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Homotopy analysis and Homotopy Padé methods for two-dimensional coupled Burgers' equations

H. Kheiri* and A. Jabbari

Faculty of Mathematical sciences, University of Tabriz-Tabriz-Iran

E-mail: h-kheiri@tabrizu.ac.ir

Abstract. In this paper, analytic solutions of two-dimensional coupled Burgers' equations are obtained by the Homotopy analysis and the Homotopy Padé methods. The obtained approximation by using Homotopy method contains an auxiliary parameter which is a simple way to control and adjust the convergence region and rate of solution series. The approximation solutions by $[m, m]$ Homotopy Padé technique is often independent of auxiliary parameter $\hbar$ and this technique accelerate the convergence of the related series.

Keywords: Homotopy analysis method, Homotopy Padé technique, Two-dimensional coupled Burgers' equations.

2000 Mathematics subject classification: 65M99, 65N99.

1. Introduction

By now, several methods have been suggested to solve nonlinear equations. These methods include the Homotopy perturbation method [1], Luapanov's artificial small parameter method, δ-expansion method, Adomian decomposition method, variational iterative method and so on [2, 3]. Homotopy analysis method(HAM), first proposed by Liao in his Ph.D dissertation [4], is an elegant method which has proved its effectiveness and efficiency in solving many types of nonlinear equations [5, 6]. Liao in his book [7] proved that HAM is a generalization of some previously used techniques such as the δ-expansion method, artificial small parameter method [8] and Adomian decomposition method. Moreover, unlike previous analytic techniques, the HAM provides

*Corresponding Author

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a convenient way to adjust and control the region and rate of convergence [9]. There exist some techniques to accelerate the convergence of a given series. Among them, the so-called Padé method is widely applied [5, 7, 10]. In this paper we apply Homotopy analysis and Homotopy Padé methods for two-dimensional coupled Burgers' equations. The Burgers' equation retains the nonlinear aspects of the governing equations in many applications, such as the mathematical model of turbulence [11] and the approximate theory of flow through a shock wave traveling in viscous fluid [12]. Fletcher using the Hopf-Cole transformation [13], gave an analytic solution of the system of two dimensional Burgers' equations. There are many numerical methods for solving the Burgers' equation, such as the cubic spline function techniques [14], applied an explicitimplicit method [15], and implicit finite-difference scheme [16]. Soliman [17] used the similarity reductions for the partial differential equations to develop a scheme for solving the Burgers' equation. Higher-order accurate schemes for solving the two-dimensional Burgers' equations have been used [18, 19]. The coupled system is derived by Esipov [19]. It is a simple model of sedimentation or evolution of scaled volume concentrations of two kinds of particles in fluid suspensions or colloids, under the effect of gravity [20].

2. Homotopy Analysis Method

For convenience of the readers, we will first present a brief description of the standard HAM. To achieve our goal, let us assume the nonlinear system of differential equations be in the form of

$$N_j[u_1(x,t), u_2(x,t), ..., u_m(x,t)] = 0, \quad j = 1...n, \tag{1}$$

where N_j are nonlinear operators, t is an independent variable, $u_i(t)$ are unknown functions. By means of generalizing the traditional homotopy method, Liao construct the zeroth-order deformation equation as follows

$$\begin{aligned} &(1-q)L_j[\phi_i(x,t,q) - u_{i0}(x,t)] \\ &= q\hbar H(t) N_j[\phi_1(x,t,q), \phi_2(x,t,q), ..., \phi_m(x,t,q)], \\ &\quad i = 1...m; \qquad j = 1,...n, \end{aligned} \tag{2}$$

where $q \in [0,1]$ is an embedding parameter, L_j are linear operators, $u_{i0}(x,t)$ are initial guesses of $u_i(x,t)$, $\phi_i(x,t;q)$ are unknown functions, $\hbar$ and $H(x,t)$ are auxiliary parameter and auxiliary function respectively. It is important to note that, one has great freedom to choose auxiliary objects such as $\hbar$ and L_j in HAM; This freedom plays an important role in establishing the keystone of validity and flexibility of HAM as shown in this paper. Obviously, when $q = 0$ and $q = 1$, both

