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Hyperbolicity of the family $f_c(x) = c(x - \frac{x^3}{3})$

Monireh Akbari[a,*] and Maryam Rabii[b]

[a]Shahid Rajaee Teacher Training University, Tehran, Iran
[b]Alzahra University, Tehran, Iran

E-mail: akbari@srttu.edu
E-mail: mrabii@alzahra.ac.ir

Abstract. The aim of this paper is to present a proof of the hyperbolicity of the family $f_c(x) = c(x - \frac{x^3}{3})$, $|c| > 3$, on an its invariant subset of $\mathbb{R}$.

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2000 Mathematics subject classification: 37E05, 37D05.

Introduction

There is extensive literature on the behavior of the unimodal map $g_\mu(x) = \mu x(1-x)$, see, e. g. [1], [2], [3] and [5]. An elementary treatment of g_μ for $\mu > 4$ can be found, among others, in [4], where the existence of an invariant hyperbolic Cantor set is established. Following the methods of [4], we treat the family $f_c(x) = c(x - \frac{x^3}{3})$ in the case $|c| > 3$. When $0 < |c| \leq 2$ the attracting fixed and periodic points of period 2, when they exist, dominate the dynamical behavior of the orbits that do not tend to infinity. When $0 < |c| \leq 3$ the orbits of the points in the interval $I_c = [-\sqrt{3(1+\frac{1}{|c|})}, \sqrt{3(1+\frac{1}{|c|})}]$ are bounded. If $|c| > 3$, there are some points in the interval I_c whose images leave this interval. The interval I_c is divided into five subintervals, two open subintervals which leave I_c after one iteration of f_c, and three closed subintervals which are

*Corresponding Author

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mapped monotonically onto I_c by f_c. Continuing this process, we determine the invariant subset of I_c under f_c. Let $\Lambda_{f_c} = \cap_{n=1}^{\infty} f_c^{-n}(I_c)$.

In this paper, we will show that Λ_{f_c} is

- repelling hyperbolic,
- totally disconnected,
- a Cantor set.

Following [4], we show in the first section that $f_c(x) = c(x - \frac{x^3}{3}),\ c > 3$, has a repelling hyperbolic set. In the second section we show the hyperbolicity of $f_{-c},\ c > 3$. It will follow from Lemma 1 and Theorem 1 that $\Lambda_{f_c},\ |c| > 3$, is a Cantor set.

Lemma 1. *If $|c| > 3$, then Λ_{f_c} is a closed perfect subset of I_c.*

Proof. It is clear that $\Lambda_{f_c} = \cap_{n=1}^{\infty} f_c^{-n}(I_c)$ is a closed set, since $I_n = f_c^{-n}(I_c)$ is a closed set. Suppose $x \in \Lambda_{f_c}$, then $x \in I_n$ for every n, and there is an interval $I_{n_k} \subset I_n$ such that $x \in I_{n_k}$. So $x \in \cap_{n=1}^{\infty} I_{n_k}$. If x is the only point of intersection, then there is a sequence of endpoints of I_{n_k}'s, $\{a_{n_k}\}$, that converges to x and $a_{n_k} \in \Lambda_{f_c}$, because these points are finally mapped to endpoints of I_c. If $\cap_{n=1}^{\infty} I_{n_k}$ contains more than one point, then it is an interval and x is a limit point of this interval. $\square$

Definition 1. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a C^1 function. A set $\Gamma \subseteq \mathbb{R}$ is a repelling hyperbolic set if Γ is a compact subset of $\mathbb{R}$ that is invariant under f, and there exists $N > 0$ such that $|(f^n)'(x)| > 1$ for all $n \geq N$ and all $x \in \Gamma$*

Lemma 2. [4] *Suppose $f : \mathbb{R} \to \mathbb{R}$ is a C^1 function, Γ is a compact subset of $\mathbb{R}$ and $f(\Gamma_)\subseteq \Gamma$. Then the following statements are equivalent.*

1. *There is an integer $N > 0$ such that $|(f^n)'(x)| > 1$ for all $n \geq N$ and all $x \in \Gamma$.*
2. *There is an integer $n_0 > 0$ such that $|(f^{n_0})'(x)| > 1$ for all $x \in \Gamma$.*
3. *For every $x \in \Gamma$, there is an integer $n_x > 0$ such that $|(f^{n_x})'(x)| > 1$.*

In order to prove the hyperbolicity of Λ_{f_c} we will show that statement 3 of Lemma 2 is satisfied. We will use the notion Schwarzian derivative and its properties.

