Annihilators of power central values of generalized skew derivations on Lie ideals

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Let R be a prime ring with center $Z(R)$ and G be a generalized α-derivation of R for $\alpha \in Aut(R)$. Let $a \in R$ be a nonzero element and n be a fixed positive integer.

(i) If $aG(x)^n \in Z(R)$ for all $x \in R$ then $aG(x) = 0$ for all $x \in R$ unless $dim_C RC = 4$.

(ii) If $aG(x)^n \in Z(R)$ for all $x \in L$, where L is a noncommutative Lie ideal of R then $aG(x) = 0$ for all $x \in R$ unless $dim_C RC = 4$.

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1. Introduction and Preliminaries

Let R be a prime ring with center $Z(R)$ and Q the two-sided Martindale quotient ring of R, Q_r the right Martindale quotient ring of R. It is known that $R \subseteq Q \subseteq Q_r$. The two overrings Q and Q_r of R are still prime rings. They have the same center, denoted by C which is a field and is called the *extended centroid* of R (for details see [2]). An additive map d of R is called a *derivation* if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. Let $\alpha \in Aut(R)$ and $f : R \to R$ be an additive map. If $f(xy) = f(x)y + \alpha(x)f(y)$ for all $x, y \in R$ then f is called an α*-derivation*. For brevity we call an α-derivation a *skew derivation*. If the derivation $d : R \to R$ assumes the form $d(x) = [a, x]$ for all $x \in R$ and for some $a \in R$, then d is called an X*-inner derivation* induced by $a \in R$ and it is denoted by d_a. A derivation is called X*-outer* if it is not X-inner. An additive map G of R is said to be a *generalized skew derivation* or *generalized* α*-derivtion* if $G(xy) = G(x)y + \alpha(x)f(y)$ for all $x, y \in R$, here f is the associated α-derivation. It is well known that generalized (α, β)-derivations are actually the same with $\alpha^{-1}\beta$-derivations.

In recent years a number of authors had a line of investigation in behaviour of the additive mappings of a ring. Particularly, they obtained many fascinating results on derivations, generalized derivations, skew derivations and generalized skew derivations. In many cases the results provide useful informations about the structure of the ring and the map. In [17], I. N . Herstein proved that there doesn't exist any nonzero derivation which is nilpotent on a prime ring R. Strictly he showed that if d is a derivation of R such that $d(x)^n = 0$ for all $x \in R$, where n is a fixed positive integer, then $d = 0$. Accordingly, in [18] I. N. Herstein generalized this result to power central case. He proved that if R is a prime ring with center $Z(R)$ and a nonzero derivation d such that $d(x)^n \in Z(R)$ for all $x \in R$ where n is a fixed positive integer then R is commutative or is an order in 4-dimensional simple algebra. Herstein's results have since been generalized by many authors. In [3], M. Brešar proved that if R is a semiprime ring, $a \in R$ and d is a derivation of R satisfying $ad(x)^n = 0$ for all $x \in R$ then $ad(R) = 0$ when R is a $(n-1)!$ torsion free ring. Laterly, T. K. Lee and J. S. Lin improved M. Brešar's result without the $(n-1)!$-torsion free assumption in [22]. They proved that if $ad(x)^n = 0$ for all $x \in L$, where L is a Lie ideal of R, then $ad(L) = 0$ unless $charR = 2$ and $dim_C RC = 4$. In addition if $[L, L] \neq 0$ then $ad(R) = 0$.

In [6], J. C. Chang generalized I. N. Herstein's result in [18] to generalized (α, β)-derivations (that is, $f(xy) = f(x)\alpha(y) + \beta(x)f(y)$). He showed that in a prime ring R with center $Z(R)$ and a nonzero generalized (α, β)-derivation f of R, if $f(x)^n \in Z(R)$ for all $x \in I$, where I is a nonzero ideal of R, then either R is commutative or R is an order in 4-dimensional simple algebra.

Afterwards, J. C. Chang handled the problem in which f is a generalized (α, β)-derivation of R, $af(x)^n = 0$ for all $x \in R$, where n is a fixed positive integer and he concluded that $af(x) = 0$ for all $x \in R$ in [7].

In [1], the authors proved the following result: Let R be a prime ring with nonzero generalized skew derivation f and $a \in R$. If $af(x)^n = 0$ for all $x \in L$, where L is a noncommutative Lie ideal of R, then $af(x) = 0$ for all $x \in R$ or R is an order in 4-dimensional simple algebra.

Motivating the results above we will treat a generalized skew derivation G of R, more precisely we will prove the following theorems:

Theorem 2.1. Let R be a prime ring with center $Z(R)$ and G be a generalized α-derivation, where α is an automorphism of R. Let $0 \neq a \in R$ and n be a fixed positive integer. If $aG(x)^n \in Z(R)$ for all $x \in R$ then $aG(x) = 0$ for all $x \in R$ or $dim_C RC = 4$.