$$\phi_i(x,t,0) = u_{i0}(x,t) \quad \text{and} \quad \phi_i(x,t,1) = u_i(x,t), \quad i = 1...m, \tag{3}$$

hold. Thus as q increases from 0 to 1, the solutions of $\phi_i(x,t;q)$ change from the initial guesses $u_{i0}(x,t)$ to the solutions $u_i(x,t)$. Expanding $\phi_i(x,t;q)$ in Taylor series with respect to q, one has

$$\phi_i(x,t,q) = u_{i0}(x,t) + \sum_{k=1}^{+\infty} u_{ik}(x,t) q^k, \quad i = 1...m, \tag{4}$$

where

$$u_{ik}(x,t) = \frac{1}{k!} \frac{\partial^k \phi_i(x,t,q)}{\partial q^k}\Big|_{q=0}, \quad i = 1...m. \tag{5}$$

If the auxiliary linear operators, the initial guesses, the auxiliary parameter $\hbar$, and the auxiliary function are so properly chosen, then the series (4) converges at $q = 1$, then one has

$$\phi_i(x,t,1) = u_{i0}(x,t) + \sum_{k=1}^{+\infty} u_{ik}(x,t), \quad i = 1...m, \tag{6}$$

which must be one of the solutions of the original nonlinear equations, as proved by Liao. Define the vectors

$$\vec{u}_{in}(t) = \{u_{i0}(x,t), u_{i1}(x,t), ..., u_{in}(x,t)\}, \quad i = 1...m. \tag{7}$$

Differentiating (2), k times with respect to the embedding parameter q and then setting $q = 0$ and finally dividing them by $k!$, we have the so-called kth-order deformation equation

$$L_j[u_{ik}(x,t) - \chi_k u_{ik-1}(x,t)] = \hbar R_{jk}(\vec{u}_{ik-1}(x,t)), \quad i = 1\cdots m;\ j = 1\cdots n, \tag{8}$$

subject to the initial conditions

$$L_j(0) = 0,$$

where

$$R_{jk}(\vec{u}_{ik-1}(x,t)) = \frac{1}{(k-1)!}\frac{\partial^{k-1} N_j[\phi_1(x,t,q),\phi_2(x,t,q),\cdots\phi_m(x,t,q)]}{\partial q^{k-1}}\Big|_{q=0}, \tag{9}$$

and

$$\chi_m = \begin{cases} 0 & m \leq 1, \\ 1 & m > 1. \end{cases} \tag{10}$$

It should be emphasized that $u_{ik}(x,t)$ is governed by the linear equations (8) and (9) with the linear boundary conditions that come from the original problem. These equations can be easily solved by symbolic computation softwares such as Maple and Mathematica.

3. Homotopy Padé Method

Traditionally the $[m,n]$ Padé for $u(x,t)$ is in the form

$$\frac{\sum_{k=0}^{m} F_k(x)t^k}{1 + \sum_{k=1}^{n} F_{m+1+k}(x)t^k},$$

or

$$\frac{\sum_{k=0}^{m} G_k(t)x^k}{1 + \sum_{k=1}^{n} G_{k+m+1}(t)x^k},$$

where $F_k(r)$ and $G_k(t)$ are functions.

In Homotopy Padé approximation, we employ the traditional Padé technique to the series (4) for the embedding parameter q to gain the $[m,n]$ Padé approximation in the form of

$$\frac{\sum_{k=0}^{m} w_k(x,t)q^k}{1+\sum_{k=1}^{n} w_{m+k+1}(x,t)q^k}, \tag{11}$$

where $w_k(t,x)$ is a function and for $i = 0,1,\cdots,m,m+2,\cdots,m+n+1$, $w_i(x,t)$ is determined by product of the denominator of the above expression in the $\sum_{i=0}^{m+n} u_i(x,t)q^i$ and equating the powers of q^i, $i = 0,1,\cdots,m+n$. Thus we have $m+n+1$ equations and $m+n+1$ unknowns $w_i(x,t)$, $\quad i =$