Suppose f is a C^3 function that has been defined in a neighborhood of x and $f'(x) \neq 0$, then Schwarzian derivative of f at x is

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}(\frac{f''(x)}{f'(x)})^2$$

It is well known that $S(f \circ g) < 0$, if $S(f) < 0$ and $S(g) < 0$ and also if $I = [a, b]$ and $Sf(x) < 0$ for all $x \in (a, b)$, then f' has neither a positive local minimum on I nor a negative local maximum on I, [2]. The following lemma holds as well:

Lemma 3. [4] *Let $I = [a, b]$ and suppose f is C^3 on I. If $Sf < 0$ on (a, b), then $|f'(x)| > \min\{|f'(a)|, |f'(b)|\}$ for all $x \in (a, b)$.*

It is easily seen that:

Lemma 4. *Let $f_c(x) = c(x - x^3/3)$, then $Sf_c(x) < 0$ for all $x \in \mathbb{R} - \{-1, 1\}$.*

1. Hyperbolicity of Λ_{f_c}, $c > 3$

Now suppose $c > 3$ and $z_0 = \sqrt{3(1 + 1/c)}$. There are three intervals that are mapped homeomorphically onto $[-z_0, z_0]$. If

$$q_1 = \big(\sqrt{3(1 + 1/c)} + \sqrt{3(1 - 3/c)}\big)/2, \; q_0 = \big(\sqrt{3(1 + 1/c)} - \sqrt{3(1 - 3/c)}\big)/2$$

then these three intervals are

$$[-z_0, -q_1], [-q_0, q_0], [q_1, z_0]$$

Fixed points of f_c are $p_1 = \sqrt{3(1 - 1/c)}$, $-p_1$ and 0. Let $p_0 = \big(-\sqrt{3(1 - 1/c)} + \sqrt{3(1 + 3/c)}\big)/2$, then $f_c(p_0) = f_c(p_1) = p_1$ and $f_c(-p_0) = f_c(-p_1) = -p_1$ (Figure 1). Also, let $J = (p_0, q_0) \cup (q_1, p_1)$ and $-J = (-q_0, -p_0) \cup (-p_1, -q_1)$, then we have the following lemma.

Lemma 5. *Suppose $f_c(x) = c(x - x^3/3)$, $c > 3$, $x \in J \cup -J$ and x is not an eventually periodic point, then there is an integer $n \geq 2$ such that $f_c^n(x) \in (-p_1, p_1)$.*

Proof. We know $f_c(p_0, q_0) = f_c(q_1, p_1) = (p_1, z_0)$ and $f_c(p_1, z_0) = (-z_0, p_1)$. Suppose $x \in J$, then $y = f_c^2(x) \in (-z_0, p_1)$. Let the orbit of y never leaves $(p_1, z_0) \cup (-z_0, -p_1)$. Now if $y \in (p_1, z_0)$, then $f_c^2(y) \in (p_1, z_0)$, so $f_c^2(y) = y$ or $\{f_c^{2n}(y)\}$ is a monotonic sequence that must converge to a periodic point of period 2, but it is easily seen that all the periodic points of period 2 are repelling and this is impossible. □

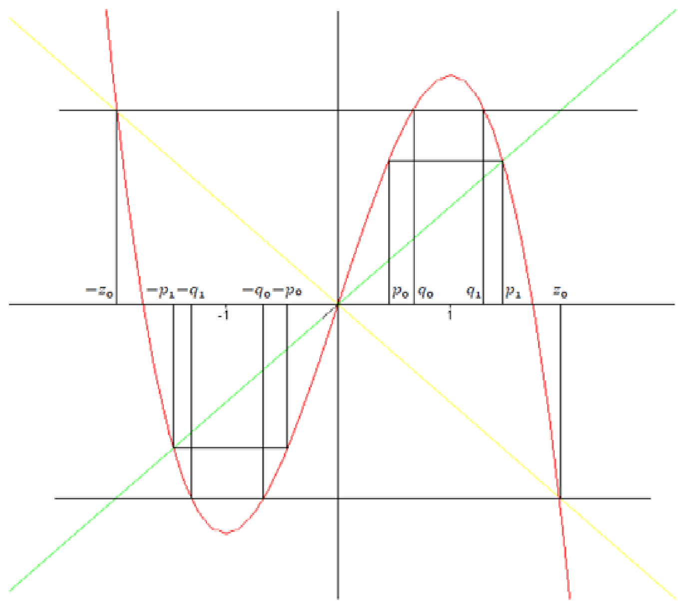

FIGURE 1. $f_c(x) = c(x - \frac{x^3}{3});\ c = \frac{7}{2}$

Lemma 6. *If* $c > 3$*, then* $p_1 - q_1 < q_0 - p_0 < z_0 - p_1$

Proof. Straightforward computation shows $p_1 - q_1 < q_0 - p_0$. In order to prove $(q_0 - p_0)^2 < (z_0 - p_1)^2$ we should show $3c - 5 < \sqrt{(c+1)(c-3)} + \sqrt{(c+1)(c+3)} + \sqrt{c^2 - 9}$. That is correct because

$c - 3 < \sqrt{(c+1)(c-3)}$
$c + 1 < \sqrt{(c+1)(c+3)}$
$c - 3 < \sqrt{c^2 - 9}$. □

Theorem 1. *Let* $c > 3$ *and* $x \in \Lambda_{f_c}$*, then there is an integer* n *such that* $|(f_c^n)'(x)| > 1$.

Proof. If $x \in [p_1, z_0]$ then $|f_c'(x)| > |f_c'(p_1)| = |-2c + 3| > 1$ and if x is an eventual fixed point, there is an integer n such that $|(f_c^n)'(x)| > 1$. If $x = q_1$, since $f_c(q_1) = z_0$ and z_0 is a repelling periodic point then there exist n such that $|(f_c^n)'(q_0)| > 1$.

