Iranian Journal of Mathematical Sciences and Informatics
Vol. 19, No. 1 (2024), pp 107-115
DOI: 10.61186/ijmsi.19.1.107

# New Characterizations of Semisimple Alternative Algebras

Said Boulmane

Department of Economic Sciences and Management
Moulay Ismail University, Polydisciplinary Faculty, Errachidia-Morocco

E-mail: said.boulmane@gmail.com

Abstract. We give some new characterizations of semisimple alternative algebras among the pseudo-Euclidean alternative algebras. These charcterizations are based on the index of a pseudo Euclidean alternative algebra, operators of Casimir type and representations of alternative algebras.

**Keywords:** Pseudo-Euclidean alternative algebras, Semisimple alternative algebras, Index of pseudo Euclidean alternative algebras, Operators of Casimir type, Representations of alternative algebras.

**2000 Mathematics subject classification:** 17A30, 17D05, 17A36.

## 1. Introduction

In this work, we consider finite-dimensional alternative algebras over a commutative field $\mathbb{F}$ of characteristic 0. An alternative algebra $A$ is called pseudo-Euclidean if it is endowed with a nondegenerate symmetric bilinear form $\psi$ which is invariant, that is,

$$\psi(xy, z) = \psi(x, yz),\ \forall x, y, z \in A.$$

A nondegenerate symmetric invariant bilinear form $\psi$ on $A$ is called an invariant scalar product. Such bilinear forms on non-associative algebras have a great value in the study of their structures. For example, the Killing form plays a key role in the theory of semi-simple Lie algebras, Albert forms in the case of semi-simple Jordan algebras and trace form in the case of semi-simple alternative

Received 26 August 2019; Accepted 20 April 2020

algebras. Let us denote by $\mathcal{F}(A)$ the linear space of all symmetric invariant bllinear forms on $A$ and let $\mathcal{B}(A)$ be the subspace of $\mathcal{F}(A)$ spanned by the set of invariant scalar products on $A$. We will say that $A$ admits a unique (up to a constant) quadratic structure if $\mathcal{B}(A)$ is one-dimensional.

I. Bajo and S. Benayadi [2] proved that any Lie algebra $g$ over $\mathbb{F}$ admitting a unique up to a constant quadratic structure is necessarily a simple Lie algebra and if the field $\mathbb{F}$ is algebraically closed, such condition is also sufficient. Afterward, A. Baklouti and S. Benayadi[1] generalized this result in case of Jordan algebras and gived some new charcterizations of semisimple Jordan algebras among the pseudo-euclidean Jordan algebras. Thus, the problem of characterizing alternative algebras with a unique quadratic structure and semisimple alternative algebras arises naturally, since alternative algebras are intimately linked with Jordan algebras and Lie algebras.

Our paper is organized as follows : The first section, we give new characterizations of semisimple alternative algebras by using the index of a pseudo Euclidean alternative algebra $A$ noted $ind(A)$, that is the dimension of $\mathcal{B}(A)$. The next one, will be devoted to characterize semisimple alternative algebras by using an operator of a pseudo-Euclidean alternative algebra called operator of Casimir type. In the last part, we will use bi-representations of alternative algebras to give another caracterization of semisimple alternative algebras.

## 2. Characterization via the index of pseudo Euclidean alternative algebras

In this section, we recall some definitions and concepts of alternative algebras and pseudo-Euclidean alternative algebras. Then, we give new characterizations of semisimple alternative algebras by using the index of a pseudo Euclidean alternative algebra. For a general theory about alternative algebras (see. [5, 6]).

