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Unification of Generalized Open Sets with Respect to an Ideal

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Abstract. This paper deals with the concepts of $\mathscr{I}_\mu$-g-closed sets, $\mathscr{I}_\mu$-semi-open sets and $\mathscr{I}_\mu$-semi-preopen sets in ideal topological spaces and investigate several properties of these sets. Some characterizations of $\mathscr{I}_\mu$-regular spaces and $\mathscr{I}_\mu$-normal spaces are discussed.

Keywords: $\mathscr{I}_\mu$-g-closed set, $\mathscr{I}_\mu$-semi-open set, $\mathscr{I}_\mu$-semi-preopen set, $\mathscr{I}_\mu$-normal space, $\mathscr{I}_\mu$-regular space.

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1. Introduction

General topology has shown its fruitfulness in both the pure and applied directions. The theory of generalized topological spaces, which was founded by Császár [1], is one of the most important development of general topology in recent years. Especially, the author defined some basic operators on generalized topological spaces. It is observed that a large number of papers is devoted to the study of generalized open like sets of a topological space containing the class of open sets and possessing properties more or less similar to those of open sets. The concept of ideals in topological spaces has been studied by Kuratowski [10] and Vaidyanathaswamy [18]. Janković and Hamlett [9] further studied ideal topological spaces and their applications to various fields. Levine [11] introduced the notions of generalized closed sets and a separation axiom called

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$T_{\frac{1}{2}}$ in topological spaces. Recently, many variations of generalized closed sets are introduced and investigated. By combining a topological space (X, τ), an ideal $\mathscr{I}$ on (X, τ) and an operation γ on τ, Dontchev et al. [6] introduced the notion of $(\tau, \mathscr{I}, \gamma)$-g-closed sets. In [15], the authors called $\mathscr{I}$-g-closed sets and investigated the further properties of $\mathscr{I}$-g-closed sets. In 2008, Renukadevi and Sivaraj [17] introduced a new class of spaces called $\mathscr{I}$-normal spaces which contains the class of all normal spaces and discussed some of their properties. Jafari and Rajesh [8] introduced and investigated the concept of generalized closed sets with respect to an ideal. Michael [13] introduced the notion of semi-open sets in terms of ideals, which generalized the usual notion of semi-open sets. Nasef et al. [14] introduced the concept of β-open sets with respect to an ideal $\mathscr{I}$, and also studied some of their properties. In the present paper, we introduce the concepts of $\mathscr{I}_\mu$-g-closed sets, $\mathscr{I}_\mu$-semi-open sets and $\mathscr{I}_\mu$-semi-preopen sets and investigate some of their fundamental properties. Finally, some characterizations of $\mathscr{I}_\mu$-nornal spaces and $\mathscr{I}_\mu$-regular spaces have been given.

2. Preliminaries

Throughout the paper (X, τ) (or simply X) will always denote a topological space on which no separation axioms are assumed unless explicitly stated. For a subset A of X, the closure, interior and complement of A in (X, τ) are denoted by $Cl(A)$, $Int(A)$ and $X - A$, respectively. A subset A of a topological space (X, τ) is called *preopen* [12] if $A \subseteq Int(Cl(A))$. An ideal $\mathscr{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies (i) $A \in \mathscr{I}$ and $B \subseteq A$ implies $B \in \mathscr{I}$ and (ii) $A \in \mathscr{I}$ and $B \in \mathscr{I}$ implies $A \cup B \in \mathscr{I}$. Let X be a non-empty set, and denote $\mathscr{P}(X)$ the power set of X. We call a class $\mu \subseteq \mathscr{P}(X)$ a *generalized topology* (briefly, GT) on X if $\emptyset \in \mu$, and an arbitrary union of elements of μ belongs to μ [2]. A set X with a GT μ on it is said to be a *generalized topological space* (briefly, GTS) and is denoted by (X, μ). For a GTS (X, μ), the elements of μ are called *μ-open* sets and the complements of μ-open sets are called *μ-closed* sets. For $A \subseteq X$, we denote by $c_\mu(A)$ the intersection of all μ-closed sets containing A and by $i_\mu(A)$ the union of all μ-open sets contained in A [4]. Then, we have $i_\mu(i_\mu(A)) = i_\mu(A)$, $c_\mu(c_\mu(A)) = c_\mu(A)$, and $i_\mu(A) = X - c_\mu(X - A)$. According to [5], for $A \subseteq X$ and $x \in X$, we have $x \in c_\mu(A)$ if and only if $x \in M \in \mu$ implies $M \cap A \neq \emptyset$. A subset A of a generalized topological space (X, μ) is called *μ-dense* [7] if $c_\mu(A) = X$. Let μ be a GT on a set $X \neq \emptyset$. Observe that $X \in \mu$ must not hold; if all the same $X \in \mu$, then we say that the GT μ is *strong* [3]. Let μ be a generalized topology on a topological space (X, τ). Then $A \subseteq X$ is called a *generalized μ-closed* (or simply *$g\mu$-closed*) set [16] if $c_\mu(A) \subseteq U$ whenever $A \subseteq U \in \tau$. The complement of a $g\mu$-closed set is called a *generalized μ-open* (or simply *$g\mu$-open*) set.

