-algebra in Example 2.9 (ii) and $\mathcal{U}$ be a topology on $\mathcal{I}$ with the base $S = \{[a, b] \cap \mathcal{I} : a, b \in \mathbb{R}\}$. Then $(\mathcal{I}, \mathcal{U})$ is a semitopological BL-algebra which is not a topological BL-algebra.*

Proof. By Proposition 3.3, it is enough to prove that $(\mathcal{I}, \{\odot, \rightarrow, \vee\}, \mathcal{U})$ is a semitopological BL-algebra. Let $x \odot y = xy \in [a, b] \in \mathcal{U}$, where $x, y \in I$.

If $y = 0$, then $x \odot y = 0 \in [a, b]$ and so $a = 0$. Hence $x \in [0, x] \in \mathcal{U}$ and $[0, x] \odot y \subseteq [0, b]$. Let $y \neq 0$. Then $x \in [a/y, x] \in \mathcal{U}$ and $[a/y, x] \odot y \subseteq [a, b]$. Thus $(\mathcal{I}, \odot, \mathcal{U})$ is a semitopological BL-algebra. Now we prove that $(\mathcal{I}, \rightarrow, \mathcal{U})$ is a semitopological BL-algebra. Let $x \rightarrow y \in [a, b]$, where $x, y, a, b \in I$. If $x \leq y$, then $x \rightarrow y = 1 \in [a, b]$ and so $b = 1$. Hence $x \in [0, x] \in \mathcal{U}$, $y \in [x, 1] \in \mathcal{U}$, $[0, x] \rightarrow y \subseteq [a, 1]$ and $x \rightarrow [x, 1] \subseteq [a, 1]$. Now, let $x > y$. Then, $x \rightarrow y = y/x \in [a, b]$. If $y = 0$, then $a = 0$ and so $x \in [0, x] \in \mathcal{U}$, $y \in \{0\} \in \mathcal{U}$, $[0, x] \rightarrow y \subseteq [0, b]$ and $x \rightarrow \{0\} \subseteq [0, b]$. If $y \neq 0$, then we can assume $a \neq 0$. Hence, $x \in [y/a, y/b] \in \mathcal{U}$, $y \in [ax, bx] \in \mathcal{U}$, $[y/a, y/b] \rightarrow y \subseteq [a, b]$ and $x \rightarrow [ax, bx] \subseteq [a, b]$. So $(\mathcal{I}, \rightarrow, \mathcal{U})$ is a semitopological BL-algebra. Moreover, it is easy to prove that $(\mathcal{I}, \vee, \mathcal{U})$ is a semitopological BL-algebra. Therefore, $(\mathcal{I}, \mathcal{U})$ is a semitopological BL-algebra. Finally, we show that $(\mathcal{I}, \mathcal{U})$ is not a topological BL-algebra. For this, let $0 \rightarrow 0 = 1 \in [1/2, 1]$. Suppose $[0, b]$ be a arbitrary open set in $\mathcal{U}$ such that $0 \in [0, b]$. There is a $n \in \mathbb{N}$ such that $1/n < b$. Now $1/(n+1), 1/n \in [0, b]$, but $1/(n+1) \rightarrow 1/n = (n+1)/n \notin [1/2, 1]$.

□

Proposition 3.9. Let $(X, *)$ be an algebra of type 2 and $(X, *, \mathcal{U})$ be a (semi) topological algebra. If B is a subset of X such that for each $x, y \in B$, $x * y \in B$, then $(B, *, \mathcal{U}_B)$ is a (semi) topological algebra, where $\mathcal{U}_B$ is the subspace topology from X.

Proof. Let $x, y \in B$ and $x * y \in U \cap B \in \mathcal{U}_B$, where $U \in \mathcal{U}$. Since $x * y \in U$ and $(X, *, \mathcal{U})$ is a semitopological algebra, there are W_1, W_2 in $\mathcal{U}$ such that $x \in W_1$, $y \in W_2$, $W_1 * y \subseteq U$ and $x * W_2 \subseteq U$. Now $x \in W_1 \cap B \in \mathcal{U}_B$, $y \in W_2 \cap B \in \mathcal{U}_B$ and

$$(W_1 \cap B) * y \subseteq (W_1 * y) \cap B \subseteq U \cap B, \quad x * (W_2 \cap B) \subseteq (x * W_2) \cap B \subseteq U \cap B$$

Hence $(B, \mathcal{U}_B)$ is a left and right topological algebra and so $(B, \mathcal{U}_B)$ is a semitopological algebra. By the similar way, we can show that $(B, \mathcal{U}_B)$ is a topological algebra. □

Corollary 3.10. *Let $\mathcal{A} = (A, \mathcal{U})$ be a (semi) topological BL-algebra.*
(i) If F is a filter in A, then F is a (semi) topological filter, when it is endowed with the subspace topology induced by topology $\mathcal{U}$ on A.
(ii) $MV(A) = \{x \in A : x'' = x\}$ is a (semi) topological algebra, when it is endowed with the subspace topology induced by topology $\mathcal{U}$ on A.

Proof. (i) Let $x, y \in F$. Since F is a filter, $x \odot y \in F$. By Proposition 2.8(ii), $x \vee y$ and $x \rightarrow y$ are in F and so by Proposition 3.9, $(F, \mathcal{U}_F)$ is a (semi)topological filter.
(ii) Let $a, b \in MV(A)$. Then by (B_{10}) and (B_{11})

$$(a \wedge b)'' = a'' \wedge b'' = a \wedge b, (a \vee b)'' = a'' \vee b'' = a \vee b,$$

$$(a \rightarrow b)'' = a'' \rightarrow b'' = a \rightarrow b, (a \odot b)'' = a'' \odot b'' = a \odot b$$

and so $a \wedge b$, $a \vee b$, $a \odot b$ and $a \to b$ are in $MV(A)$. Hence by Proposition 3.9, $MV(A)$ is a (semi)topological algebra. $\square$

Theorem 3.11. *Let A be a BL-algebra which has no zero divisor i.e. for each $x, y \in A$ ($x \odot y = 0 \Rightarrow x = 0$ or $y = 0$). Let $\mathcal{U}$ be a topology on A and $\{0\}$ be an open and closed subset of A. Then $(A,\mathcal{U})$ is a semitopological BL-algebra if and only if*
(I) the negation map c is continuous.
(II) there exists a family $\mathcal{F} = \{F_i\}_{i \in I}$ of open proper filters in A such that
(i) $A \setminus \{0\} = \bigcup_{i \in I} F_i$,
(ii) for each $i, j \in I$, $F_i \subseteq F_j$ or $F_j \subseteq F_i$,
(iii) for each $F \in \mathcal{F}$, $(F,\mathcal{U}_F)$ is a semitopological filter, where $\mathcal{U}_F$ is the subspace topology on F.