Theorem 2.2. Let R be a prime ring with center $Z(R)$, L be a noncommutative Lie ideal of R and G be a generalized α-derivation of R, where α is an automorphism of R. Let $a \in R$ be a nonzero element and n be a fixed positive integer. If $aG(x)^n \in Z(R)$ for all $x \in L$ then $aG(x) = 0$ for all $x \in R$ unless $dim_C RC = 4$.

We give the following conclusions related to the above theorems. Since every α-derivation is a generalized α-derivation, the following two corollaries are direct consequences of Theorem 2.1 and Theorem 2.2, respectively:

Corollary 1. Let R be a prime ring with center $Z(R)$ and $a \in R$. Suppose that α is an automorphism of R and f is a nonzero α-derivation of R such that $af(x)^n \in Z(R)$ for all $x \in R$, where n is a fixed positive integer. Then $a = 0$ unless $dim_C RC = 4$.

Corollary 2. Let R be a prime ring with center $Z(R)$, L be a noncommutative Lie ideal of R and $a \in R$. Suppose that α is an automorphism of R and f is a nonzero α-derivation of R such that $af(x)^n \in Z(R)$ for all $x \in L$, where n is a fixed positive integer. Then either $a = 0$ or $dim_C RC = 4$.

If α is an automorphism of R such that $\alpha \neq I$, the identity automorphism of R, then $I - \alpha$ is a skew derivation of R. Hence,

Corollary 3. Let R be a prime ring with center $Z(R)$, L be a noncommutative Lie ideal of R and $a \in R$. Suppose that $\alpha \neq I$ is an automorphism of R and such that $a\big(x - \alpha(x)\big)^n \in Z(R)$ for all $x \in L$, where n is a fixed positive integer. Then either $a = 0$ or $dim_C RC = 4$.

Let R be a unital ring and $u \in R$ be an invertible element in R. If $\alpha_u(x) = uxu^{-1}$ for all $x \in R$ and d is a nonzero derivation of R, then ud is an α_u-derivation of R. In this manner, if G is a nonzero generalized derivation with an associated derivation d of R, then uG is a generalized α_u-derivation associated with the α_u-derivation ud of R. Thereby we have following two conclusions:

Corollary 4. Let R be a unital ring and $u \in R$ be an invertible element in R. If d is a nonzero derivation of R such that $a(ud(x))^n \in Z(R)$ for all $x \in L$, a noncommutative Lie ideal of R, then $a = 0$ or $dim_C RC = 4$.

Corollary 5. Let R be a unital ring and $u \in R$ be an invertible element in R. Let G be a nonzero generalized derivation of R, associated with the derivation d of R. If $a(uG(x))^n \in Z(R)$ for all $x \in L$, a noncommutative Lie ideal of R, then $a = 0$ or $G(x) = sx$ for all $x \in R$ and some $s \in Q$, unless $dim_C RC = 4$.

We will frequently use the following facts in the proofs:

Fact 1 ([15]) Let R be a prime ring with *char* $R \neq 2$ and L be a noncentral Lie ideal of R. Then there exists a nonzero ideal $I = R[L, L]R$ of R such that $0 \neq [I, R] \subseteq L$.

Fact 2 ([2]) Let R be a semiprime ring and X be a countable set of noncommuting indeterminates. The elements of the free product $T = Q * C\{X\}$ are called *generalized polynomials*. Let $q_i \in Q$ and $y_i \in X$, then the elements of the form $m = q_0 y_1 q_1 y_2 q_2 y_3 \ldots$ are called *monomials* where q_i 's are the *coefficients*. For all $f \in T$, f is the finite sum of the monomials and uniquely determined. Let $f = f(x_1, \ldots, x_n)$ be generalized polynomial in T. If $f(r_1, \ldots, r_n) = 0$ for all $r_1, \ldots, r_n \in R$ then f is called a *generalized polynomial identity* and R is called a *generalized polynomial identity ring.*

Fact 3. ([13]) Let R be a prime ring with an X-outer α-derivation δ. Then any generalized polynomial identity of R in the form $\phi(x_i, \delta(x_i)) = 0$ yields a generalized polynomial identity $\phi(x_i, y_i) = 0$ of R, where x_i, y_i are distinct indeterminates.

Fact 4. ([11]) Let R be a prime ring with an X-outer α-derivation δ. Suppose that R satisfies a generalized polynomial identity $\phi(x_i, \alpha(x_i)) = 0$, where $\phi(x_i, y_i)$ is a nontivial generalized polynomial in distinct indeterminates x_i, y_i. Then R is a GPI-ring.

