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Left Jordan derivations on Banach algebras

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Abstract. In this paper we characterize the left Jordan derivations on Banach algebras. Also, it is shown that every bounded linear map $d : \mathcal{A} \to \mathcal{M}$ from a von Neumann algebra $\mathcal{A}$ into a Banach $\mathcal{A}-$module $\mathcal{M}$ with property that $d(p^2) = 2pd(d)$ for every projection p in $\mathcal{A}$ is a left Jordan derivation.

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1. Introduction

Let $\mathcal{A}$ be a unital Banach algebra. We denote the identity of $\mathcal{A}$ by 1. A Banach $\mathcal{A}-$module $\mathcal{M}$ is called unital provided that $1x = x = x1$ for each $x \in \mathcal{M}$. A linear (additive) mapping $d : \mathcal{A} \to \mathcal{M}$ is called a left derivations (left ring derivation) if

$$d(ab) = ad(b) + bd(a) \quad (a, b \in \mathcal{A}). \tag{1}$$

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Also, d is called a left Jordan derivation (or Jordan left derivation) if

$$d(a^2) = 2ad(a) \quad (a \in \mathcal{A}). \tag{2}$$

Bresar, Vukman [4], Ashraf et al. [1, 2], Jung and Park [8], Vukman [13, 14] studied left Jordan derivations and left derivations on prime rings and semiprime rings, which are in a close connection with so-called commuting mappings (see also [7, 10, 11, 12]).

Suppose that $\mathcal{A}$ is a Banach algebra and $\mathcal{M}$ is an $\mathcal{A}-$module. Let S be in A. We say that S is a left separating point of $\mathcal{M}$ if the condition $Sm = 0$ for $m \in \mathcal{M}$ implies $m = 0$.

We refer to [3] for the general theory of Banach algebras.

Theorem 1.1. *Let $\mathcal{A}$ be a unital Banach algebra and $\mathcal{M}$ be a Banach $\mathcal{A}-$module. Let S be in $\mathcal{Z}(\mathcal{A})$ such that S is a left separating point of $\mathcal{M}$. Let $f : \mathcal{A} \to \mathcal{M}$ be a bounded linear map. Then the following assertions are equivalent*

a) $f(ab) = af(b) + bf(a)$ for all $a, b \in \mathcal{A}$ with $ab = ba = S$.

b) f is a left Jordan derivation which satisfies $f(Sa) = Sf(a) + af(S)$ for all $a \in \mathcal{A}$.

Proof. First suppose that (a) holds. Then we have

$$f(S) = f(1S) = f(S)1 + f(1)S = f(S) + f(1)S$$

hence, by assumption, we get that $f(1) = 0$. Let $a \in \mathcal{A}$. For scalars λ with $|\lambda| < \frac{1}{\|a\|}$, $1 - \lambda a$ is invertible in $\mathcal{A}$. Indeed, $(1 - \lambda a)^{-1} = \sum_{n=0}^{\infty} \lambda^n a^n$. Then

$$\begin{aligned} f(S) &= f[(1-\lambda a)(1-\lambda a)^{-1}S] = ((1-\lambda a)^{-1}S)f((1-\lambda a)) \\ &\quad + (1-\lambda a)f((1-\lambda a)^{-1}S) \\ &= -\lambda(\sum_{n=0}^{\infty}\lambda^n a^n S)f(a) + (1-\lambda a)f(\sum_{n=1}^{\infty}\lambda^n a^n S) \\ &= f(S) + \sum_{n=1}^{\infty}\lambda^n[f(a^nS) - a^{n-1}Sf(a) - af(a^{n-1}S)]. \end{aligned}$$

So

$$\sum_{n=1}^{\infty}\lambda^n[f(a^nS) - a^{n-1}Sf(a) - af(a^{n-1}S)] = 0$$

for all λ with $|\lambda| < \frac{1}{\|a\|}$. Consequently

$$f(a^nS) - a^{n-1}Sf(a) - af(a^{n-1}S) = 0 \tag{3}$$

for all $n \in \mathbb{N}$. Put $n = 1$ in (3) to get

$$f(Sa) = f(aS) = af(S) + Sf(a). \tag{4}$$

for all $a \in \mathcal{A}$.

Now, put $n = 2$ in (3) to get

$$f(a^2S) = aSf(a) + af(aS) = aSf(a) + a(af(S) + Sf(a)) = a^2f(S) + 2Saf(a). \tag{5}$$

Replacing a by a^2 in (4), we get

$$f(a^2S) = a^2f(S) + Sf(a^2). \tag{6}$$

It follows from (5), (6) that

$$S(f(a^2) - 2af(a)) = 0. \tag{7}$$

On the other hand S is right separating point of $\mathcal{M}$. Then by (7), f is a left Jordan derivation.