Proof. ($\Rightarrow$) Let $(A,\mathcal{U})$ be a semitopological BL-algebra. Then by Proposition 3.3, the negation map $c : x \longrightarrow x'$ is continuous. This proves I. Now we prove II. Let $\mathcal{F} = \{A \setminus \{0\}\}$. It is easy to see $\mathcal{F}$ satisfies in (i), (ii) and $A \neq A \setminus \{0\}$. Since $\{0\}$ is closed, the set $A \setminus \{0\}$ is an open set in A. Now we prove that $A \setminus \{0\}$ is a filter. For this let $x \in A \setminus \{0\}$ and $x \to y \in A \setminus \{0\}$. Then since A has no zero divisor, $x \wedge y = x \odot (x \to y) \in A \setminus \{0\}$, which implies that $y \neq 0$. By Proposition 2.8 (i), the set $A \setminus \{0\}$ is a filter. By Corollary 3.10 (i), the set $A \setminus \{0\}$ is a semitopological filter, when it is endowed with the subspace topology.

($\Leftarrow$) Let us have I and II. We show that $(A,\mathcal{U})$ is a semitopological BL-algebra. By Proposition 3.4, it is enough to prove that $(A, \{\vee, \odot, \to\},\mathcal{U})$ is a semitopological BL-algebra. So, we consider the following cases:

Case 1: $(A, \odot,\mathcal{U})$ is a semitopological BL-algebra:
Let $x \odot y \in U \in \mathcal{U}$, where $x, y \in A$. We consider the following cases:
(1-1): If $x \odot y = 0$, then $x = 0$, *or* $y = 0$. If $x = 0$, then $\{0\}$ is an open set of 0 and $\{0\} \odot y = \{0\} \subseteq U$. If $y = 0$, then A is an open set of x and $A \odot y = \{0\} \subseteq U$.
(1-2): If $x \odot y \neq 0$, then by (i) there is a $F \in \mathcal{F}$ such that $x \odot y \in F$. By (B_1), $x \odot y \leq x$, which implies that $x \in F$. Since $x \odot y \in U \cap F$ and $(F,\mathcal{U}_F)$ is a semitopological filter, there is a $V \in \mathcal{U}$ such that $x \in V \cap F \in \mathcal{U}_F$ and $(V \cap F) \odot y \subseteq (U \cap F) \subseteq U$. Since $F \in \mathcal{U}$, then $x \in V \cap F \in \mathcal{U}$ and so $(A, \odot,\mathcal{U})$ is a left topological BL-algebra. Now, by Proposition 2.4 (ii), $(A, \odot,\mathcal{U})$ is a semitopological BL-algebra.

Case 2: $(A, \to,\mathcal{U})$ is a semitopological BL-algebra:
Let $x \to y \in U \in \mathcal{U}$ where $x, y \in A$. We consider the following cases:
(2-1): If $x = 0$, then $1 = 0 \to y = x \to y \in U$. Now $\{0\}$ and A are two open sets such that $x \in \{0\}$, $y \in A$, $\{0\} \to y = \{1\} \subseteq U$ and $x \to A = 0 \to A = \{1\} \subseteq U$.
(2-2): If $y = 0$, since the negation map $c : A \to A$ by $c(x) = x'$ is continuous, there is an open neighborhood V of x such that $V' \subseteq U$. Now $V \to 0 = V' \subseteq U$

and $x \to \{0\} \subseteq U$.
(2-3): If $x \neq 0$ and $y \neq 0$, then by $(i),(ii)$ there exist an $F \in \mathcal{F}$ such that $x, y \in F$. Since $x \to y \in U \cap F \in \mathcal{U}_F$ and $(F, \to, \mathcal{U}_F)$ is a semitopological filter, there are $V, W \in \mathcal{U}$ such that $x \in V \cap F$, $y \in W \cap F$, $(V \cap F) \to y \subseteq U \cap F \subseteq U$ and $x \to (W \cap F) \subseteq U \cap F \subseteq U$. Since $F \in \mathcal{U}$, the sets $V \cap F$ and $W \cap F$ are in $\mathcal{U}$ containing x, y, respectively. Hence $(A, \to, \mathcal{U})$ is a semitopological BL-algebra.

Case 3: $(A, \vee, \mathcal{U})$ is a semitopological BL-algebra:
Let $x \vee y \in U \in \mathcal{U}$, where $x, y \in A$. We consider the following cases:
(3-1): If $x = 0$, then $y = x \vee y \in U$. Now $\{0\}$ and U are two open sets such that $x \in \{0\}$ and $y \in U$ and $\{0\} \vee y = y \in U$.
(3-2): If $y = 0$, the proof is similar to the proof of *(3-1)*.
(3-3): If $x \neq 0$ and $y \neq 0$, then by $(i),(ii)$ there is a $F \in \mathcal{F}$ such that $x, y \in F$. Since $x \vee y \in U \cap F \in \mathcal{U}_F$ and $(F, \vee, \mathcal{U}_F)$ is a semitopological filter, there is a $V \in \mathcal{U}$ such that $x \in V \cap F$ and $(V \cap F) \vee y \subseteq U \cap F$. Now, since $F \in \mathcal{U}$ and $V \cap F \in \mathcal{U}$, $(A, \vee, \mathcal{U})$ is a left topological BL-algebra. Therefore, by Proposition 2.4 (ii), $(A, \vee, \mathcal{U})$ is a semitopological BL-algebra. □