Fact 5.([20]) Let R be a ring with extended centroid C and α be an automorphism of R. Let n be a fixed positive integer. If

$$\alpha(\lambda) = \lambda \text{ for all } \lambda \in C, \text{ when } char\ R = 0,$$
$$\alpha(\lambda) = \lambda^{p^n} \text{ for all } \lambda \in C, \text{ when } char\ R = p \geq 2,$$

then α is called a *Frobenius automorphism* of R.

Fact 6. ([12]) Let R be a prime ring with an automorphism α and suppose that α is not a Frobenius automorphism of R. Then any generalized polynomial identity of R in the form $\phi(x_i, \alpha(x_i)) = 0$ yields the generalized polynomial identity $\phi(x_i, y_i) = 0$ of R, where x_i, y_i are distinct indeterminates.

Fact 7. ([24]) Let R be a prime ring, I be a nonzero ideal of R, $a, b \in U/\{0\}$, n a fixed positive integer and δ a nonzero generalized derivation of R.
(i) Suppose that $a(\delta(x)b)^n = 0$ for all $x \in I$. Then there exist $a_1, b_1 \in U$ such that $\delta(x) = a_1x + xb_1$ for all $x \in R$ and $b_1b = 0$. Moreover, either $ba_1 = 0$ or $aa_1 = 0$.
(ii) Suppose that $a(\delta(x)b)^n \in C$ for all $x \in R$. If $a(\delta(x_0)b)^n \neq 0$ for some $x_0 \in I$, then $dim_C RC = 4$.

2. Results

Lemma 2.1. *Let R be a noncommutative prime ring with center $Z(R)$ and $a, b, c, q \in R$ with $q \in R$ invertible. Suppose that $a \neq 0$. If $a(bx - qxq^{-1}c)^n \in Z(R)$ for all $x \in R$ then either $q^{-1}c \in Z(R)$ and $a(b - c) = 0$ or $dim_C RC = 4$.*

Proof. Suppose that $dim_C RC > 4$. If $Z(R) = 0$, then $a(bx - qxq^{-1}c)^n = 0$ for all $x \in R$. By Lemma 3 in [7], $a(bx - qxq^{-1}c) = 0$ for all $x \in R$. Applying Martindale's Lemma (Lemma 7.41 in [4]), we see that $ab = \lambda aq$ for some $\lambda \in C$. So $aqR(\lambda - q^{-1}c) = 0$ and by the primeness of R, we have $aq = 0$ or $q^{-1}c \in C$. Since $a \neq 0$ then $q^{-1}c \in C$. By the initial assumption $a((b - c)x)^n = 0$ for all $x \in R$ and we have $a(b - c) = 0$ via Lemma 1 in [7].

Thereby we may assume that $Z(R) \neq 0$. If $q^{-1}c \in Z(R)$ then $a((b-c)x)^n \in Z(R)$ for all $x \in R$. In view of Fact 7, $a(b - c) = 0$. Now assume that $q^{-1}c \notin Z(R)$. In this case R satisfies the GPI

$$a(bx - qxq^{-1})^n y - ya(bx - qxq^{-1}c)^n = 0.$$

By Martindale's result (for details see [2]), Q is a primitive ring having nonzero socle H and its associated division ring D is finite over C. Hence Q is isomorphic to a dense subring of $End(_DV)$. If $dim_D V = \infty$ then $H \cap C = (0)$. Hence

$$a(bx - qxq^{-1}c)^n = 0 \tag{2.1}$$

for all $x \in H$ and (2.1) holds for all $x \in Q$. Using Lemma 3 in [7], $a(bx - qxq^{-1}c) = 0$ for all $x \in R$ and there exists some $\lambda \in C$ such that $ab = \lambda aq$ by Martindale's

Lemma. Thus owing to the primeness of R we have $a = 0$ or $q^{-1}c \in C$, a contradiction.

Now suppose that $dim_D V < \infty$. Hence Q is isomorphic to D_m, the matrix ring over D for some positive integer m. If C is finite, then D (being finite dimensional over C) is a finite ring and thus is a field by Wedderburn's theorem. In this case $Q \cong C_m$. In other hand if C is infinite and F is the maximal subfield of D, then by a standard argument, $a(bx - qxq^{-1}c)^n = 0$ for all $x, y \in Q \otimes_C F$ (see, for instance proposition in [21]). But $Q \otimes_C F \cong D_m \otimes_C F \cong (D \otimes_C F)_m \cong F_k$ for some k. In either case, we may suppose that $R \cong F_k$ for some $k > 1$.