**Definition 2.1.** Let $A$ be a non-associative algebra (i.e. not necessarily associative), $A$ is called alternative if:

$$x^2y = x(xy) \quad and \quad yx^2 = (yx)x, \ \forall x, y \in A.$$

The left and right equations are known, respectively, as the left and right alternative laws. They are equivalent in terms of left and right multiplications to:

$$L_{x^2} = L_x^2 \quad and \quad R_{x^2} = R_x^2, \ \forall x \in A.$$

**Definition 2.2.** Let $A$ be an alternative algebra and $\psi : A \times A \longrightarrow \mathbb{F}$ be a bilinear form. $\psi$ will be called:

(1) symmetric if $\forall x, y \in A, \ \psi(x, y) = \psi(y, x)$;

(2) nondegenerate if $\psi(x,y)=0,\ \forall y\in A \Rightarrow x=0$ and if $\psi(x,y)=0,\ \forall x\in A \Rightarrow y=0$;
(3) invariant if $\forall x,y,z\in A\ \psi(xy,z)=\psi(x,yz)$.

If $\psi$ is symmetric, nondegenerate and invariant, $(A,\psi)$ will be called a pseudo-Euclidean alternative algebra and $\psi$ will be called an invariant scalar product.

Now we recall the bilinear form on an alternative algebra which characterizes the semisimplicity case. This form plays the role of the Killing form in the case of Lie algebras. The following Proposition can be found in ([5], p. 44).

**Proposition 2.3.** *Let $A$ be an alternative algebra and $\psi : A\times A\to \mathbb{F}$ be the bilinear form defined by:*

$$\psi(x,y)=tr\ (R_xR_y),\ \forall x,y\in A,$$

*where $tr\ (R_xR_y)$ is the trace of $R_xR_y$.*

*The radical of $A$ is the radical of the form $\psi$. (i.e $Rad(A)=\{x\in A,\ \psi(x,y)=0,\ \forall y\in A\}$. This form is called the trace form on $A$.*

**Corollary 2.4.** *Let $A$ be an alternative algebra, then $A$ is semisimple (that is the radical of $A$ is 0) if and only if the trace form of $A$ is nondegenerate.*

Since the trace form of an alternative algebra is invariant, we can deduce from Corollary (2.4) that any semisimple alternative algebra is pseudo-Euclidean.

Another interesting pseudo-Euclidean alternative algebra is giving as follows:

Let $A$ be an alternative algebra and $A^*$ be the dual vector space of the underlying vector space of $A$. An easy computation prove that the following product $\star$ define an alternative algebra structure on the vector space $\tilde{A}=A\oplus A^*$:

$$(x+f)\star(y+h):=xy+f\circ L_y+h\circ R_x,\ \forall (x,f),(y,h)\in A\oplus A^*.$$

Moreover, if we consider the bilinear form $\psi : (A\oplus A^*)\times(A\oplus A^*)\to\mathbb{F}$ defined by:

$$\psi(x+f,y+h)=f(y)+h(x),\ \forall (x,f),(y,h)\in A\oplus A^*,$$

then $(A\oplus A^*,\psi)$ is a pseudo-Euclidean alternative algebra called the trivial $T^*$-extension of $A$ and noted by $T_0^*A$ ([4]). For more details about pseudo-Euclidean alternative algebras see [3].

**Definition 2.5.** Let $(A,\psi)$ be a pseudo-Euclidean alternative algebra and $I$ an arbitrary vector subspace of $A$.

**(a):** $I$ is called an ideal (resp. a subalgebra) of $A$ if and only if $AI+IA\subset I$ (resp. $II\subset I$ ).
**(b):** $I$ is called nondegenerate if the restriction of $\psi$ to $I\times I$ is nondegenerate, otherwise, it is called degenerate.

**(c):** We say that $(A, \psi)$ is irreducible if every ideal of $A$ is degenerate.
**(d):** $A$ is called simple if it has no nonzero proper ideal.

**Definition 2.6.** Let $(A, \psi)$ be a pseudo-Euclidean alternative algebra. The dimension of $\mathcal{B}(A)$ is called the index of $A$ and will be denoted by $ind(A)$.