Theorem 3.12. *Let $F \neq 1$ be a filter in A such that for any $x, y \in A$, $((F \odot x) \to (F \odot y)) \subseteq F \odot (x \to y)$, and let for each $x, y \in A \setminus \{0, 1\}$, $x \odot y \neq x, y$. Then, there is a non-trivial topology $\mathcal{U}$ on A such that $\mathcal{A} = (A, \mathcal{U})$ is a topological BL-algebra.*

Proof. Let $\mathcal{U} = \{U \subseteq A : \forall x \in U, \quad F \odot x \subseteq U\}$. For each $x \in A$, $F \odot x \in \mathcal{U}$, because if $y \in F \odot x$, then $F \odot y \subseteq F \odot F \odot x \subseteq F \odot x$. It is easy to see that $\mathcal{U}$ is a topology on A. We show that $\mathcal{U}$ is a non-trivial topology. If $\{x\} \in \mathcal{U}$ for some $x \in A \setminus \{0\}$, then $F \odot x = \{x\}$. Since $F \neq \{1\}$ and $x \neq 0$, there is a $y \in F$ such that $y \odot x = x$ and $y \neq 0, 1$, which is a contradiction with hypothesis. Also $F \neq A$, because for each $y < 1$, $(A \to (A \odot y)) \nsubseteq (A \odot y)$. This shows that $F \odot x \neq A$, for each $x \neq 1$. Now, in the following, we prove that $\mathcal{A} = (A, \mathcal{U})$ is a topological BL-algebra. By Theorem 3.5 it is enough to prove that $(A, \{\odot, \to\}, \mathcal{U})$ is a topological BL-algebra. For this we consider the following cases:

Case 1: $(A, \odot, \mathcal{U})$ is a topological BL-algebra:
Let $x \odot y \in U \in \mathcal{U}$, where $x, y \in A$. Then $F \odot (x \odot y) \subseteq U$. Now, $x \in F \odot x \in \mathcal{U}$, $y \in F \odot y \in \mathcal{U}$ and $x \odot y \in (F \odot x) \odot (F \odot y) \subseteq F \odot (x \odot y) \subseteq U$. Therefore, $(A, \odot, \mathcal{U})$ is a topological BL-algebra.

Case 2: $(A, \to, \mathcal{U})$ is a topological BL-algebra:
Let $x \to y \in U \in \mathcal{U}$, where $x, y \in A$. $F \odot x$ and $F \odot y$ are two open neighborhoods of x, y respectively and $x \to y \in (F \odot x) \to (F \odot y) \subseteq F \odot (x \to y) \subseteq U$. Hence $(A, \to, \mathcal{U})$ is a topological BL-algebra. □

Theorem 3.13. *Let $\mathcal{A} = (A, \mathcal{U})$ be a semitopological BL-algebra and the negation map c be onto. Then $\mathcal{A} = (A, \mathcal{U})$ is a topological BL-algebra if and only if $(A, \odot, \mathcal{U})$ or $(A, \to, \mathcal{U})$ is a topological BL-algebra.*

Proof. First we prove that for any $x \in A$, $c(c(x)) = x$. For this, let $x \in A$. Since c is onto, there is a $a \in A$ such that $c(a) = x$. Now by (B_{14}), $c(c(x)) = c(c(c(a))) = a''' = a' = c(a) = x$. Also, let $I : A \to A$ by $I(x) = x$ be identity map. Then by (B_{12})

$$c(\to (I\times c))(x, y) = c(\to (x, y')) = c(x \to y') = c((x\odot y)') = c(c(x\odot y)) = x\odot y$$

By the similar way, we get that $c(\odot(I\times c))(x, y) = x \to y$. Now, by Proposition 3.3, c is continuous, so $\odot$ is continuous if and only if $\to$ is continuous. Now, by Theorem 3.5, $\mathcal{A} = (A,\mathcal{U})$ is a topological BL-algebra if and only if $(A, \odot,\mathcal{U})$ or $(A, \to,\mathcal{U})$ is a topological BL-algebra. □

Theorem 3.14. *Let $(A, \odot,\mathcal{U})$ be a semitopological BL-algebra and the negation map c be continuous and one to one. Then $\mathcal{A} = (A,\mathcal{U})$ is a semitopological BL-algebra.*

Proof. First note that by proof of Proposition 3.3, c is homeomorphism. We prove that $\mathcal{A} = (A,\mathcal{U})$ is a semitopological BL-algebra. It is enough to prove that $(A, \{\to, \vee\},\mathcal{U})$ is a semitopological BL-algebra. For this, we consider the following cases:

Case 1: $(A, \to,\mathcal{U})$ is a semitopological BL-algebra:
Let $x \to y \in U \in \mathcal{U}$ where $x, y \in A$. Since c is onto, there is a $z \in A$ such that $y = c(z) = z'$. By (B_{12}), $(x \odot z)' = x \to z' = x \to y \in U$. Since the negation map c is continuous, there exists an open neighborhood V of $x \odot z$ such that $V' \subseteq U$ and since $(A, \odot,\mathcal{U})$ is a semitopological BL-algebra, there are two open neighborhoods V_1, V_2 of x and z, respectively, such that $V_1 \odot z \subseteq V$ and $x \odot V_2 \subseteq V$. Now by (B_{12}), $V_1 \to y = V_1 \to z' = (V_1 \odot z)' \subseteq V' \subseteq U$ and so $(A, \to,\mathcal{U})$ is a left topological BL-algebra. Since $z \in V_2$ and the negation map c is open, so V_2' is an open neighborhood of $y = z'$. Now, since $x \to V_2' = (x \odot V_2)' \subseteq V' \subseteq U$, hence $(A, \to,\mathcal{U})$ is a right topological BL-algebra and so $(A, \to,\mathcal{U})$ is a semitopological BL-algebra.