Suppose that $k \geq 3$. If x is an element of Q, such that rank$(x) = 1$, then bx and qxq^{-1} are of rank at most 1. Through using this we see that $a(bx - xc)$ and $a(bx - xc)^n$ are of rank at most 2. In connection with $\text{rank}\big(a(bx - xc)^n\big) \leq 2$ and $k \geq 3$, then $a(bx - xc)^n = 0$ for any element x of rank 1. Since $q^{-1}c \notin F$ then there exists $v \in V$ such that v and $q^{-1}cv$ are linearly independent over F. Thus, $xv = 0$ and $xq^{-1}cv = q^{-1}v$ for some $x \in Q$ of rank 1. Therefore

$$0 = a(bx - qxq^{-1}c)^n v = (-1)^n av$$

which implies $a = 0$, contradiction. So $k = 2$ and $Q \cong F_2$, that is, R is an order in 4-dimensional simple algebra. □

Lemma 2.2. *([1], Lemma 3.1) Let R be a noncommutative prime ring, $a, b, c \in R$ and n a fixed positive integer.*

(i) If $a([x, y]b)^n = 0$ for all $x, y \in R$ then $a = 0$ or $b = 0$.
(ii) If $a(b[x, y])^n = 0$ for all $x, y \in R$ then $ab = 0$.

Lemma 2.3. *Let R be a noncommutative prime ring with $dim_C RC > 4$, $a, b, c \in R$ and n is a fixed positive integer.*

(i) If $a([x, y]b)^n \in Z(R)$ for all $x, y \in R$ then $a = 0$ or $b = 0$.
(ii) If $a(b[x, y])^n \in Z(R)$ for all $x, y \in R$ then $ab = 0$.

Proof. Suppose that $a \neq 0$ and $b \neq 0$. If R is not a PI-ring $a\,(xb)^n \in Z(R)$ for all $x \in R$ by Lemma 2 in [23]. Since $dim_C RC > 4$ then in view of Fact 7 $a\,(xb)^n = 0$ for all $x \in R$. Hence we obtain either $a = 0$ or $b = 0$ by Lemma 2 in [7], a contradiction.

Now suppose that R is a PI-ring. Then RC is a finite dimensional central simple algebra over C. Let $\bar{C}$ be the central closure of C. We may take $F = \bar{C}$ or $F = C$, in case C is infinite or finite respectively. So $RC \otimes_C F = M_k(F)$ for some $k > 1$ and

$$a\,([x, y]b)^n \in C \tag{2.2}$$

for all $x, y \in RC \otimes_C F$. If $a\,([x, y]b)^n = 0$ for all $x, y \in RC \otimes_C F$ then by Lemma 2.2 we have $a = 0$ or $b = 0$, which leads a contradiction. Hence there exist $x_0, y_0 \in RC \otimes_C F$ such that $a\,([x_0, y_0]b)^n \neq 0$. Since C is a field, $a\,([x_0, y_0]b)^n$ is invertible and so is a.

Let $e \in RC \otimes_C F$ be an element of rank 1. Substituting x by e and y by $ey(1-e)$ in (2.2) we obtain

$$a\left(ey(1-e)b\right)^n \in C$$

for all $y \in RC \otimes_C F$ and

$$\text{rank}\left(a(eyb(1-e))^n\right) \leq 2.$$

Since $dim_C RC > 4$ then

$$a\left(ey(1-e)b\right)^n = 0 \tag{2.3}$$

for all $y \in RC \otimes_C F$. Right multiplying (2.3) by e we have

$$ae\left(y(1-e)be\right)^n = 0.$$

Hence either $ae = 0$ or $(1-e)be = 0$. Since a is invertible then $ae = 0$ implies $e = 0$. Therefore $(1-e)be = 0$ for any idempotent element $e \in RC \otimes_C F$. Then $eb(1-e) = 0$ for $1-e \in RC \otimes_C F$. In this case we have $eb = ebe = be$. Let E be the additive subgroup of R generated by all idempotent elements in R. It is well known that E is a noncommutative Lie ideal of R. Then $[b, E] = 0$ and hence $b \in C$. Since we assume $b \neq 0$ then $b \in C$ is invertible. So

$$a\left([x,y]b\right)^n = b^n a\left([x,y]\right)^n \in C$$

for all $x, y \in RC \otimes_C F$ and we have $a\left([x,y]\right)^n \in C$. Then $a\left([x,y_0]\right)^n \in C$ for $y_0 \in RC \otimes_C F$ and we have

$$ad\left(x\right)^n \in C$$

for all $x \in RC \otimes_C F$ where $d = [-, y_0]$ is a derivation. In that case we obtain $a = 0$ or $d = 0$ by Theorem 2 in [5]. Since we assume $a \neq 0$ then $d = 0$ and $y_0 \in C$. Repeating this process for any $y \in RC \otimes_C F$ we conclude that RC is commutative and hence R is commutative, a contradiction. Analogously, (ii) is obtained. □

Lemma 2.4. *Let R be a noncommutative prime ring with center $Z(R)$ and $a, b, c, q \in R$ with q invertible. Suppose that a is not zero. If $a(bx - qxq^{-1}c)^n \in Z(R)$ for all $x \in [R, R]$ then either $q^{-1}c \in Z(R)$ and $a(b-c) = 0$ or $dim_C RC = 4$.*