$0, 1, \cdots, m, m+2, \cdots, m+n+1$. By setting $q = 1$ in (11) the so-called $[m, n]$ Homotopy Padé approximation in the following form is yield.

$$\frac{\sum_{k=0}^{m} w_k(x,t)}{1+\sum_{k=1}^{n} w_{m+k+1}(x,t)}. \tag{12}$$

It is found that the $[m, n]$ Homotopy Padé approximation often converges faster than the corresponding traditional $[m, n]$ Padé approximation and in many cases the $[m, m]$ Homotopy Padé approximation is independent of the auxiliary parameter $\hbar$. In these cases, even if the corresponding solution series diverge, utilizing the Homotopy-Padé technique will result in a convergent series [22]. However, up to now, It has not seen a mathematical proof about it in general cases in literature[7, 10].

The Homotopy-Padé technique can greatly enlarge the convergence region of the solution series. Besides, the results solutions of Homotopy Padé often converge faster than solutions calculated by Homotopy analysis method.

4. Applications

In this section we apply HAM and HPadéM to solve the two-dimensional coupled Burgers' equations. In all cases, we assume that the initial guesses are $u_0(x, t) = u(x, 0)$ and $v_0(x, t) = v(x, 0)$ i.e. the initial conditions. We use the auxiliary linear operator $L_j = \frac{\partial}{\partial t}$ and the auxiliary function $H(x, t) = 1$. We give approximations of computed error terms to show the efficiency of HAM and HPadéM.

4.1. Two-dimensional coupled Burgers' equations.

Let us consider the two-dimensional coupled Burgers' equations [18, 21, 22]

$$\begin{aligned} u_t + uu_x + vu_y &= \frac{1}{Re}(u_{xx} + u_{yy}), \\ v_t + uv_x + vv_y &= \frac{1}{Re}(v_{xx} + v_{yy}), \end{aligned} \tag{13}$$

where Re is the Reynolds number, with the exact solutions

$$\begin{aligned} u(x, y, t) &= \frac{3}{4} - \frac{1}{4(1+\exp(\frac{Re(-t-4x+4y)}{32}))}, \\ v(x, y, t) &= \frac{3}{4} + \frac{1}{4(1+\exp(\frac{Re(-t-4x+4y)}{32}))}. \end{aligned} \tag{14}$$

Employing HAM with mentioned parameters in section 2, we have the following zero-order deformation equations

$$\begin{aligned} (1-q)L_1[\phi_{1t} - u(x, y, 0)] &= q\hbar[\phi_{1t} + \phi_1\phi_{1x} + \phi_2\phi_{1y} - \frac{1}{Re}(\phi_{1xx} + \phi_{1yy})], \\ (1-q)L_2[\phi_{2t} - v(x, y, 0)] &= q\hbar[\phi_{2t} + \phi_1\phi_{2x} + \phi_2\phi_{2y} - \frac{1}{Re}(\phi_{2xx} + \phi_{2yy})]. \end{aligned} \tag{15}$$

Subsequently solving the Nth order deformation equations one has

$$u_0(x,y,t) = \frac{3}{4} - \frac{1}{4(1+\exp(\frac{Re(-x+y)}{8}))},$$
$$v_0(x,y,t) = \frac{3}{4} + \frac{1}{4(1+\exp(\frac{Re(-x+y)}{8}))},$$
$$u_1(x,y,t) = \frac{Re\hbar t}{128}\frac{\exp(\frac{Re(-x+y)}{8})}{(1+\exp(\frac{Re(-x+y)}{8}))^2},$$
$$v_1(x,y,t) = \frac{-Re\hbar t}{128}\frac{\exp(\frac{Re(-x+y)}{8})}{(1+\exp(\frac{Re(-x+y)}{8}))^2},$$
$$u_2(x,y,t) = \frac{-Re\hbar t\exp(\frac{Re(-x+y)}{8})}{8192}(\frac{-64-64\exp(\frac{Re(-x+y)}{8})}{(1+\exp(\frac{Re(-x+y)}{8}))^3}$$
$$+\frac{Re\hbar t\exp(\frac{Re(-x+y)}{8}) - Re\hbar t - 64\hbar - 64\hbar\exp(\frac{Re(-x+y)}{8})}{(1+\exp(\frac{Re(-x+y)}{8}))^3}),$$
$$v_2(x,y,t) = \frac{Re\hbar t\exp(\frac{Re(-x+y)}{8})}{8192}(\frac{-64-64\exp(\frac{Re(-x+y)}{8})}{(1+\exp(\frac{Re(-x+y)}{8}))^3}$$
$$+\frac{Re\hbar t\exp(\frac{Re(-x+y)}{8}) - Re\hbar t - 64\hbar - 64\hbar\exp(\frac{Re(-x+y)}{8})}{(1+\exp(\frac{Re(-x+y)}{8}))^3}),$$