Case 2: $(A, \vee,\mathcal{U})$ is a semitopological BL-algebra:
Let $x \vee y \in U \in \mathcal{U}$, where $x, y \in A$. Then $(x \vee y)' \in U' \in \mathcal{U}$. By (B_9), $x' \wedge y' = (x \vee y)' \in U'$. Since $(A, \wedge,\mathcal{U})$ is a semitopological BL-algebra, there is a $V \in \mathcal{U}$ such that $x' \in V$ and $V \wedge y' \in U'$. Since the negation map c is continuous, there is an open set W of x such that $W' \subseteq V$. Since c is one-to-one and by (B_9), $(W\vee y)' = W'\wedge y' \subseteq V\wedge y' \subseteq U'$, then $W\vee y \subseteq U$ and so $(A, \vee,\mathcal{U})$ is a left topological BL-algebra. Hence by Proposition 2.4 (ii), $(A, \vee,\mathcal{U})$ is a semitopological BL-algebra.

Therefore, $\mathcal{A} = (A,\mathcal{U})$ is a semitopological BL-algebra. □

Theorem 3.15. *Let $\mathcal{U}$ be a topology on BL-algebra A such that the negation map c be continuous and one to one. Then $(A, \odot,\mathcal{U})$ is a topological BL-algebra iff $\mathcal{A} = (A,\mathcal{U})$ is a topological BL-algebra.*

Proof. ($\Rightarrow$) Let $(A, \odot, \mathcal{U})$ be a topological BL-algebra. By Proposition 3.3, c is homeomorphism. Let $x \rightarrow y \in U \in \mathcal{U}$, where $x, y \in A$. Since c is onto, there is a $z \in A$ such that $y = c(z) = z'$. By (B_{12}), $(x \odot z)' = x \rightarrow z' = x \rightarrow y \in U$. Since c is continuous and $(A, \odot, \mathcal{U})$ is a topological BL-algebra, there are open neighborhoods V, V_1 and V_2 of $x \odot z$, x and z, respectively, such that $V' \subseteq U$ and $V_1 \odot V_2 \subseteq V$. Since c is open, V_2' is an open set of $y = z'$. Now, $x \in V_1$ and $y \in V_2'$ and $x \rightarrow y \in V_1 \rightarrow V_2' = (V_1 \odot V_2)' \subseteq V' \subseteq U$, so $(A, \rightarrow, \mathcal{U})$ is a topological BL-algebra. By Theorem 3.5, $\mathcal{A} = (A, \mathcal{U})$ is a topological BL-algebra.

($\Leftarrow$) The proof is clear.

□

Example 3.16. *Let $I = [0, 1]$ and binary operations "$\odot$" and "$\rightarrow$" on I are defined as follows:*

$$x \odot y = max\{0, x + y - 1\} \ , \ x \rightarrow y = min\{1, 1 - x + y\}.$$

Then $\mathcal{I} = (I, min, max, \odot, \rightarrow, \leq, 0, 1)$ is a BL-algebra. Let $\mathcal{U}$ be a topology on $\mathcal{I}$ with the base $\{[a, b] \cap \mathcal{I} : a, b \in \mathbb{R}\}$. Then $(\mathcal{I}, \mathcal{U})$ is a semitopological BL-algebra and the negation map c is homeomorphism.

Proof. Let $x \odot y \in U \in \mathcal{U}$, where $x, y \in I$. If $x \odot y = 0$, then $[0, x]$ is a open set of x such that $[0, x] \odot y \subseteq U$. Let $x \odot y = x + y - 1$. Then there exists a $a \in I$ such that $[a, x + y - 1] \subseteq U$. Now $[a - y + 1, x]$ is an open set of x such that $([x - y + 1, x] \odot y) \subseteq U$. Hence $(\mathcal{I}, \odot, \mathcal{U})$ is a semitopological BL-algebra. Now we prove that c is continuous and one-to-one. Obviously c is one-to-one, because for each $x \in I$, $c(x) = 1 - x$. Suppose $x \rightarrow 0 \in U \in \mathcal{U}$. There exists a $a \in I$ such that $[a, 1 - x] \subseteq U$. Now $[x, 1 - a]$ is an open set of x that $c([x, 1 - a]) \subseteq [a, 1 - x] \subseteq U$. Hence, c is continuous. Therefore, by Theorem 3.14, $(\mathcal{I}, \mathcal{U})$ is a semitopological BL-algebra and By Proposition 3.3, c is a homeomorphism. It is easy to prove that $\odot$ is not continuous at $(1, 1)$, so $(\mathcal{I}, \odot, \mathcal{U})$ is not a topological BL-algebra. By Theorem 3.15, $(\mathcal{I}, \mathcal{U})$ is not a toplogical BL-algebra. □

4. Hausdorff BL-algebras

Proposition 4.1. Let $1 \in U \in \mathcal{U}$ and for all $a \in A$, $a \odot U$ or $U \rightarrow a$ be a neighborhood of a. Then $\mathcal{A} = (A, \mathcal{U})$ is a T_0-space.

Proof. Let $x, y \in A$ and $x \neq y$. We consider the following cases:

Case 1: Let U be a neighborhood of 1 such that for every $a \in A$, the set $a \odot U$ be a neighborhood of a. Then $x \odot U$ and $y \odot U$ are two neighborhoods of x and y, respectively. We claim that $x \notin y \odot U$ or $y \notin x \odot U$. If $x \in y \odot U$ and $y \in x \odot U$, then there are z_1 and $z_2 \in U$ such that $x = z_1 \odot y$ and $y = z_2 \odot x$. By (B_1), $x = z_1 \odot y \leq y$ and $y = z_2 \odot x \leq x$ and so $x = y$, a contradiction. Hence $\mathcal{A} = (A, \mathcal{U})$ is a T_0-space.

Case 2: The proof is similar to the proof of Case 1. □

Proposition 4.2. Let $(A, \rightarrow, \mathcal{U})$ be a right (left) topological BL-algebra. If for each $1 \neq x \in A$, there exists a neighborhood U of x such that $1 \notin U$, then $\mathcal{A} = (A, \mathcal{U})$ is a T_0-space.