**Lemma 2.7.** *If $A$ is an alternative algebra admitting an invariant scalar product, then $\mathcal{B}(A) = \mathcal{F}(A)$.*

*Proof.* Let $\psi$ be an invariant scalar product on $A$ and $\phi \in \mathcal{F}(A)$. Let $M(\psi)$ and $M(\phi)$ be associated matrices of $\psi$ and $\phi$ in some fixed basis of $A$. Then, for $\lambda \in \mathbb{F}$ the determinant $det(M(\phi) - \lambda M(\psi))$ is a polinomial in $\lambda$. Hence, we can find $\lambda_0 \in \mathbb{F}$ such that $det(M(\phi) - \lambda_0 M(\psi)) \neq 0$. This proves that $\phi - \lambda_0\psi$ is nondegenerate and thus $\phi = (\phi - \lambda_0\psi) + \lambda_0\psi$ is nondegenerate. $\square$

**Proposition 2.8.** *Let $\mathbb{F}$ be an algebraically closed field of characteristic zero and let $A$ be a simple alternative algebra. If $\psi_1, \psi_2$ are two invariant scalar products on $A$, then there is a nonzero scalar $\lambda$ such that $\psi_1 = \lambda\psi_2$, that is, $ind(A) = 1$.*

*Proof.* Since $A$ is simple alternative algebra, then the trace form $\psi$ of $A$ is nondegenerate.

If we consider $\varphi$ another invariant scalar product on $A$, then there exists an endomorphism $D$ of $A$ such that:

$$\psi(x, y) = \varphi(D(x), y),\ \forall x, y \in A.$$

Let $\lambda \in \mathbb{F}$ and $\varphi'$ be the bilinear form of $A$ defined by $\varphi'(x, y) = \varphi(D(x) - \lambda x, y)$, $\forall x, y \in A$. Since $D$ has at least one eigenvalue, we can suppose $\lambda$ an eigenvalue of $D$ and $v$ is an eigenvector for $\lambda$, then $v \in A^{\perp} = \{x \in A;\ \varphi'(x, y) = 0,\ \forall y \in A\}$, it follows that $A^{\perp} \neq \{0\}$. Moreover, since $\varphi'$ is invariant on $A$, then $A^{\perp}$ is an ideal of $A$. Therefore, $A^{\perp} = A$, i.e, $D(x) = \lambda x$, $\forall x \in A$. Then, $\psi(x, y) = \lambda\varphi(x, y)$, $\forall x, y \in A$. $\square$

Example 2.9. Let $\mathbb{F}$ be an algebraically closed field of characteristic zero and $\mathbb{O}$ be the Cayley-Dickson algebra over $\mathbb{F}$. It is known that $\mathbb{O}$ is a simple alternative algebra, then $ind(\mathbb{O}) = 1$.

**Lemma 2.10.** *Let $A$ be an alternative algebra, if $ind(A) = 1$ then $A$ is irreducible.*

*Proof.* Suppose that $ind(A) = 1$, then by Lemma (2.7), every nonzero symmetric invariant bilinear form on $A$ is nondegenerate. Moreover, since $A$ is a pseudo-Euclidean alternative algebra, then, (see. [4])

$$A = \bigoplus_{i=1}^{n} I_i$$

where for all $1 \leq i \leq n$, $I_i$ is a nondegenerate irreducible ideal, and for all $i \neq j$, $I_i$ and $I_j$ are orthogonal. If $\psi_1$ denotes the trace form of $I_1$ then the bilinear form $\varphi$ on $A$ defined by $\varphi(x,y) = \psi_1(x,y)$ whenever $x, y \in I_1$ and $\varphi(x,y) = 0$ otherwise, is a degenerate invariant symmetric bilinear form, which contradicts the result in Lemma 2.7. Then $A$ is irreducible. □

**Proposition 2.11.** *Let $(A,\psi)$ be a pseudo-Euclidean alternative algebra. If $ind(A) = 1$, then $A$ is either a simple alternative algeba or $A$ is the one-dimensional algebra with zero product.*

*Proof.* By Lemma 2.10 we deduce that $A$ is irreducible. If we assume that $A$ is neither simple nor the one-dimensional algebra with zero product, then by Corollary (2.3) and Corollary (3.3) in [3], $A$ is either a double extension of a pseudo-Euclidean alternative algebra by a simple alternative algebra or a generalized double extension of a nilpotent pseudo-Euclidean alternative algebra by the one-dimensional algebra with zero product.