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3. Properties of $\mathscr{I}_\mu$ generalized open sets

We begin this section by introducing the concept of $\mathscr{I}_\mu$-g-closed sets.

Definition 3.1. Let μ be a generalized topology on an ideal topological space $(X, \tau, \mathscr{I})$. A subset A of X is said to be *$\mathscr{I}_\mu$-g-closed* if $c_\mu(A) - U \in \mathscr{I}$ whenever U is open and $A \subseteq U$. A subset A of X is said to be *$\mathscr{I}_\mu$-g-open* if $X - A$ is $\mathscr{I}_\mu$-g-closed.

Remark 3.2. Every $g\mu$-closed set is $\mathscr{I}_\mu$-g-closed, but the converse need not be true, as this may be seen from the following example.

Example 3.3. Let $X = \{a, b, c\}$ with topology $\tau = \{\emptyset, \{a\}, \{a, c\}, X\}$, generalized topology $\mu = \{\emptyset, \{a\}, \{a, c\}\}$ and ideal $\mathscr{I} = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}$. Then, the set $\{c\}$ is $\mathscr{I}_\mu$-g-closed but it is not $g\mu$-closed.

Proposition 3.4. *Let μ be a generalized topology on an ideal topological space $(X, \tau, \mathscr{I})$. If A is $\mathscr{I}_\mu$-g-closed and $A \subseteq B \subseteq c_\mu(A)$, then B is $\mathscr{I}_\mu$-g-closed.*

Proof. Suppose that A is $\mathscr{I}_\mu$-g-closed. Let $B \subseteq U$ and U is open. Then $A \subseteq U$. Since A is $\mathscr{I}_\mu$-g-closed, we have $c_\mu(A) - U \in \mathscr{I}$. Since $B \subseteq c_\mu(A)$, $c_\mu(B) \subseteq c_\mu(A)$. This implies that $c_\mu(B) - U \subseteq c_\mu(A) - U \in \mathscr{I}$ and hence, B is $\mathscr{I}_\mu$-g-closed. □

Remark 3.5. The intersection of two $\mathscr{I}_\mu$-g-closed sets need not be $\mathscr{I}_\mu$-g-closed as shown by the following example.

Example 3.6. Let $X = \{a, b, c\}$ with topology $\tau = \{\emptyset, \{a\}, \{a, c\}, X\}$, generalized topology $\mu = \{\emptyset, \{a\}, \{a, c\}\}$ and ideal $\mathscr{I} = \{\emptyset, \{b\}\}$. Then $\{a, b\}$ and $\{a, c\}$ are $\mathscr{I}_\mu$-g-closed sets but their intersection $\{a\}$ is not $\mathscr{I}_\mu$-g-closed.

Theorem 3.7. *Let μ be a generalized topology on an ideal topological space $(X, \tau, \mathscr{I})$. A set A is $\mathscr{I}_\mu$-g-open if and only if $F - U \subseteq i_\mu(A)$, for some $U \in \mathscr{I}$, whenever $F \subseteq A$ and F is closed.*

Proof. Suppose that $F \subseteq A$ and F is closed. We have $X - A \subseteq X - F$. By assumption, $c_\mu(X - A) \subseteq (X - F) \cup U$, for some $U \in \mathscr{I}$. This implies

$$X - [(X - F) \cup U] \subseteq X - [c_\mu(X - A)]$$

and hence, $F - U \subseteq i_\mu(A)$.