Proof. Let $(A, \rightarrow, \mathcal{U})$ be a right (left) topological BL-algebra and $x, y \in A$ such that $x \neq y$. Then $x \rightarrow y \neq 1$ or $y \rightarrow x \neq 1$. W.O.L.G, let $x \rightarrow y \neq 1$. Then there exists a neighborhood U of $x \rightarrow y$ such that $1 \notin U$. Since $(A, \rightarrow, \mathcal{U})$ is a right (left) topological BL-algebra, there exists a neighborhood V of x (V of y) such that $V \rightarrow y \subseteq U$ ($x \rightarrow V \subseteq U$). We claim that $y \notin V$ ($x \notin V$). If $y \in V$ ($x \in V$), then by (B_4), $1 = y \rightarrow y \in U$ ($1 = x \rightarrow x \in U$), which is a contradiction. Hence $\mathcal{A} = (A, \mathcal{U})$ is a T_0-space. □

Proposition 4.3. Let $\mathcal{A} = (A, \mathcal{U})$ be a topological (semitopological) BL-algebra and for each $x \in A$ and each neighborhood U of 1, the sets $x \odot U$ and $U \rightarrow x$ be neighborhoods of x. Then $\mathcal{A} = (A, \mathcal{U})$ is a T_1 space if and only if for each $x \neq 1$ there exists a neighborhood U of 1 such that $x \notin U$.

Proof. ($\Rightarrow$) The proof is clear.

($\Leftarrow$) Let for each $x \neq 1$ there exists a neighborhood U of 1 such that $x \notin U$. We prove that A is a T_1-space. Let $x, y \in A$ and $x \neq y$. We consider the following cases:

Case 1: Let $x = 1$. Then $y \neq 1$. Hence there is a neighborhood U of $x = 1$ such that $y \notin U$. By hypothesis $y \odot U$ is a neighborhood of y. We claim that $1 \notin y \odot U$. If $1 \in y \odot U$, then there is a $z \in U$ such that $1 = y \odot z$. By (B_1), $1 = y \odot z \leq y$ and by (B_4) $y \leq 1$ which implies that $1 = y$, a contradiction.

Case 2: Let $x, y \neq 1$ and $x < y$. Let U be a neighborhood of 1 such that $y \notin U$. Then $y \in U \rightarrow y$ and $x \in U \odot x$. We claim that $y \notin U \odot x$ and $x \notin U \rightarrow y$. If $y \in U \odot x$ or $x \in U \rightarrow y$, then $y = z_1 \odot x$ or $x = z_2 \rightarrow y$ for some $z_1, z_2 \in U$. By (B_1) and (B_5), $y = z_1 \odot x = x \odot z_1 \leq x$ and $y \leq z_2 \rightarrow y = x$, a contradiction. If $y < x$, then the proof is similar.

Case 3: Let $x, y \neq 1$ and $x \not< y$ and $y \not< x$. Let U be a neighborhood of 1 such that $y \notin U$. Then $x \in x \odot U$ and $y \in y \odot U$. We claim that $x \notin y \odot U$ and $y \notin x \odot U$. If $x \in y \odot U$ or $y \in x \odot U$, then by (B_1), $x \leq y$ or $y \leq x$, which is a contradiction. □

Proposition 4.4. Let $(A, \rightarrow, \mathcal{U})$ be a semitopological BL-algebra. Then $\mathcal{A} = (A, \mathcal{U})$ is a T_1-space if and only if for any $x \neq 1$ there are neighborhoods U and V of x and 1, respectively, such that $1 \notin U$ and $x \notin V$.

Proof. ($\Rightarrow$) The proof is clear.

($\Leftarrow$) Let for any $x \neq 1$ there are neighborhoods U and V of x and 1, respectively, such that $1 \notin U$ and $x \notin V$. We prove that A is a T_1-space. Let $x, y \in A$ and $x \neq y$. Then $x \rightarrow y \neq 1$ or $y \rightarrow x \neq 1$. W.O.L.G, let $x \rightarrow y \neq 1$.

Let U be a neighborhood of $x \to y$ such that $1 \notin U$. Since $(A, \to, \mathcal{U})$ is a semitopological BL-algebra, then there exist two neighborhoods V and W of x and y, respectively, such that $V \to y \subseteq U$ and $x \to W \subseteq U$. We claim that $x \notin W$ and $y \notin V$. If $y \in V$ or $x \in W$, then $1 = y \to y \in U$ or $1 = x \to x \in U$ which are contradiction. Hence $\mathcal{A} = (A, \mathcal{U})$ is a T_1-space. □

Proposition 4.5. Let $(A, \to, \mathcal{U})$ be a topological BL-algebra. Then $\mathcal{A} = (A, \mathcal{U})$ is a Hausdorff space if and only if for each $x \neq 1$ there exist two open neighborhoods U and V of x and 1, respectively, such that $U \cap V = \phi$.

Proof. ($\Rightarrow$) The proof is clear.

($\Leftarrow$) Let for each $x \neq 1$ there exist open neighborhoods U and V of x and 1, respectively, such that $U \cap V = \phi$. Let $x, y \in A$ and $x \neq y$. Then $x \to y \neq 1$ or $y \to x \neq 1$. W.O.L.G, let $x \to y \neq 1$. By the hypothesis there are neighborhoods U and V of $x \to y$ and 1. Since $(A, \to, \mathcal{U})$ is a topological BL-algebra, there are two neighborhoods W_1 and W_2 of x, y , respectively, such that $W_1 \to W_2 \subseteq U$. We claim that $W_1 \cap W_2 = \phi$. If $z \in W_1 \cap W_2$, then $1 = z \to z \in U$, which shows that $1 \in U \cap V$, a contradiction . □

Lemma 4.6. *Let $(X, *)$ be a monoid with identity 1, $\mathcal{U}$ be a topology on X and $\mathcal{U} = \{U\}$ be a fundamental system of open neighborhoods of 1 in X. If $(X, \mathcal{U})$ is a T_1-space or Hausdorff space, then $\bigcap_{U \in \mathcal{U}} U = 1$.*

Theorem 4.7. *Let $(A, \to, \mathcal{U})$ be a regular topological BL-algebra. Then the following statements are equivalent.*
(i) $(A, \mathcal{U})$ is a Hausdorff space.
(ii) $(A, \mathcal{U})$ is a T_1 space.
(iii) $\bigcap_{U \in \mathcal{U}} U = 1$, where $\mathcal{U}$ is a fundamental system of neighborhoods of 1.

Proof. ($i \Rightarrow ii$) The proof is clear.