Now suppose $x \in \Lambda_{f_c}$ and $x \in (q_1, p_1)$. According to Lemma 5, there exists $n \geq 2$ such that $f_c^n(x) \in (-p_1, p_1)$. Since $x \in \Lambda_{f_c}$, there is n such that $x \in I_n$ and there is an interval $I_{n_j} \subset I_n$ such that $x \in I_{n_j}$ and f_c^n maps I_{n_j} monotonically onto $[-z_0, z_0]$.

First we suppose $I_{n_j} \subseteq [q_1, p_1)$. We divide I_{n_j} to three subintervals, $I_{n_j} = L_{n_j} \cup K_{n_j} \cup R_{n_j}$, such that

$$f_c^n(L_{n_j}) = [-z_0, -p_1], \ f_c^n(K_{n_j}) = (-p_1, p_1), \ f_c^n(R_{n_j}) = [p_1, z_0]$$

According Lemma 6, $|f_c^n(L_{n_j})| > |L_{n_j}|$ and $|f_c^n(R_{n_j})| > |R_{n_j}|$. By using Mean Value Theorem, there exists $y \in L_{n_j}$ such that $|(f_c^n)'(y)| > 1$ and there is $z \in R_{n_j}$ such that $|(f_c^n)'(z)| > 1$. Since x is between y and z and since $Sf_c(I_{n_j}) < 0$, then according to Lemma 3, $|(f_c^n)'(x)| > 1$.

Now suppose $I_{n_j} \nsubseteq [q_1, p_1)$, so $x < p_1$, $x \in I_{n_j}$ and $p_1 \in I_{n_j}$. As before we define R_{n_j}, K_{n_j} and L_{n_j}. Again, $x \in K_{n_j}$ and L_{n_j} or R_{n_j} is a subset of $[q_1, p_1)$. Suppose L_{n_j} has this property. As before there exists y with this property that $|(f_c^n)'(y)| > 1$ and p_1 is a repelling fixed point, therefore $|(f_c^n)'(p_1)| > 1$. x is between y and p_1 and we conclude that $|(f_c^n)'(x)| > 1$.

The other cases are proved similarly. □

2. Hyperbolicity of Λ_{f_c}, $|c| > 3$

In this section we describe how the case $c < -3$ can be deduced from the case $c > 3$.

Lemma 7. *$\Lambda_{f_c} = \Lambda_{f_c^k}$ for all $k \in \mathbb{N}$.*

Proof. It is clear that $\Lambda_{f_c} \subseteq \Lambda_{f_c^k}$. Let $x \in \Lambda_{f_c^k}$, but $x \notin \Lambda_{f_c}$. Then $\lim_{n\to\infty} |f_c^n(x)| = \infty$, especially $\lim_{n\to\infty} |f_c^{kn}(x)| = \infty$ whereas $\{|f_c^{kn}(x)|\}_{n\geq 0}$ is bounded. □

Corollary 1. *For any c, $\Lambda_{f_c} = \Lambda_{f_c^2} = \Lambda_{f_{-c}^2} = \Lambda_{f_{-c}}$.*

Lemma 8. *$(f_c^n)'(-x) = (f_c^n)'(x)$ and $(f_{-c}^n)'(x) = (-1)^n (f_c^n)'(x)$.*

Proof. We know $f'_{-c}(x) = -f'_c(x)$, $f'_c(-x) = f'_c(x)$, $f_c(-x) = -f_c(x)$. Now lemma is proved by induction. □

Now let $c > 3$, the following lemma shows that f_{-c} on $\Lambda_{f_{-c}}$ is repelling hyperbolic.

Lemma 9. *For any $x \in \Lambda_{f_{-c}}$, there exists $n_x \in \mathbb{N}$ such that $|(f_{-c}^{n_x})'(x)| > 1$.*

Proof. Let $x \in \Lambda_{f_{-c}} = \Lambda_{f_c}$. Therefore, by Theorem 1 there exists $n_x \in \mathbb{N}$ such that $|(f_c^{n_x})'(x)| > 1$. By Lemma 8, $|(f_{-c}^{n_x})'(x)| = |(-1)^n (f_c^{n_x})'(x)| = |(f_c^{n_x})'(x)| > 1$. □

Theorem 2. *If $|c| > 3$, Λ_{f_c} is totally disconnected.*

Proof. Suppose $[x, y] \subseteq \Lambda_{f_c}$. According to Theorem 1 and Lemma 2 there is $N > 0$ such that $|(f_c^n)'(z)| > 1$ for all $z \in \Lambda_{f_c}$ and $n \geq N$. Let $|(f_c^N)'(z)| \geq \lambda > 1$. By Mean Value Theorem we have $|f_c^{kN}(x) - f_c^{kN}(y)| \geq \lambda^k |x - y|$ and $f_c^{kN}(x),\ f_c^{kN}(y) \in \Lambda_{f_c}$, for $k \in \mathbb{N}$ and this is a contradiction. □

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