and so on.
We use an 11-term approximation and set
$app10{=}u_0 + u_1 + u_2 + ... + u_8 + u_9 + u_{10}$, and
$app10{=}v_0 + v_1 + v_2 + ... + v_8 + v_9 + v_{10}$.

We declare the results for 10th order HAM approximations in Tables 1 and 2. The results obtained with $\hbar = -1.1$ are better than $\hbar = -1$. Hence, the outputs of HAM are better than the Homotopy perturbation method. The influence of $\hbar$ on the convergence of the solution series are given in Figure 1. This figure was obtained by using 10th order HAM approximation for various values of x and t. It is easy to see that in order to have a good approximation, $\hbar$ has to be chosen in $-1.7 < \hbar < -0.7$. This means that for these values of $\hbar$ the series (6) converges to the exact solution (13). In Table 3, the absolute error of approximation results is given by $[5,5]$ HPad*é*M. These results highlighting high accuracy and high accelerate of HPad*é*M.

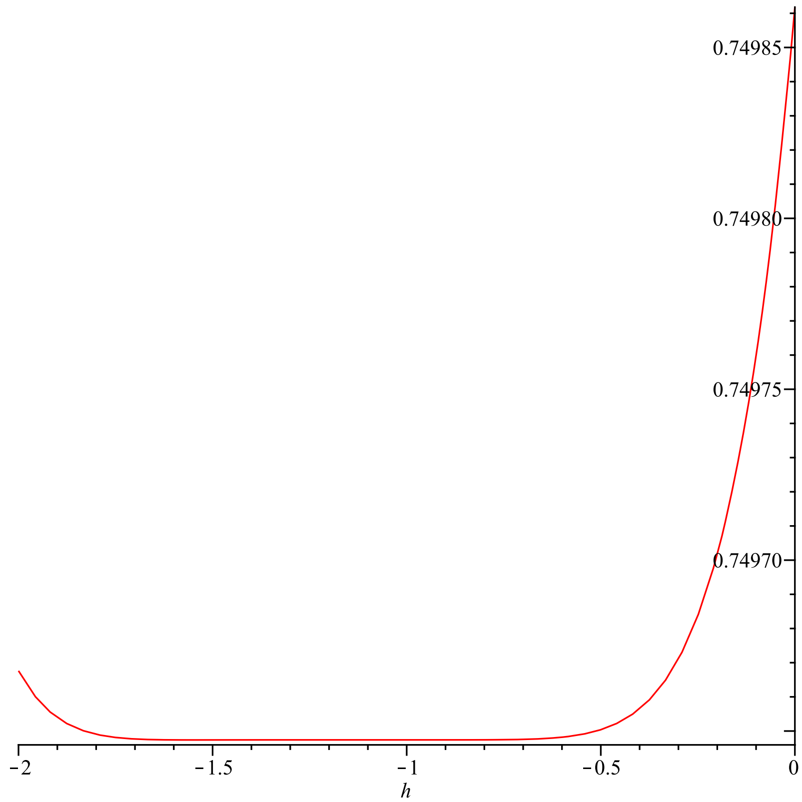

FIGURE 1. The $\hbar$ curve of two-dimensional coupled Burgers' equations for u(0.3,1,0.2) obtained from the 10th order HAM.