($ii \Rightarrow iii$) The proof come from Lemma 4.6.

($iii \Rightarrow i$) Let $\bigcap\limits_{U \in \mathcal{U}} U = 1$, $x, y \in A$ and $x \neq y$. Hence $x \to y \neq 1$ or $y \to x \neq 1$. W.O.L.G, let $x \to y \neq 1$. Then, there exists a $U \in \mathcal{U}$ such that $x \to y \notin U$. Since $(A, \mathcal{U})$ is a regular space, there is a $V \in \mathcal{U}$ such that $1 \in V \subseteq \overline{V} \subseteq U$. Now $A \setminus \overline{V}$ is an open set of $x \to y$. Since $(A, \to, \mathcal{U})$ is a topological BL-algebra, there exist two open neighborhoods W_1, W_2 of x, y respectively, such that $W_1 \to W_2 \subseteq A \setminus \overline{V}$. We claim that $W_1 \cap W_2 = \phi$. Let $z \in W_1 \cap W_2 = \phi$. Then by (B_4), $1 = z \to z \in W_1 \to W_2 \subseteq A \setminus \overline{V}$, which is a contradiction. □

Proposition 4.8. Let $(A, \mathcal{U})$ be a Hausdorff topological BL-algebra and $B = A \setminus \{0\}$ be compact filter. Let for each $a, b \in B, D \subseteq B$, if $a \wedge b \in a \wedge D$, then $b \in D$. Then for each $a, b \in A \setminus \{0\}$, the equation $a \to x = b$ has solution.

Proof. Let $a \in B = A \setminus \{0\}$. We claim that $a \to B \subseteq B$. Let $a \to b \notin B$, for some $b \in B$. Then $a \to b = 0$. Hence, by (B_5), $1 = b \to (a \to b) = b \to 0 = b'$

and so by (B_8), $b = 0 \notin B$, a contradiction. Now we construct a decreasing sequence of compact closed subsets B. Let $a \in B$. Since B is a filter, for each $n \in \mathbb{N}$, $a^n \in B$. Put $B_0 = B$ and for each $n \in \mathbb{N}$, $B_n = a^n \to B$. We have just shown $B_1 \subseteq B$. Hence $a \to B_1 \subseteq a \to B$ and so by (B_6),

$$B_2 = a^2 \to B = (a \odot a) \to B = a \to (a \to B) = a \to B_1 \subseteq a \to B = B_1 \subseteq B$$

With a similar argument as above, we can show that for each $n \in \mathbb{N}$, $B_{n+1} \subseteq B_n \subseteq B$. Hence the sequence $\{B_n : n \in \mathbb{N}\}$ is decreasing. Since B is compact, Hausdorff, and $\to$ is continuous, all B_n are closed compact subsets of B. Consider the sequence $Q = \{a^n : n \in \mathbb{N}\}$. Since B is compact, Q has an accumulation point $y \in B$. Take any open set V such that $y \to (a \to B) \subseteq V$. Suppose b is an arbitrary element of B. Since $(y \odot a) \to b = y \to (a \to b) \in V$ and $(A, \to, \mathcal{U})$ is a topological BL-algebra, there are two open sets U_b, W_b of $y \odot a$ and b, respectively, such that $(y \odot a) \to b \in U_b \to W_b \subseteq V$. Now since $\{W_b\}_{b \in B}$ is an open cover of B and B is compact, hence there are elements $b_1, \cdots, b_n \in B$ such that $B \subseteq \bigcup_{i=1}^{n} W_{b_i}$. Let $U_1 = \bigcap_{i=1}^{n} U_{b_i}$ and $W = \bigcup_{i=1}^{n} W_{b_i}$. Then $U_1 \to W \subseteq V$. Since $y \odot a \in U_1$ and l_a is continuous, there is an open neighborhood U of y such that $y \odot a \in U \odot a \subseteq U_1$. Now,

$$y \to (a \to B) \subseteq U \to (a \to W) \subseteq (U \odot a) \to W \subseteq U_1 \to W \subseteq V.$$

Hence $y \in U$ and $U \to (a \to B) \subseteq U \to (a \to W) \subseteq V$. Since y is an accumulation point of Q, for each m, there is a $a^k \in U \cap \{a^n : n \geq m\}$. For each $n \geq k + 1$

$$B_n \subseteq B_{k+1} = a^{k+1} \to B = a^k \to (a \to B) \subseteq U \to (a \to B) \subseteq V.$$

Now, we prove that $a \to B = B$. For this, take any $b \in B$. Suppose U is an open neighborhood of $y \to b$. Since R_b is continuous, there is an open neighborhood W of y such that $W \to b \subseteq U$. Since y is accumulation point Q, for each $n \geq k + 1$, there is a $a^m \in W \cap \{a^n : n \geq m\}$. Hence $a^m \to b \in W \to b \subseteq U$. This shows that $a^m \to b \in V \cap U$. Therefore, $y \to b \in \overline{V}$. Thus we have shown that $y \to b$ belongs to the closure of any neighborhood of the set $y \to (a \to B)$. Hence $y \to b \in y \to (a \to B)$, because if $y \to b \notin y \to (a \to B)$, then since B is compact Hausdorff and so normal, there is an open neighborhood V of $y \to (a \to B)$ such that $y \to b \notin \overline{V}$, a contradiction. Now $y \to b \in y \to (a \to B)$ implies that $y \wedge b \in y \wedge (a \to B)$. By hypothesis $b \in a \to B$. Therefore, we can prove that $a \to B = B$. Now obviously for each $b \in B$, the equation $a \to x = b$ has solution in B. □

Lemma 4.9. [1] *Let* $(X, *)$ *be an algebra of type* 2 *and* $\mathcal{U}$ *be a topology on* X *which satisfies the following conditions:*
(i) *For each* $x, y \in X$*, there are two open sets* U *and* V *of* x *and* y *respectively such that* $\overline{U * V}$ *is compact.*

(ii) For each $x, y \in X$ and for each open set W of $x * y$ and each $z \in X \setminus W$ there exist two open sets U and V of x, y respectively such that $z \notin \overline{U * V}$.
Then $(X, *, \mathcal{U})$ is a topological algebra.