Conversely, let G be a open set such that $X - A \subseteq G$. This implies that $X - G \subseteq A$. By assumption, $(X - G) - U \subseteq i_\mu(A) = X - c_\mu(X - A)$. This gives that $X - (G \cup U) \subseteq X - c_\mu(X - A)$. Then $c_\mu(X - A) \subseteq G \cup U$, for some $U \in \mathscr{I}$. This shows that $c_\mu(X - A) - G \in \mathscr{I}$. Therefore, we have $X - A$ is $\mathscr{I}_\mu$-g-closed and hence, A is $\mathscr{I}_\mu$-g-open. □

Proposition 3.8. *Let μ be a generalized topology on an ideal topological space $(X, \tau, \mathscr{I})$. If A is $\mathscr{I}_\mu$-g-open and $i_\mu(A) \subseteq B \subseteq A$, then B is $\mathscr{I}_\mu$-g-open.*

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Proof. Suppose that $i_\mu(A) \subseteq B \subseteq A$ and A is $\mathscr{I}_\mu$-g-open. Then $X - A \subseteq X - B \subseteq c_\mu(X - A)$ and $X - A$ is $\mathscr{I}_\mu$-g-closed. By Proposition 3.4, $X - B$ is $\mathscr{I}_\mu$-g-closed and so B is $\mathscr{I}_\mu$-g-open. □

Definition 3.9. Let μ be a generalized topology on a topological space (X, τ). A subset A of X is said to be *τ_μ-semi-open* if there exists an open set U such that $U \subseteq A \subseteq c_\mu(U)$.

Proposition 3.10. *Let μ be a generalized topology on a topological space (X, τ). A subset A of X is τ_μ-semi-open if and only if $A \subseteq c_\mu(Int(A))$.*

Definition 3.11. Let μ be a strong generalized topology on an ideal topological space $(X, \tau, \mathscr{I})$. A subset A of X is said to be *$\mathscr{I}_\mu$-semi-open* if there exists a open set U such that $U - A \in \mathscr{I}$ and $A - c_\mu(U) \in \mathscr{I}$. A subset A of X is said to be *$\mathscr{I}_\mu$-semi-closed* if $X - A$ is $\mathscr{I}_\mu$-semi-open.

Remark 3.12. Every τ_μ-semi-open set is $\mathscr{I}_\mu$-semi-open, but the converse need not be true, as this may be seen from the following example.

Example 3.13. In Example 3.3, $\{b\}$ is $\mathscr{I}_\mu$-semi-open but it is not τ_μ-semi-open.

Theorem 3.14. *Let μ be a generalized topology on an ideal topological space $(X, \tau, \mathscr{I})$. For a subset A of X, the following properties are equivalent:*

(1) *A is $\mathscr{I}_\mu$-semi-closed.*
(2) *There exists a closed set F such that $i_\mu(F) - A \in \mathscr{I}$ and $A - F \in \mathscr{I}$.*

Proof. (1) $\Rightarrow$ (2): Suppose that A is $\mathscr{I}_\mu$-semi-closed. Then $X - A$ is $\mathscr{I}_\mu$-semi-open. There exists an open set V such that $V - (X - A) \in \mathscr{I}$ and $(X - A) - c_\mu(V) \in \mathscr{I}$. Since $V - (X - A) = A - (X - V)$ and $(X - A) - c_\mu(V) = i_\mu(X - V) - A$, we have (2) by choosing the closed set $X - V$ as F.