Proof. Suppose that $dim_C RC > 4$. If R is not a PI-ring, then $a(bc - qxq^{-1}c)^n \in Z(R)$ for all $x \in R$ by Lemma 2 in [23] . In this case, we are done by Lemma 2.1. If R is a PI-ring then RC is a finite dimensional central simple C-algebra and the ring of all linear transformations of a k-dimensional vector space V over a division ring D, for $k > 1$. In the light of [11],

$$a(bx - qxq^{-1}c)^n \in C \tag{2.4}$$

for all $x \in \left[RC, RC\right]$. Let $e \in RC$ be an idempotent such that rank$(e) = 1$. Substituting $\left[q^{-1}(1-e)xe, q^{-1}(1-e)ye\right]$ into x in (2.4), we obtain

$$a\big(b\left[q^{-1}(1-e)xe, q^{-1}(1-e)ye\right] - q\left[q^{-1}(1-e)xe, q^{-1}(1-e)ye\right]q^{-1}c\big)^n \in C$$

for all $x, y \in RC$. It is clear that

$$\mathrm{rank}\Big(a\big(b\big[q^{-1}(1-e)xe, q^{-1}(1-e)ye\big] - q\big[q^{-1}(1-e)xe, q^{-1}(1-e)ye\big]q^{-1}c\big)^n\Big) \leq 4.$$

Since we assume that $dim_C RC > 4$, then

$$a\big(b\big[q^{-1}(1-e)xe, q^{-1}(1-e)ye\big] - q\big[q^{-1}(1-e)xe, q^{-1}(1-e)ye\big]q^{-1}c\big)^n = 0$$

for all $x, y \in RC$. Multipliying on the right by $(1-e)$ we obtain

$$a(1-e)\big((yeq^{-1}(1-e)x - xeq^{-1}(1-e)y)eq^{-1}c(1-e)\big)^n = 0 \tag{2.5}$$

for all $x, y \in RC$. In view of Fact 7 one of the following holds:

(i) $a(1-e) = 0$,
(ii) $eq^{-1}c(1-e)$,
(iii) $eq^{-1}(1-e)yeq^{-1}c(1-e) = -\lambda eq^{-1}c(1-e)$ and $eq^{-1}c(1-e)yeq^{-1}(1-e) = -\lambda eq^{-1}c(1-e)$,
(iv) $eq^{-1}(1-e)yeq^{-1}c(1-e) = -\lambda eq^{-1}c(1-e)$ and $a(1-e)yeq^{-1}(1-e) = -\lambda a(1-e)$

for all $y \in RC$ and some $\lambda \in C$. Using (iii) in (2.5) we have

$$\lambda^n a(1-e)\big((x-y)eq^{-1}c(1-e)\big)^n = 0$$

for all $x, y \in RC$. In particular

$$\lambda^n a(1-e)\big(xeq^{-1}c(1-e)\big)^n = 0$$

for all $x \in RC$. Since RC is a prime ring then either $\lambda = 0$ or $a(1-e) = 0$ or $eq^{-1}c(1-e) = 0$. If $\lambda = 0$ then $eq^{-1}(1-e) = 0$. In like manner, using (iv) in (2.5) we obtain either $a(1-e) = 0$ or $eq^{-1}(1-e) = 0$ or $eq^{-1}c(1-e) = 0$ for any idempotent of rank 1. Now assume that $e \in RC$ is an idempotent of rank 1 such that $eq^{-1}(1-e) = 0$. Substituting $\big[q^{-1}(1-e)xe, ye\big]$ into x in (2.4), we have

$$a\big(b\big[q^{-1}(1-e)xe, ye\big] - q\big[q^{-1}(1-e)xe, ye\big]q^{-1}c\big)^n \in C$$

which implies

$$a\big(bq^{-1}(1-e)xeye - (1-e)xeyeq^{-1}c\big)^n = 0 \tag{2.6}$$

for all $x, y \in RC$, by familiar calculations. Right multipliying (2.6) by $(1-e)$ we have

$$a(1-e)\big(xeyeq^{-1}c(1-e)\big)^n = 0$$

for all $x, y \in RC$. In light of [14], $a(1-e) = 0$ or $eyeq^{-1}c(1-e) = 0$ for all $y \in RC$ which yields $eq^{-1}c(1-e) = 0$ owing to the primeness of RC. Hence either $a(1-e) = 0$ or $eq^{-1}c(1-e) = 0$. Assume that $a(1-e) = 0$ for some nontrivial idempotent $e \in RC$. Since $(1-e) + ex(1-e)$ is also an idempotent for all $x \in RC$ and $a(e - ex(1-e)) \neq 0$, then $\big((1-e)+ex(1-e)\big)q^{-1}c\big(e - ex(1-e)\big)$ for all $x \in RC$. In particular $(1-e)q^{-1}ce = 0$. Hence $eq^{-1}c = eq^{-1}ce = q^{-1}ce$ for any idempotent $e \in RC$ of rank 1. Let E be the additive subgroup of idempotents of R generated by all idempotents of rank 1 in R. Hence $\big[e, q^{-1}c\big] = 0$ for all $e \in E$. Since E is a noncommutative Lie ideal of R and $q^{-1}c \in C$ by Lemma 1 in [8]. Eventually, $a\big((b-c)x\big)^n \in C$ for all $x \in [RC, RC]$ and we are done by Lemma 2.3 (ii). $\square$