Lemma 4.10. *Let $(A,\mathcal{U})$ be a locally compact Hausdorff space. Let for any $a \in A$, $l_a : A \to A$ by $l_a(x) = a \odot x$ and $L_a : A \to A$, by $L_a(x) = a \to x$ be continuous maps and for each neighborhood U of 1, $a \odot U$ be a neighborhood of a. If for each $b \in A$ the operations $\odot$ and $\to$ are continuous at $(1, b)$, then for any $x, y \in A$, there exist $U, V \in \mathcal{U}$ such that $x \in U$ and $y \in V$ and $\overline{U \odot V}$ and $\overline{U \to V}$ are compact.*

Proof. Let $x, y \in A$ and $y \in W \in \mathcal{U}$. Since A is locally compact Hausdorff space, we can assume that $\overline{W}$ is compact. Since $1 \odot y = y$ and $1 \to y = y$ are in W and $\odot$ and $\to$ are continuous at $(1, y)$, there exist $U_1, V \in \mathcal{U}$ such that $1 \in U_1$, $y \in V$, $U_1 \odot V \subseteq W$ and $U_1 \to V \subseteq W$. Let $U = x \odot U_1$. Then $x \in U \in \mathcal{U}$. Now we show that $\overline{U \odot V}$ and $\overline{U \to V}$ are compact. For this we consider the following cases:

Case 1: $\overline{U \odot V}$ is compact:
Since $(A, \odot)$ is a monoid, then $U \odot V = x \odot U_1 \odot V \subseteq x \odot W$. Since $\overline{W}$ is compact and $l_x : z \to x \odot z$ of A into A is continuous, $x \odot \overline{W}$ is compact. Since $(A,\mathcal{U})$ is Hausdorff, $x \odot \overline{W}$ is closed. Now by $W \subseteq \overline{W}$ we get that $x \odot W \subseteq x \odot \overline{W}$ and so $\overline{x \odot W} \subseteq \overline{x \odot \overline{W}} = x \odot \overline{W}$. Also since l_x is continuous,

$$x \odot \overline{W} = l_x(\overline{W}) \subseteq \overline{l_x(W)} = \overline{x \odot W}.$$

Therefore $\overline{x \odot W} = x \odot \overline{W}$. This prove that $\overline{x \odot W}$ is compact. Since $\overline{U \odot V} \subseteq \overline{x \odot W}$, the set $\overline{U \odot V}$ is compact.

Case 2: $\overline{U \to V}$ is compact:
By (B_6), We get that $U \to V = (x \odot U_1) \to V = x \to (U_1 \to V) \subseteq x \to W$. Since $\overline{W}$ is compact and $L_x : z \to (x \to z)$ of A into A is continuous, so $x \to \overline{W}$ is compact and closed. Similar to the proof of Case 1, we can prove that $x \to \overline{W} = \overline{x \to W}$. Now $\overline{x \to W}$ is compact and so $\overline{U \to V}$ is compact. □

Lemma 4.11. [1] *Let X and Y be locally compact Hausdorff spaces, f be a separately continuous mapping of $X \times Y$ to a regular space Z and $(x, y) \in X \times Y$. Let W be an open set of $f(x, y)$ and U be an open set of x, then there exists a non-empty open set U_1 in X and an open set V in Y such that $U_1 \subseteq U$ and $y \in V$ and $f(U_1 \times V) \subseteq W$.*

Theorem 4.12. *Let $(A, \to, \mathcal{U})$ be a locally compact Hausdorff BL-algebra, for each $a \in A$ the mappings $l_a : x \to (a \odot x)$ and $L_a : x \to (a \to x)$ of A into A be continuous and for any $a \in A$ and each open set U of 1, $a \odot U$ be an open set of a. If for any $b \in A$, the operations $\to$ and $\odot$ are continuous at $(1, b)$, then $\mathcal{A} = (A,\mathcal{U})$ is a topological BL-algebra.*

Proof. By Theorem 3.5, it is enough to show that $(A,\odot,\mathcal{U})$ and $(A,\to,\mathcal{U})$ are topological BL-algebras. For this we consider the following cases:

Case 1: $(A,\odot,\mathcal{U})$ is a topological BL-algebra:
First we prove that $(A,\odot,\mathcal{U})$ is a semitopological BL-algebra. Let $x\odot y\in U\in\mathcal{U}$. Since $\odot$ is continuous at $(1,x\odot y)$, there is a $V\in\mathcal{U}$ such that $1\in V$ and $V\odot x\odot y\subseteq U$. Now by hypothesis $W=V\odot x$ is an open neighborhood of x and $W\odot y\subseteq U$. This shows that $(A,\odot,\mathcal{U})$ is a left topological BL-algebra. By Proposition 2.4, $(A,\odot,\mathcal{U})$ is a semitopological BL-algebra. Now, We prove that $(A,\odot,\mathcal{U})$ satisfies in conditions (i) and (ii) of Lemma 4.9. By Lemma 4.10, $(A,\odot,\mathcal{U})$ satisfies in conditions (i) of Lemma 4.9. Now, let $x\odot y\in W\in\mathcal{U}$ and $z\in A\setminus W$. Since A is a locally compact and so regular we can assume that $z\notin\overline{W}$. Since $1\odot z=z\in A\setminus\overline{W}$ and operation $\odot$ is continuous at $(1,z)$, there is an open set H of 1 such that $(H\odot z)\cap\overline{W}=\phi$. By hypothesis, $H\odot x$ is an open neighborhood of x and by Lemma 4.11, there are two non-empty sets U_1 and V such that $U_1\subseteq H\odot x$, $y\in V$ and $U_1\odot V\subseteq W$. Since $\phi\neq U_1\subseteq H\odot x$, there is a $h\in H$ such that $h\odot x\in U_1$. Since l_h is continuous, there is a $U\in\mathcal{U}$ such that $x\in U$ and $h\odot U\subseteq U_1$. Therefore, $(h\odot U)\odot V\subseteq U_1\odot V\subseteq W$ and so

$$h\odot\overline{U\odot V}=l_h(\overline{U\odot V})\subseteq\overline{l_h(U\odot V)}=\overline{h\odot U\odot V}\subseteq\overline{W}.$$

But $z\notin\overline{U\odot V}$, because if $z\in\overline{U\odot V}$, then $h\odot z\in h\odot\overline{U\odot V}$ and so $h\odot z\in\overline{W}\cap(H\odot z)$, a contradiction. Hence $(A,\odot,\mathcal{U})$ satisfies in condition (ii) of Lemma 4.9. Therefore, $(A,\odot,\mathcal{U})$ is a topological BL-algebra.