(2) $\Rightarrow$ (1): Suppose that (2) hold, then the choice of the open set $U = X - F$ shows that $X - A$ is $\mathscr{I}_\mu$-semi-open and so A is $\mathscr{I}_\mu$-semi-closed. □

Proposition 3.15. *Let μ be a strong generalized topology on an ideal topological space $(X, \tau, \mathscr{I})$. If A and B are $\mathscr{I}_\mu$-semi-closed, then so is their intersection $A \cap B$.*

Proof. Let the given conditions hold. There are closed sets F_1 and F_2 such that $i_\mu(F_1) - A$, $A - F_1 \in \mathscr{I}$ and $i_\mu(F_2) - B$, $B - F_2 \in \mathscr{I}$. Put $F = F_1 \cap F_2$, we have that

$$\begin{aligned} i_\mu(F_1 \cap F_2) - A \cap B &\subseteq i_\mu(F_1) \cap i_\mu(F_2) - A \cap B \\ &= [(i_\mu(F_1) - A) \cap i_\mu(F_2)] \cup [(i_\mu(F_2) - B) \cap i_\mu(F_1)] \in \mathscr{I} \end{aligned}$$

and $(A \cap B) - (F_1 \cap F_2) = [(A - F_1) \cap B] \cup [(B - F_2) \cap A] \in \mathscr{I}$. Therefore, $A \cap B$ is $\mathscr{I}_\mu$-semi-closed. □

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Definition 3.16. Let μ be a generalized topology on a topological space (X, τ). A subset A of X is said to be *τ_μ-semi-preopen* if there exists a preopen set U such that $U \subseteq A \subseteq c_\mu(U)$.

Proposition 3.17. *Let μ be a generalized topology on a topological space (X, τ). If A is a τ_μ-semi-preopen set, then $A \subseteq c_\mu(Int(Cl(A)))$.*

The converse of above proposition is not true in general, which follows from the following example.

Example 3.18. Let $X = \{a, b, c\}$ with topology $\tau = \{\emptyset, \{a\}, \{a, c\}, \{a, b\}, X\}$ and generalized topology $\mu = \{\emptyset, \{a\}\}$. Then $\{b, c\} \subseteq c_\mu(Int(Cl(\{b, c\})))$ but $\{b, c\}$ is not τ_μ-semi-preopen.

Definition 3.19. Let μ be a generalized topology on an ideal topological space $(X, \tau, \mathscr{I})$. A subset A of X is said to be *$\mathscr{I}_\mu$-semi-preopen* if there exists a preopen set U such that $U - A \in \mathscr{I}$ and $A - c_\mu(U) \in \mathscr{I}$. A subset A of X is said to be *$\mathscr{I}_\mu$-semi-preclosed* if $X - A$ is $\mathscr{I}_\mu$-semi-preopen.

Remark 3.20. From the definitions one may deduce the following implications:

$$\begin{array}{ccccc}
\mu\text{-open} & \Longrightarrow & g\mu\text{-open} & \Longrightarrow & \mathscr{I}_\mu\text{-g-open} \\
\Downarrow & & & & \\
\tau_\mu\text{-semi-open} & \Longrightarrow & \mathscr{I}_\mu\text{-semi-open} & & \\
\Downarrow & & \Downarrow & & \\
\tau_\mu\text{-semi-preopen} & \Longrightarrow & \mathscr{I}_\mu\text{-semi-preopen} & &
\end{array}$$

However, none of these implications is reversible as shown by the following examples:

Example 3.21. Let $X = \{a, b, c\}$ with topology $\tau = \{\emptyset, \{a, b\}, X\}$ and generalized topology $\mu = \{\emptyset, \{a, b\}, \{b, c\}, X\}$. Then, the set $\{a, c\}$ is τ_μ-semi-preopen but it is not τ_μ-semi-open.

Example 3.22. Let $X = \{a, b, c\}$ with topology $\tau = \{\emptyset, \{a\}, \{a, c\}, X\}$, generalized topology $\mu = \{\emptyset, \{a\}, \{a, c\}\}$ and ideal $\mathscr{I} = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}$. Then, the set $\{a, b\}$ is τ_μ-semi-open but it is not μ-open. The set $\{b\}$ is both $g\mu$-open and $\mathscr{I}_\mu$-semi-preopen but it is neither μ-open nor τ_μ-semi-preopen.

Example 3.23. Let $X = \{a, b, c\}$ with topology $\tau = \{\emptyset, X\}$, generalized topology $\mu = \{\emptyset, \{a, b\}, \{b, c\}, X\}$ and ideal $\mathscr{I} = \{\emptyset\}$. Then, the set $\{a, b\}$ is $\mathscr{I}_\mu$-semi-preopen but it is not $\mathscr{I}_\mu$-semi-open.