If $A$ is not nilpotent, then $A$ is a double extension of a pseudo-Euclidean alternative algebra $(\mathcal{W},T)$ by a simple alternative algebra $\mathcal{S}$. Moreover, if $\sigma$ is an invariant symmetric bilinear form on $\mathcal{S}$, then ([3], Theorem 2.1) the bilinear form $\widetilde{\psi}_\sigma$ defined by:

$$\begin{array}{cccll} \widetilde{\psi}_\sigma & : & (\mathcal{S}\oplus\mathcal{W}\oplus\mathcal{S}^*)\times(\mathcal{S}\oplus\mathcal{W}\oplus\mathcal{S}^*) & \longrightarrow & \mathbb{F}, \\ & & (x+y+f, x'+y'+f') & \longmapsto & \sigma(x,x')+\psi(y,y')+f(x')+f'(x), \end{array}$$

is an invariant scalar product on $A = \mathcal{S}\oplus\mathcal{W}\oplus\mathcal{S}^*$. If we consider $\sigma_1$ the trace form on $\mathcal{S}$ and $\sigma_2 = 0$, an invariant symmetric bilinear form on $\mathcal{S}$, then $\widetilde{\psi}_{\sigma_1}$ and $\widetilde{\psi}_{\sigma_2}$ are two linearly independant elements of $\mathcal{B}(A)$, which contradicts the fact that $ind(A) = 1$.

Now, if $A$ is nilpotent, then, by ([3]. Theorem 3.2 ) $\mathcal{A} := \mathbb{F}e\oplus\mathcal{W}\oplus\mathbb{F}b$ is the generalized double extension of the pseudo-Euclidean alternative algebra $(\mathcal{W},T)$ by the one dimensional alternative algebra $\mathbb{F}e$ with null product. Where, $b \in Ann(\mathcal{A})\backslash\{0\}$ and $e \in \mathcal{A}$ such that $\psi(b,b) = 0$, $\psi(e,b) = 1$, $\psi(e,e) = 0$, $\mathcal{W} := (\mathbb{F}b\oplus\mathbb{F}e)^\perp$ and $T := \psi_{|\mathcal{W}\times\mathcal{W}}$. Let us consider the invariant symmetric bilinear form $\varphi$ on $A$ defined by: $\varphi(b,b) = 1$, and $\varphi(x,y) = 0$, $\forall(x,y) \in A\times A\backslash\mathbb{F}b\times\mathbb{F}b$. It is clear that $\varphi$ and $\psi$ are two linearly independant elements of $\mathcal{B}(A)$. Therefore, $ind(A) \geqslant 2$, which contradicts the hypothesis $ind(A) = 1$.

We conclude that if $ind(A) = 1$, then $A$ is either a simple alternative algebra or $A$ is the one-dimensional algebra with zero product. □

**Corollary 2.12.** *Let $\mathbb{F}$ be an algebraically closed field of characteristic zero and let $(A,\psi)$ be a pseudo-Euclidean alternative algebra. Then, the two following assertions are equivalent:*

(1) $ind(A) = 1$

(2) *$A$ is either a simple alternative algeba or $A$ is the one-dimensional algebra with zero product.*

Recall that an alternative algebra $A$ is said to be reductive if $A = \mathcal{S} \oplus Ann(A)$ is a direct sum of a semisimple alternative algebra $\mathcal{S}$ and its annulator $Ann(A)$.