Case 2: $(A,\to,\mathcal{U})$ is a topological BL-algebra:
First, we show that $(A,\to,\mathcal{U})$ is a semitopological BL-algebra. By hypothesis, since for each $a\in A$, L_a is continuous, then $(A,\to,\mathcal{U})$ is a right topological BL-algebra. Let $x\to y\in U\in\mathcal{U}$. Since by (B_4), $1\to(x\to y)=x\to y\in U$ and by hypothesis $\to$ is continuous at $(1,x\to y)$, there is an open set V of 1 such that $V\to(x\to y)\subseteq U$. Now by hypothesis $W=V\odot x$ is an open set of x and by (B_6)

$$W\to y=(V\odot x)\to y=V\to(x\to y)\subseteq U.$$

This show that $(A,\to,\mathcal{U})$ is a left topological BL-algebra. Now, We prove that $(A,\to,\mathcal{U})$ satisfies in conditions (i) and (ii) of Lemma 4.9. By Lemma 4.10, $(A,\to,\mathcal{U})$ satisfies in condition (i) of Lemma 4.9. Similar to the proof of Case 1, let $x\to y\in W\in\mathcal{U}$ and $z\in A\setminus\overline{W}$. Since $1\to z=z$ and $1\odot z=z$ are in open set $A\setminus\overline{W}$ and since l_z and L_z are continuous, there are two open sets H_1 and H_2 of 1 such that $H_1\to z\subseteq A\setminus\overline{W}$ and $H_2\odot z\subseteq A\setminus\overline{W}$. Let $H=H_1\cap H_2$. Then H is an open set of 1 such that

$$(H\to z)\cap\overline{W}=(H\odot z)\cap\overline{W}=\phi.$$

The set $H \odot x$ is also an open set of x. By Lemma 4.11, there are two non-empty subsets U_1 and V such that

$$U_1 \subseteq H \odot x \ , \ y \in V \ , \ U_1 \to V \subseteq W.$$

Since $U_1 \subseteq H \odot x$ and U_1 is non-empty, there is a $h \in H$ such that $h \odot x \in U_1$. Since l_h is continuous, there is an open set U of x such that $h \odot U \subseteq U_1$. Now $(h \odot U) \to V \subseteq U_1 \to V \subseteq W$ and by (B_6) and continuity of l_h,

$$h \to (\overline{U \to V}) \subseteq \overline{h \to (U \to V)} = \overline{h \odot U \to V} \subseteq \overline{W}.$$

But $z \notin \overline{U \to V}$, because if $z \in \overline{U \to V}$, then $h \to z \in h \to (\overline{U \to V} \subseteq \overline{W}$ and so $h \to z$ is in $(H \to z) \cap \overline{W}$, a contradiction. Hence $(A, \to, \mathcal{U})$ satisfies in condition (ii) of Lemma 4.9. Therefore, $(A, \to, \mathcal{U})$ is a topological BL-algebra. □

5. Conclusion

In this paper we have studied the relationships between the topology and BL-algebra operations and have introduced (semi)topological BL-algebras. We have also shown the influences the topological properties on the underlying BL-algebra structure and vice versa.

Next researches can study (semi)topological quotient BL-algebras, the relationships between homomorphism and homeomorphism in (semi)topological BL-algebras. Also they can discuss metrizability, convergency, and many of the other concepts of topology.

Acknowledgements. The authors would like to express their sincere thanks to the referees for their valuable suggestions and comments

References

[1] A. Arhangel'skii, M. Tkachenko, *Topological groups and related structures*, Atlantis press, 2008.

[2] N. Bourbaki, *Elements of mathematics general topology*, Addison-Wesley Publishing Company, 1966.

[3] D. Busneag, D. Piciu, Boolean BL-algebra of fractions, Annals of University of Craiova, *Math. Comp. Soi.*, Ser. **31**, (2004), 1-19.

[4] L. C. Ciungu, Convergences in Perfect BL-algebras, *Mathware Soft Computing*, **14**, (2007), 67-80.

[5] A. Di Nola, A. Lettieri, Finite BL-algebras, *Discrete Mathematics*, **269**(1-3), (2003), 93-112.

[6] A. Di Nola, L. Leustean, Compact Representations of BL-algebra, *Arch. Math. Logic*, **42**, (2003), 737-761.

[7] E. Eslami, F. Kh. Haghani, Pure Filters and Stable Topology on BL-algebras, *Kybernetika*, **45**(3), (2009), 491506.

[8] P. Hájek, *Metamathematics of Fuzzy Logic*, Kluwer academic publishers, 1998.

[9] M. Haveshki, A. Borumand saeid, E. Eslami, Some Types of Filters in BL-algebras, *Soft computing*, **10**, (2006), 657-664.

[10] M. Haveshki, E. Eslami, A. Borumand Saeid, A topology induced by uniformity on BL-algebras, *Math. Log. Quart.*, **53**(2), (2007), 162169.

[11] T. Husain, *Introduction to topological groups*, W. B. Sunders Company, 1966.

[12] M. Kondo, W. A. Dudek, Filter theory of BL-algebras, *Soft Computing*, **12**, (2008), 419-423.

[13] J. Mi Ko, Y. C. Kim, Closure operators on BL-algebras, *Commun. Korean Math. Soc.*, **19**(2), (2004), 219-232.

[14] A. Nagoorgania, P. Palaniammalb, A fuzzy production model with probabilistic resalable returns, *Iranian Journal of Mathematical Sciences and Informatics*, **3**(1), (2008), 77-86.

[15] T. Roudabri, L. Torkzadeh, A topology on BCK-algebras via left and right stabilizers, *Iranian Journal of Mathematical Sciences and Informatics*, **4**(2), (2009), 1-8.