Theorem 3.24. *Let μ be a generalized topology on an ideal topological space $(X, \tau, \mathscr{I})$, where every non-empty preopen subset of X is μ-dense, and the collection of preopen subsets of X satisfies the finite intersection property.*

(1) *If A is $\mathscr{I}_\mu$-semi-preopen and $A \subseteq B$, then B is $\mathscr{I}_\mu$-semi-preopen.*
(2) *If A is $\mathscr{I}_\mu$-semi-preopen, then so is $A \cup B$, for any subset B of X.*

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(3) *If A and B are both $\mathscr{I}_\mu$-semi-preopen, then so is their intersection $A \cap B$.*

Proof. (1) Suppose that A is $\mathscr{I}_\mu$-semi-preopen, and that $A \subseteq B$. There is a preopen set U such that $U - A \in \mathscr{I}$ and $A - c_\mu(U) \in \mathscr{I}$. Since $A \subseteq B$, we have that $U - B \subseteq U - A \in \mathscr{I}$; moreover, $B - c_\mu(U) = B - X = \emptyset \in \mathscr{I}$. Thus, B is $\mathscr{I}_\mu$-semi-preopen.

(2) Since $A \subseteq B \Leftrightarrow A \cup B = B$, (2) immediately follows from (1).

(3) Suppose that both A and B are $\mathscr{I}_\mu$-semi-preopen. By assumption, there are preopen sets U and V such that $U - A, A - c_\mu(U) \in \mathscr{I}$ and $V - B$, $B - c_\mu(V) \in \mathscr{I}$. Consider the preopen set $U \cap V$, which is non-empty (by the finite intersection property). Since

$$(U \cap V) - (A \cap B) = [((U - A) \cap V)] \cup [(U \cap (V - B))] \in \mathscr{I}$$

and $(A \cap B) - c_\mu(U \cap V) = (A \cap B) - X = \emptyset \in \mathscr{I}$, it follows that $A \cap B$ is $\mathscr{I}_\mu$-semi-preopen. □

Theorem 3.25. *Under the condition of Theorem 3.24, we have that A is $\mathscr{I}_\mu$-semi-preopen if and only if $c_\mu(A)$ is $\mathscr{I}_\mu$-semi-preopen.*

Proof. If A is $\mathscr{I}_\mu$-semi-preopen, then $c_\mu(A)$ is $\mathscr{I}_\mu$-semi-preopen, by Theorem 3.24(2).

Conversely, suppose that $c_\mu(A)$ is $\mathscr{I}_\mu$-semi-preopen. Then there is a preopen set U such that $U - c_\mu(A) \in \mathscr{I}$ and $c_\mu(A) - c_\mu(U) \in \mathscr{I}$. To show that A is $\mathscr{I}_\mu$-semi-preopen, consider the μ-open set $V = U - c_\mu(A) = U \cap [X - c_\mu(A)] \in \mathscr{I}$, by assumption. We have that $V - A = U \cap [X - c_\mu(A)] \cap (X - A) \in \mathscr{I}$, because of the heredity property; moreover, $A - c_\mu(V) = A - c_\mu(U \cap [X - c_\mu(A)]) = A - X = \emptyset \in \mathscr{I}$. This shows that A is $\mathscr{I}_\mu$-semi-preopen. □

Theorem 3.26. *Let μ be a generalized topology on an ideal topological space $(X, \tau, \mathscr{I})$. For a subset A of X, the following properties are equivalent:*

(1) *A is $\mathscr{I}_\mu$-semi-preclosed.*
(2) *There exists a preclosed set F such that $i_\mu(F) - A \in \mathscr{I}$ and $A - F \in \mathscr{I}$.*

Proof. The proof is similar to that of Theorem 3.14. □

Proposition 3.27. *Let μ be a strong generalized topology on an ideal topological space $(X, \tau, \mathscr{I})$. If A and B are $\mathscr{I}_\mu$-semi-preclosed, then so is their intersection $A \cap B$.*

Proof. The proof is similar to that of Proposition 3.15. □

4. Some separation axioms in ideal topological spaces

In this section, we investigate some characterizations of $\mathscr{I}_\mu$-normal spaces and $\mathscr{I}_\mu$-regular spaces.