Lemma 2.5. *Let R be a prime ring with center $Z(R)$, $a, b, c \in R$ and $a \neq 0$. Let α be an automorphism of R. If $a\big(bx - \alpha(x)c\big)^n \in Z(R)$ for all $x \in R$, where n is a fixed positive integer then either $dim_C RC = 4$ or $a\big(bx - \alpha(x)c\big) = 0$ for all $x \in R$.*

Proof. Assume that $dim_C RC > 4$ and $a\big(bx - \alpha(x)c\big)^n \in Z(R)$ for all $x \in R$. If $b = 0$ or $c = 0$ then we are done by Fact 7. So we may assume that $b \neq 0$ and $c \neq 0$. If $Z(R) = 0$ then $a\big(bx - \alpha(x)c\big)^n = 0$ for all $x \in R$ and the proof is finished by Lemma 4 in [7]. Suppose that $Z(R) \neq 0$. If α is an X-inner automorphism of R, then there exists an invertible element $q \in Q$ such that $\alpha(x) = qxq^{-1}$ for all $x \in R$. Through the hypothesis, we have $a\big(bx - qxq^{-1}c\big)^n \in Z(R)$ for all $x \in R$. In view of Lemma 2.1, we obtain $q^{-1}c \in C$ and $a(b-c) = 0$. Hence we are finished for this case.

Now suppose that α is an X-outer derivation of R. Since $a\big(bx-\alpha(x)c\big)^n \in Z(R)$ for all $x \in R$ then

$$a\big(bx - \alpha(x)c\big)^n y - ya\big(bx - \alpha(x)c\big)^n = 0 \tag{2.7}$$

for all $x, y \in R$. By Theorem 1 in [11], (2.7) holds for all $x, y \in Q$ and is a GPI for Q. Hence Q is a primitive ring with nonzero socle H and Q is isomorphic to a dense subring of $End_D(V)$, where V is a vector space over the division ring D.

First suppose that $dim_D V = \infty$. Since H contains finite rank elements, then

$$a\big(bx - \alpha(x)c\big)^n = 0$$

for all $x \in H$ and thereby for all $x \in Q$. Hence using Lemma 4 in [7], we have $a\big(bx - \alpha(x)c\big) = 0$ for all $x \in R$. So we may consider that $dim_D V < \infty$. Thus, $Q \cong End({}_D V)$ and it is isomorphic to the $k \times k$ matrix ring D_k over the division ring D. In the light of [19] there exists a semi-linear automorphism $T \in End({}_D V)$ such that $\alpha(x) = TxT^{-1}$ for all $x \in Q$. Thus $a(bx - TxT^{-1}c)^n \in C$ for all $x \in Q$.

Suppose that $k > 2$. First assume that v and $T^{-1}cv$ are D-dependent for all $v \in V$. In this manner, there exists some $\lambda \in C$ such that

$$T^{-1}cv = \lambda v.$$

This yields

$$\begin{aligned}\big(bx - \alpha(x)c\big)v =& \big(bx - TxT^{-1}c\big)v\\ =& bxv - TxT^{-1}cv\\ =& bxv - Tx\lambda v\\ =& bxv - TT^{-1}cxv\\ =& (b-c)xv\end{aligned}$$

for all $x \in Q$ and $v \in V$. Since the action of Q on V is faithful, then

$$bx - TxT^{-1}c = (b-c)x$$

for all $x \in Q$. Using this in the initial assumption we have $a\big((b-c)x\big)^n \in C$ for all $x \in Q$. By Fact 7, we see that $a(b-c) = 0$ and

$$a\big(bx - \alpha(x)c\big)v = a\big(bx - TxT^{-1}c\big)v = a(b-c)xv = 0.$$

Hence $a\big(bx - \alpha(x)c\big) = 0$ for all $x \in R$.