Now suppose that the condition (b) holds. We denote $aob := ab + ba$ for all $a, b \in \mathcal{A}$. It follows from left Jordan derivation identity that

$$f(aob) = 2(bf(a) + af(b)) \tag{8}$$

for all $a, b \in \mathcal{A}$ (see proposition 1.1 of [4]). On the other hand, we have

$$a \circ (a \circ b) = a \circ (ab + ba) = a^2 \circ b + 2aba$$

for all $a, b \in \mathcal{A}$. Then

$$\begin{aligned} 2f(aba) &= f(a \circ (a \circ b)) - f(a^2 \circ b) \\ &= 2[(a \circ b)f(a) + af(a \circ b)] - 2[bf(a^2) + a^2f(b)] \\ &= 2[(ab + ba)f(a) + 2a(bf(a) + af(b))] - 2[2baf(a) + a^2f(b)] \\ &= 6abf(a) + 2a^2f(b) - 2baf(a). \end{aligned}$$

Hence,

$$f(aba) = 3abf(a) + a^2f(b) - baf(a) \tag{9}$$

for all $a, b \in \mathcal{A}$. Now suppose that $ab = ba = S$, then

$$f(Sa) = 3abf(a) + a^2f(b) - baf(a) = 2abf(a) + a^2f(b). \tag{10}$$

On the other hand $S \in \mathcal{Z}(\mathcal{A})$. Then by multiplying both sides of (10) by b to get

$$Sf(S) - Sbf(a) - Saf(b) = 0 \tag{11}$$

since $S \in \mathcal{Z}(\mathcal{A})$, then it follows from (11) that

$$[f(S) - f(a)b - f(b)a]S = 0$$

then we have

$$f(S) = f(a)b + f(b)a.$$

□

Now, we characterize the left Jordan derivations on von Neumann algebras.

Theorem 1.2. *Let $\mathcal{A}$ be a von Neumann algebra and let $\mathcal{M}$ be a Banach $\mathcal{A}-$module and $d : \mathcal{A} \to \mathcal{M}$ be a bounded linear map with property that $d(p^2) = 2pd(p)$ for every projection p in $\mathcal{A}$. Then d is a left Jordan derivation.*

Proof. Let $p, q \in \mathcal{A}$ be orthogonal projections in $\mathcal{A}$. Then $p+q$ is a projection wherefore by assumption,

$$\begin{aligned} 2pd(p) + 2qd(q) &= d(p) + d(q) = d(p+q) = 2(p+q)d(p+q) \\ &= 2[pd(p) + pd(q) + qd(q) + qd(p)]. \end{aligned}$$

It follows that

$$pd(q) + qd(p) = 0. \tag{12}$$

Let $a = \sum_{j=1}^{n} \lambda_j p_j$ be a combination of mutually orthogonal projections $p_1, p_2, ..., p_n \in \mathcal{A}$. Then we have

$$p_i d(p_j) + p_j d(p_i) = 0 \tag{13}$$

for all $i, j \in \{1, 2, ..., n\}$ with $i \neq j$. So

$$d(a^2) = d(\sum_{j=1}^{n} \lambda_j^2 p_j) = \sum_{j=1}^{n} \lambda_j^2 d(p_j). \tag{14}$$

On the other hand by (13), we obtain that

$$\begin{aligned} ad(a) = (\sum_{i=1}^{n} \lambda_i p_i) \sum_{j=1}^{n} \lambda_j d(p_j) &= \lambda_1 p_1 \sum_{j=1}^{n} \lambda_j d(p_j) \\ + \lambda_2 p_2 \sum_{j=1}^{n} \lambda_j d(p_j) &+ ... + \lambda_n p_n \sum_{j=1}^{n} \lambda_j d(p_j) \\ = \sum_{j=1}^{n} \lambda_j^2 p_j d(p_j). \end{aligned}$$

It follows from above equation and (14) that $d(a^2) = 2ad(a)$. By the spectral theorem (see Theorem 5.2.2 of [9]), every self adjoint element $a \in \mathcal{A}_{sa}$ is the norm–limit of a sequence of finite combinations of mutually orthogonal projections. Since d is bounded, then

$$d(a^2) = 2ad(a) \tag{15}$$

for all $a \in \mathcal{A}_{sa}$. Replacing a by $a+b$ in (15), we obtain

$$\begin{aligned} d(a^2 + b^2 + ab + ba) &= 2(a+b)(d(a) + d(b) \\ &= 2ad(a) + 2bd(b) + 2ad(b) + 2bd(a), \end{aligned}$$

$$d(ab + ba) = 2ad(b) + 2bd(a) \tag{16}$$

for all $a, b \in \mathcal{A}_{sa}$. Let $a \in \mathcal{A}$. Then there are $a_1, a_2 \in \mathcal{A}_{sa}$ such that $a = a_1 + ia_2$. Hence,

$$\begin{aligned} d(a^2) &= d(a_1^2 + a_2^2 + i(a_1a_2 + a_2a_1)) \\ &= 2a_1d(a_1) + 2a_2d(a_2) + i[2a_1d(a_2) + 2a_2d(a_1)] \\ &= 2ad(a). \end{aligned}$$

This completes the proof of theorem. □

Corollary 1.3. *Let $\mathcal{A}$ be a von Neumann algebra and let $\mathcal{M}$ be a Banach $\mathcal{A}-$module and $d : \mathcal{A} \to \mathcal{M}$ be a bounded linear map. Then the following assertions are equivalent*

a) $ad(a^{-1}) + a^{-1}d(a) = 0$ for all invertible $a \in \mathcal{A}$.
b) d is a left Jordan derivation.
c) $d(p^2) = 2pd(p)$ for every projection p in $\mathcal{A}$.

Proof. $(a) \Leftrightarrow (b)$ follows from Theorem 1.1, and $(b) \Leftrightarrow (c)$ follows from Theorem 1.2. □

In 1996, Johnson [6] proved the following theorem (see also Theorem 2.4 of [5]).

Theorem 1.4. *Suppose $\mathcal{A}$ is a C^*- algebra and $\mathcal{M}$ is a Banach $\mathcal{A}-$module. Then each Jordan derivation $d : \mathcal{A} \to \mathcal{M}$ is a derivation.*

We do not know whether or not every left Jordan derivation on a C^*-algebra is a left derivation.

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