**Proposition 2.13.** *Let $(A, \psi)$ be a pseudo-Euclidean alternative algebra and $I_1, ..., I_n$, $(n \in \mathbb{N})$ be nondegenerate $\psi$-irreducible ideals of $A$ such that*

$$A = \bigoplus_{i=1}^{n} I_i$$

*and for all $i, j \in \{1, ..., n\}$, $I_i$ and $I_j$ are $\psi$-orthogonal for $(i \neq j)$. Then the following assertions are equivalent:*

(1) *$ind(A) = n$ and $A^2 = A$*
(2) *$A$ is semisimple.*

*Proof.* Assume that $A$ is semisimple, then $Ann(A) = \{0\}$ and it follows by ([1]. Corollary 9.14) that $ind(A) = n$, moreover, we have $A^2 = Ann(A)^{\perp} = \{0\}^{\perp} = A$.

Conversely, suppose that $ind(A) = n$, then we deduce by ([1]. Corollary 9.14) that $A$ is reductive and $dim(Ann(A)) \leqslant 1$. Assume that $dim(Ann(A)) = 1$, then $(\mathbb{F}a)^{\perp} = A^2 = A$, for $a \in Ann(A) \backslash \{0\}$ which is absurd. It follows that $A$ is semisimple. □

## 3. Characterization of semisimple alternative algebras by means of operator of Casimir

Now, we are going to give a characterization of semi-simple alternative algebras by using an operator of a pseudo-Euclidean alternative algebra called operator of Casimir type.

Let $(A, \psi)$ be a pseudo-Euclidean alternative algebra of dimension $n$. We consider $\{e_1, ., ., .e_n\}$ and $\{e'_1, ., ., .e'_n\}$ two basis of $A$ such that $\psi(e_i, e'_j) = \delta_{ij}$, $\forall i, j \in \{1, ..., n\}$, where $\delta_{ij}$ is Kronecker's symbol and we denote by $C_{e_i, e'_i}$ the operator of $A$ defined by:

$$C_{e_i, e'_i} = \sum_{i=1}^{n} R_{e_i} R_{e'_i}.$$

$C_{e_i, e'_i}$ is called the operator of Casimir type of the pseudo-Euclidean alternative algebra $(A, \psi)$.

**Lemma 3.1.** *Let $(A, \psi)$ be a pseudo-Euclidean alternative algebra. Then,*

**(i):** *$tr(R_x R_y) = \psi(C_{e_i, e'_i}(x), y)$, $\forall x, y \in A$;*
**(ii):** *$C_{e_i, e'_i} = C_{f_i, f'_i}$;*
**(iii):** *$C_{e_i, e'_i} \circ R_x = R_x \circ C_{e_i, e'_i}$, $\forall x \in A$.*

*Where* $\{f_1,.,.,.f_n\}$ *and* $\{f'_1,.,.,.f'_n\}$ *are two basis of* $A$ *such that* $\psi(f_i,f'_j)=\delta_{ij},\ \forall i,j\in\{1,...,n\}$.

*Proof.* (i) For all $x,y\in A$, we have

$$\psi(C_{e_i,e'_i}(x),y)=\psi(\sum_{i=1}^{n}R_{e_i}R_{e'_i}(x),y)=\sum_{i=1}^{n}\psi((xe'_i)e_i,y)=\sum_{i=1}^{n}\psi((e_iy)x,e'_i)$$
$$=\sum_{i=1}^{n}\psi(R_xR_y(e_i),e'_i)=tr(R_xR_y).$$

(ii) By using (i), it is clear that
$tr(R_xR_y)=\psi(C_{e_i,e'_i}(x),y)=\psi(C_{f_i,f'_i}(x),y),\ \forall x,y\in A$.
Moreover, since $\psi$ is nondegenerate, then $C_{e_i,e'_i}=C_{f_i,f'_i}$.
(iii) Let $x,a,b\in A$,

$$\begin{aligned}\psi((C_{e_i,e'_i}\circ R_x-R_x\circ C_{e_i,e'_i})(a),b)&=\psi(C_{e_i,e'_i}(ax),b)-\psi(C_{e_i,e'_i})(a),xb)\\&=tr(R_{ax}R_b)-tr(R_aR_{xb})\\&=tr(R_{ax}R_b)-tr(R_{ax}R_b)\\&=0.\end{aligned}$$