Now consider that there exists $v_0 \in V$ such that v_0 and $T^{-1}cv_0$ are D-independent. Then there exists some $x \in Q$ of rank 1 such that

$$\begin{aligned} xv_0 &= 0 \\ xT^{-1}cv_0 &= T^{-1}v_0 \end{aligned}$$

by the density of Q. Thus, $a\big(bx - TxT^{-1}c\big)v_0 = a(bxv_0 - TxT^{-1}cv_0) = -av_0$ and $a\big(bx - TxT^{-1}c\big)^n v_0 = (-1)^n av_0$. It is easy to see that $a(bx - TxT^{-1})^n$ is of rank at most 2. Since we assume $k > 2$, then $a(bx - TxT^{-1})^n = 0$ for all $x \in Q$. Eventually, $av_0 = 0$ implies $a = 0$, which is a contradiction. Therefore $dim_D V \leq 2$.

If C is finite then D is finite (being finite dimensional over C). By Wedderburn's Theorem in [19], D is a field. Hence, Q is commutative, a contradiction. If C is infinite then we need to consider two cases of the automorphism α, for being Frobenius or not. If α is not a Frobenius automorphism of R then $a\big(bx - yc\big)^n \in C$ for all $x, y \in Q$ by [12]. In particular we have $a\big(bx - xc\big)^n \in C$ for all $x \in Q$. In that case $a\big(bx - xc\big)^n = 0$ and hence $c = 0$ and either $b = 0$ or $ab = 0$ by Fact 7, a contradiction.

Now suppose that α is a Frobenius automorphism of R. If *char* $Q = 0$ then by the definition of the Frobenius automorphism, $\alpha(\lambda) = \lambda$ for all $\lambda \in C$. In the light of Theorem 4.7.4 in [2], α is an inner automorphism, which leads a contradiction. Hence, *char* $Q = p \geq 2$ and $\alpha(\lambda) = \lambda^{p^k}$ for all $\lambda \in C$ and some $k \neq 0$. Substituting λx into x in the main identity with $\lambda \neq 0$, we obtain

$$a\big(\lambda bx - \alpha(\lambda x)c\big)^n = \lambda^n a\big(bx - \lambda^{p^k-1}\alpha(x)c\big)^n \in C$$

for all $x \in Q$. Thus we have

$$a\big(bx - \lambda^{p^k-1}\alpha(x)c\big)^n \in C \tag{2.8}$$

for all $x \in Q$. Expanding (2.8) we obtain

$$\sum_{i=0}^{n} \Big(\sum_{(i,n-i)} z_1 z_2 \dots z_n \Big) \lambda^{i(p^k-1)} \in C \tag{2.9}$$

in which each term of this summation has $n - i$ (bx)'s and i $\big(\alpha(x)c\big)$'s in permutational order. Set $t = \lambda^{p^k-1}$ and

$$y_i = a\Big(\sum_{(i,n-i)} z_1 z_2 \dots z_n \Big)$$

for $i \in \{0, 1, \dots, n\}$. Then we can reinscribe (2.9) as

$$y_0 + ty_1 + \dots + t^n y_n \in C. \tag{2.10}$$

Substituting λ into $1, \lambda, \ldots \lambda^n$ respectively in (2.10), leads us to the system of equations

$$\begin{aligned} y_0 + y_1 + \cdots + y_n &= \gamma_0 \\ y_0 + ty_1 + \cdots + t^n y_n &= \gamma_1 \\ &\vdots \\ y_0 + t^n y_1 + \cdots + t^{n^2} y_n &= \gamma_n \end{aligned} \tag{2.11}$$

where $\gamma_i \in C$ for all $i = 0, 1, \ldots, n$. In this case there exist infinitely many $\lambda \in C$ such that $\lambda^{m(p^k-1)} \neq 1$ for $m = 1, 2, \ldots, n$, due to the fact that C is infinite. Thus the van der Monde determinant

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & t & \cdots & t^n \\ \vdots & \vdots & & \vdots \\ 1 & t & \cdots & t^{n^2} \end{vmatrix} = \prod_{\substack{i,j=0 \\ i<j}}^{n} (t^i - t^j) = \prod_{\substack{i,j=0 \\ i<j}}^{n} \big(\lambda^i(p^k-1) - \lambda^j(p^k-1)\big)$$

is not zero. Particularly, using $y_0 = a\big(bx\big)^n \in C$ and $y_n = a\big(\alpha(x)c\big)^n \in C$ for all $x \in Q$, in view of Fact 7 we see that $ab = 0$ and either $a = 0$ or $c = 0$, a contradiction. □

Lemma 2.6. *([1], Lemma 3.4) Let R be a prime ring and L be a noncommutative Lie ideal of R. Let $a, b, c \in R$ and $\alpha \in Aut(R)$. Suppose that $a(bx - \alpha(x)c)^n = 0$ for all $x \in L$, where n is a fixed positive integer. Then either $a = 0$ or $a(bx - \alpha(x)c) = 0$ for all $x \in R$.*

Lemma 2.7. *Let R be a prime ring with center $Z(R)$ and $a, b, c \in R$ with $a \neq 0$. Suppose that*

$$a(bx - \alpha(x)c)^n \in Z(R) \tag{2.12}$$

for all $x \in [R, R]$ where α is an automorphism of R and n is a fixed positive integer. Then either $a(bx - \alpha(x)c) = 0$ for all $x \in R$ or $dim_C RC = 4$.