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Definition 4.1. Let μ be a generalized topology on an ideal topological space $(X, \tau, \mathscr{I})$. An ideal generalized topological space $(X, \mu, \mathscr{I})$ is said to be $\mathscr{I}_\mu$-*normal* if for every pair of disjoint closed sets A and B of X, there exist disjoint μ-open sets U and V such that $A - U \in \mathscr{I}$ and $B - V \in \mathscr{I}$.

Theorem 4.2. *Let μ be a generalized topology on an ideal topological space $(X, \tau, \mathscr{I})$. Then the following are properties equivalent:*

(1) *$(X, \tau, \mathscr{I})$ is $\mathscr{I}_\mu$-normal.*
(2) *For every closed set F and open set G containing F, there exists a μ-open set V such that $F - V \in \mathscr{I}$ and $c_\mu(V) - G \in \mathscr{I}$.*
(3) *For each pair of disjoint closed sets A and B, there exists a μ-open set U such that $A - U \in \mathscr{I}$ and $c_\mu(U) \cap B \in \mathscr{I}$.*

Proof. (1) $\Rightarrow$ (2): Let F be closed and G be open such that $F \subseteq G$. Then $X - G$ is a closed set such that $(X - G) \cap F = \emptyset$. By hypothesis, there exist disjoint μ-open sets U and V such that $(X - G) - U \in \mathscr{I}$ and $F - V \in \mathscr{I}$. Now $U \cap V = \emptyset$ implies that $c_\mu(V) \subseteq X - U$ and so $(X - G) \cap c_\mu(V) \subseteq (X - G) \cap (X - U)$ which in turn implies that $c_\mu(V) - G \subseteq (X - G) - U \in \mathscr{I}$ and hence, $c_\mu(V) - G \in \mathscr{I}$.

(2) $\Rightarrow$ (3): Let A and B be disjoint closed subsets of X. Then $A \subseteq X - B$ and $X - B$ is open. There exists a μ-open set U such that $A - U \in \mathscr{I}$ and $c_\mu(U) - (X - B) \in \mathscr{I}$ which implies that $A - U \in \mathscr{I}$ and $c_\mu(U) \cap B \in \mathscr{I}$.

(3) $\Rightarrow$ (1): Let A and B be disjoint μ-closed subsets of X. Then there exists a μ-open set U such that $A - U \in \mathscr{I}$ and $c_\mu(U) \cap B \in \mathscr{I}$. Now $c_\mu(U) \cap B \in \mathscr{I}$ implies that $B - [X - c_\mu(U)] \in \mathscr{I}$. Put $V = X - c_\mu(U)$, then V is a μ-open set such that $B - V \in \mathscr{I}$ and $U \cap V = U \cap [X - c_\mu(U)] = \emptyset$. This shows that $(X, \tau, \mathscr{I})$ is $\mathscr{I}_\mu$-normal. □

Definition 4.3. Let μ be a generalized topology on an ideal topological space $(X, \tau, \mathscr{I})$. An ideal generalized topological space $(X, \tau, \mathscr{I})$ is said to be $\mathscr{I}_\mu$-*regular* if for each closed set F and a point $x \notin F$, there exist disjoint μ-open sets U and V such that $x \in U$ and $F - V \in \mathscr{I}$.

Theorem 4.4. *Let μ be a generalized topology on an ideal topological space $(X, \tau, \mathscr{I})$. Then the following properties are equivalent:*

(1) *$(X, \tau, \mathscr{I})$ is $\mathscr{I}_\mu$-regular.*
(2) *For each $x \in X$ and an open set U containing x, there is a μ-open set V containing x such that $c_\mu(V) - U \in \mathscr{I}$.*
(3) *For each $x \in X$ and closed sets F not containing x, there is a μ-open set U containing x such that $c_\mu(V) \cap F \in \mathscr{I}$.*

Proof. (1) $\Rightarrow$ (2): Let $x \in X$ and U be an open set containing x. Then, there exist disjoint μ-open sets V and W such that $x \in V$ and $(X - U) - W \in \mathscr{I}$. Since $(X - U) - W \in \mathscr{I}$, $X - U \subseteq W \cup [(X - U) - W]$. Now $V \cap W = \emptyset$ implies

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that $V \subseteq X - W$ and so $c_\mu(V) \subseteq X - W$. Hence,

$$\begin{aligned} c_\mu(V) - U &\subseteq (X - W) \cap [W \cup ((X - U) - W)] \\ &= (X - W) \cap [(X - U) - W] \\ &\subseteq (X - U) - W \in \mathscr{I}. \end{aligned}$$

(2) $\Rightarrow$ (3): Let F be a closed subset of X such that $x \notin F$. Then, there exists a μ-open set V containing x such that $c_\mu(V) - (X - F) \in \mathscr{I}$ which implies that $c_\mu(V) \cap F \in \mathscr{I}$.