Moreover, since $\psi$ is nondegenerate it follows that $C_{e_i,e'_i}\circ R_x=R_x\circ C_{e_i,e'_i}$. □

**Proposition 3.2.** *Let* $(A,\psi)$ *be a pseudo-Euclidean alternative algebra,* $A$ *is semisimple if and only if the operateur of Casimir type* $C_{e_i,e'_i}$ *of* $A$ *is inversible.*

*Proof.* By ([5]. Proposition 3.13) we have $A$ is semisimple if and only if the trace form on $A$ is nondegenerate. From Lemma (3.1) we deduce that $tr(R_xR_y)=\psi(C_{e_i,e'_i}(x),y),\ \forall x,y\in A$, then the trace form on $A$ is nondegenerate if and only if $C_{e_i,e'_i}$ is inversible. Consequently, $A$ is semisimple if and only if the operateur of Casimir type $C_{e_i,e'_i}$ of $A$ is inversible. □

## 4. Characterization via bi-representations of alternative algebra

Let $A$ be an alternative algebra and $M$ be a vector space. Then, $M$ is a bimodule over $A$ in case there are two linear maps $\pi:A\to End(M)$, $\Pi:A\to End(M)$ satisfying:

(i) $\pi(a^2)=\pi(a)^2$,
(ii) $\Pi(a^2)=\Pi(a)^2$,
(iii) $\Pi(aa')-\Pi(a')\Pi(a)=[\Pi(a'),\pi(a)]$,
(iv) $\pi(aa')-\pi(a)\pi(a')=[\pi(a),\Pi(a')]$,

for all $a, a' \in A$.
The vector space direct sum $\tilde{A} = A \oplus M$ is made into an alternative algebra, by defining multiplication by:

$$(a+m).(a'+m') = aa' + \pi(a)m' + \Pi(a')m, \tag{4.1}$$

for all $a, a' \in A$ et $m, m' \in M$.
$(\pi, \Pi)$ is called (bi)-representation of $A$ associated to $M$.

Consider the bilinear form $\psi : A \times A \to \mathbb{F}$ defined by $\psi(a,b) = tr(\pi(a)\pi(b))$, for all $a, b \in A$. In this case, we say that $\psi$ is the bilinear form of $A$ associate to the bi-representation $(\pi, \Pi)$.

In this section we study the structure of $A$ such that $\psi$ is non-degenerate.

**Proposition 4.1.** *Let $A$ be an alternative algebra, $M$ a vector space and $(\pi, \Pi)$ a (bi)-representation of $A$ associated to $M$. If $A$ is nilpotent then the algebra $\tilde{A} = A \oplus M$ defined by the multiplication (4.1) is nilpotent.*

*Proof.* We deduce by the multiplication (4.1) that $M$ is a nilpotent ideal, then $M$ is contained in the radical $Rad(\tilde{A})$ of $\tilde{A}$. If we suppose that $\tilde{A}$ is not nilpotent, then $Rad(\tilde{A}) \neq \tilde{A}$ and according to Wedderburn decomposition, we have $\tilde{A} = S \oplus Rad(\tilde{A})$, where $S$ is a semi-simple subalgebra of $\tilde{A}$. Let $\varphi : \tilde{A} \longrightarrow \tilde{A}/M$ be the canonical surjection, since $S \cap M = \{0\}$, then $\varphi : S \longrightarrow \varphi(S)$ is an isomorphism of alternative algebras. Consequently $\varphi(S)$ is semi-simple. Moreover, $A$ is nilpotent implies that $\tilde{A}/M$ is nilpotent. It follows that $\varphi(S)$ is also nilpotent, hence $Rad(\varphi(S)) = \varphi(S) = 0$. Therfore, $S \subset M$ which contradicts $S \neq \{0\}$, which completes the proof. $\square$