TABLE 1. Absolute error of two-dimensional coupled Burgers' equations for 10th order HAM with $\hbar = -1.1$ and $y = 1$

$R = 100$					
x	t=0.1	t=0.2	t=0.3	t=0.4	t=0.5
$\|u_a - u_e\|$					
0.1	3.47E-12	2.33E-11	1.51E-10	3.35E-10	6.07E-09
0.2	1.21E-11	8.15E-11	5.33E-10	1.24E-09	2.08E-08
0.3	4.24E-11	2.85E-10	1.92E-09	5.13E-09	6.75E-08
0.4	1.48E-10	1.00E-09	7.47E-09	2.78E-08	1.76E-07
0.5	5.22E-10	3.58E-09	3.47E-08	2.12E-07	8.81E-08
$\|v_a - v_e\|$					
0.1	3.47E-12	2.33E-11	1.51E-10	3.35E-10	6.07E-09
0.2	1.21E-11	8.15E-11	5.33E-10	1.24E-09	2.08E-08
0.3	4.24E-11	2.85E-10	1.92E-09	5.13E-09	6.75E-08
0.4	1.48E-10	1.00E-09	7.47E-09	2.78E-08	1.76E-07
0.5	5.22E-10	3.58E-09	3.47E-08	2.12E-07	8.81E-08

TABLE 2. Absolute error of two-dimensional coupled Burgers' equations for 10th order HAM $\hbar = -1$ and $y = 1$

$R = 100$					
x	t=0.1	t=0.2	t=0.3	t=0.4	t=0.5
$\|u_a - u_e\|$					
0.1	4.40E-12	2.95E-10	3.53E-09	2.08E-08	8.38E-08
0.2	1.53E-11	1.03E-09	1.23E-08	7.26E-08	2.92E-07
0.3	5.31E-11	3.56E-09	4.25E-08	2.51E-07	1.01E-06
0.4	1.80E-10	1.21E-08	1.44E-07	8.48E-07	3.40E-06
0.5	5.71E-10	3.80E-08	4.50E-07	2.63E-06	1.04E-05
$\|v_a - v_e\|$					
0.1	4.40E-12	2.95E-10	3.53E-09	2.08E-08	8.38E-08
0.2	1.53E-11	1.03E-09	1.23E-08	7.26E-08	2.92E-07
0.3	5.31E-11	3.56E-09	4.25E-08	2.51E-07	1.01E-06
0.4	1.80E-10	1.21E-08	1.44E-07	8.48E-07	3.40E-06
0.5	5.71E-10	3.80E-08	4.50E-07	2.63E-06	1.04E-05

5. CONCLUSION

In this paper, we approximate the solutions of the two-dimensional coupled Burgers' equations by the HAM and HPadéM. The convergence region for our approximation, are determined by the parameter $\hbar$, which provides us a great freedom to choose convenient value for it. It is illustrated efficiency and accuracy of proposed methods by implementing on the mentioned equation. It is shown the HPadéM accelerate the convergence of the related series.

Table 3. Absolute error of two-dimensional coupled Burgers' equations for $[5, 5]$ HPad$\acute{e}$M

$R = 100$					
x	t=0.1	t=0.2	t=0.3	t=0.4	t=0.5
$\lvert u_a - u_e \rvert$					
0.1	1.23E-21	3.46E-18	4.14E-16	1.36E-14	2.21E-13
0.2	4.29E-21	1.21E-17	1.45E-15	4.75E-14	7.72E-13
0.3	1.50E-20	4.22E-17	5.04E-15	1.66E-13	2.69E-12
0.4	5.22E-20	1.47E-16	1.76E-14	5.77E-13	9.36E-12
0.5	1.81E-19	5.11E-16	6.09E-14	2.00E-12	3.22E-11
$\lvert v_a - v_e \rvert$					
0.1	1.23E-21	3.46E-18	4.14E-16	1.36E-14	2.21E-13
0.2	4.29E-21	1.21E-17	1.45E-15	4.75E-14	7.72E-13
0.3	1.50E-20	4.22E-17	5.04E-15	1.66E-13	2.69E-12
0.4	5.22E-20	1.47E-16	1.76E-14	5.77E-13	9.36E-12
0.5	1.81E-19	5.11E-16	6.09E-14	2.00E-12	3.22E-11

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