(3) $\Rightarrow$ (1): Let F be a closed subset of X such that $x \notin F$. Then, there exists a μ-open set V containing x such that $c_\mu(V) \cap F \in \mathscr{I}$. Put $W = X - c_\mu(V)$, then W is a μ-open set such that $F - W \in \mathscr{I}$ and $V \cap W = V \cap (X - c_\mu(V)) = \emptyset$. This shows that $(X, \tau, \mathscr{I})$ is $\mathscr{I}_\mu$-regular. □

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References

1. Á. Császár, Generalized Open Sets, *Acta Mathematca Hungarica*, **75**(1-2), (1997), 65–87.
2. Á. Császár, Generalized Topology, Generalized Continuity, *Acta Mathematica Hungarica*, **96**(4), (2002), 351–357.
3. Á. Császár, Extremally Disconnected Generalized Topologies, *Annales Universitatis Scientiarium Budapestinensis de Rolando Eötvös Nominatae Sectio Mathematica*, **47**, (2004), 91–96.
4. Á. Császár, Generalized Open Sets in Generalized Topologies, *Acta Mathematica Hungarica*, **106**(1-2), (2005), 53–66.
5. Á. Császár, δ- and θ- modifications of Generalized Topologies, *Acta Mathematica Hungarica*, **120**(3), (2008), 275–279.
6. J. Dontchev, M. Ganster, T. Noiri, Unified Operations Approach of Generalized Closed Sets via Topological Ideals, *Mathematica Japonica*, **49**, (1999), 395–401.
7. E. Ekici, Generalized Submaximal Spaces, *Acta Mathematica Hungarica*, **134**(1-2), (2012), 132–138.
8. S. Jafari, N. Rajesh, Generalized Closed Sets with Respect to an Ideal, *European Journal of Pure and Applied Mathematics*, **4**(2), (2011), 147–151.
9. D. Janković, T. R. Hamlett, New Topologies from Old via Ideals, *The American Mathematical Monthly*, **97**, (1990), 295–310.
10. K. Kuratowski, *Topology*, Vol. I, New York, Academic Press, 1966.
11. N. Levine, Generalized Closed Sets in Topology, *Rendiconti del Circolo Matematico di Palermo Series 2*, **19**, (1970), 89–96.
12. A. S. Mashhour, M. E. Abd El-Monsef, S. N. El-Deeb, On Precontinuous and Weak Precontinuous Mappings, *Proceedings of the Mathematical and Physical Society Egypt*, **53**, (1982), 47–53.
13. F. I. Michael, On Semi-open Sets with Respect to an Ideal, *European Journal of Pure and Applied Mathematics*, **6**(1), (2013), 53–58.
14. A. A. Nasef, R. Mareay, E. I. Michael, Idealization of Some Topological Concepts, *European Journal of Pure and Applied Mathematics*, **8**(3), (2015), 389–394.

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[DOI: 10.61186/ijmsi.19.1.85]

15. M. Navaneethakrishnan, J. P. Joseph, g-closed Sets in Ideal Topological Spaces, *Acta Mathematica Hungarica*, **119**(4), (2008), 365–371.
16. T. Noiri, B. Roy, Unification of Generalized Open Sets on Topological Spaces, *Acta Mathematica Hungarica*, **130**(4), (2011), 349–357.
17. V. Renukadevi, D. Sivaraj, A Generalized of Normal Spaces, *Archivum Mathematicum*, **44**, (2008), 265–270.
18. R. Vaidyanathaswamy, The Localization Theory in Set Topology, *Proceedings of the Indian Academy Sciences Section A*, **20**, (1944), 51–61.

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[DOI: 10.61186/ijmsi.19.1.85]