**Corollary 4.2.** *Let $A$ be an alternative algebra, $M$ a vector space and $(\pi, \Pi)$ a (bi)-representation of $A$ associated to $M$. If $A$ is nilpotent then $\pi(a)$ and $\Pi(a)$ are nilpotent for all $a \in A$.*

*Proof.* Assume that $A$ is nilpotent, then by Proposition (4.1) the algebra $\tilde{A}$ is nilpotent. Then for all $a \in A$ there is an integer $r$ such that $a^r = 0$. Consequently,

$$a^r.m = m.a^r = \pi(a)^r(m) = \Pi(a)^r(m) = 0, \ \forall m \in M.$$

Hence $\pi(a)$ and $\Pi(a)$ are nilpotent. $\square$

*Remark* 4.3. One can deduce that for all integer $n$,

$$a^n.m = \pi(a)^n(m) = \pi(a^n)(m) \text{ and } m.a^n = \Pi(a)^n(m) = \Pi(a^n)(m).$$

That is, $\pi(a)^n = \pi(a^n)$ and $\Pi(a)^n = \Pi(a^n)$.

The main result of this section is contained in the following Theorem:

**Theorem 4.4.** *Let $A$ be an alternative algebra and $M$ be a vector space. Then the following assertions are equivalent:*

(i) *$A$ is semi-simple;*

(ii) *There exists a finite-dimensional bi-representation $\pi : A \to End(M)$, $\Pi : A \to End(M)$ such that the bilinear form $\psi : A \times A \to \mathbb{F}$ defined by $\psi(a, b) = tr(\pi(a)\pi(b))$, for all $a, b \in A$ is non-degenerate.*

*Proof.* Suppose that $A$ is semi-simple, if we consider $(\pi, \Pi) = (R, L)$, where $R(a) = R_a$ and $L(a) = L_a$ for all $a, b \in A$, then by ([5]. Proposition 3.13) the bilinear form $\psi : A \times A \to \mathbb{F}$ defined by $\psi(a, b) = tr(R_aR_b)$, for all $a, b \in A$ is non-degenerate.

Conversely, Let $a \in A$ and $r \in Rad(A)$, where $Rad(A)$ is the radical of $A$. Since $ra \in Rad(A)$ and $Rad(A)$ is nilpotent, then by Corollary (4.2), $\pi(ra)$ is nilpotent. It follows that $\psi(r, a) = tr(\pi(r)\pi(a)) = tr(\pi(ra)) = 0$. Then, $r = 0$ since $\psi$ is non-degenerate. Which proves that $Rad(A) = \{0\}$. Consequently, $A$ is semi-simple. $\square$

## Acknowledgments

The author wish to thank the referee for their careful reading and for valuable comments.

## References

1. A. Baklouti, S. Benayadi, Pseudo-Euclidean Jordan Algebras, *Comm. in. Alg.*, **43**, (2015), 2094-2123.
2. I. Bajo, S. Benayadi, Lie Algebras Admitting a Unique Quadratic Structure, *Comm. in. Alg.*, **25**(9), (1997), 2795-2805.
3. M. Ait Ben Haddou, S. Boulmane, Pseudo-Euclidean Alternative Algebras, *Comm. in. Alg.*, **44**(12), (2016), 5199–5222.
4. M. Bordemann, Non-degenerate invariant bilinear forms on nonassociative algebras, *Acta Math. Univ. Comenian.*, **LXVI**(2), (1997), 151-201.
5. R. D. Schafer, *An Introduction to Nonassociative Algebras*, Academic Press, New York, 1996.
6. K. A. Zhevlakov, A. M. Slin'ko, I. P. Shestakov, A. I. Shirshov, *Rings that Are Nearly Associative. Translated from the Russian by Harry F. Smith.*, Pure and Applied Mathematics, 104. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1982.