Proof. Assume that $dim_C RC > 4$. If $b = 0$ or $c = 0$ then we are done by Lemma 2.3. So we may assume that $b \neq 0$ and $c \neq 0$. Suppose first that α is an X-inner automorphism of R, then there exists an invertible element $q \in Q$ such that $\alpha(x) = qxq^{-1}$ for all $x \in R$. Hence $a\big(bx - qxq^{-1}c\big)^n \in Z(R)$ for all $x \in \big[R, R\big]$ and the proof is finished by Lemma 2.4. Now suppose that α is an X-outer automorphism of R. Since $b \neq 0$ and $c \neq 0$ then by [10], R is a GPI-ring. Thus RC is a primitive ring with nonzero socle H. If $H \cap Z(R) = (0)$ then

$$a\big(bx - \alpha(x)c\big)^n = 0 \tag{2.13}$$

for all $x \in \big[H, H\big]$ and in view of Lemma 2.6 we see that $a\big(bx - \alpha(x)c\big) = 0$ for all $x \in H$. The last identity holds for all $x \in R$ and in that case we are done by Lemma 2.6. In turn we may assume that $H \cap Z(R) \neq (0)$. Hence H is a central simple $Z(R)$-algebra and so is R. Therefore we may consider that $H = R = Q$ is a finite dimensional central simple $Z(R)$-algebra by Wedderburn-Artin Theorem and R is the ring of all linear transformations of a k-dimensional vector space V over a division ring D, for $k > 1$. Let e be an idempotent of R such that $rank(e) = 1$ and $x, y \in R$. Substituting $\big[\alpha^{-1}(1-e)xe, \alpha^{-1}(1-e)ye\big]$ into x in (2.12) we have

$$a\Big(b\big[\alpha^{-1}(1-e)xe, \alpha^{-1}(1-e)ye\big] - \alpha\Big(\big[\alpha^{-1}(1-e)xe, \alpha^{-1}(1-e)ye\big]\Big)\Big]c\Big)^n \in Z(R). \tag{2.14}$$

The rank of (2.14) is at most 4 and since we assume $dim_C RC > 4$ then

$$a\Big(b\big[\alpha^{-1}(1-e)xe, \alpha^{-1}(1-e)ye\big] - \alpha\Big(\big[\alpha^{-1}(1-e)xe, \alpha^{-1}(1-e)ye\big]\Big)\Big]c\Big)^n = 0$$

for all $x, y \in R$. Multiplying by $(1-e)$ on the right we obtain

$$a(1-e)\big(\alpha(y)\alpha(e)(1-e)\alpha(x)\alpha(e)c(1-e) - \alpha(x)\alpha(e)(1-e)\alpha(y)\alpha(e)c(1-e)\big)^n = 0$$

and since α is an X-outer derivation of R then

$$a(1-e)\big(x\alpha(e)(1-e)y\alpha(e)c(1-e) - y\alpha(e)(1-e)x\alpha(e)c(1-e)\big)^n = 0 \tag{2.15}$$

for all $x, y \in R$. By virtue of Fact 7, we see that one of the following holds:

(i) $a(1-e) = 0$,
(ii) $\alpha(e)c(1-e)$,
(iii) $\big(\alpha(e)(1-e)y\alpha(e)c(1-e)\big) = -\lambda\alpha(e)c(1-e)$ and
$\alpha(e)c(1-e)y\alpha(e)(1-e) = -\lambda\alpha(e)c(1-e)$
(iv) $\big(\alpha(e)(1-e)y\alpha(e)c(1-e)\big) = -\lambda\alpha(e)c(1-e)$ and $a(1-e)y\alpha(e)(1-e) = -\lambda a(1-e)$

for all $y \in R$ and some $\lambda \in C$. Using (iii) in (2.15) we have

$$\lambda^n a(1-e)\big((x-y)\alpha(e)c(1-e)\big)^n = 0$$

for all $x, y \in R$. In particular,

$$\lambda^n a(1-e)\big(x\alpha(e)c(1-e)\big)^n = 0$$

for all $x \in R$. By the primeness of R, either $\lambda = 0$ or $a(1-e) = 0$ or $\alpha(e)c(1-e) = 0$. If $\lambda = 0$ then $\alpha(e)(1-e) = 0$. Accordingly, using (iv) in (2.15) we get either $a(1-e) = 0$ or $\alpha(e)(1-e) = 0$ or $\alpha(e)c(1-e) = 0$. Consider that there exists an idempotent $e \in R$ such that $\alpha(e)(1-e) = 0$. Substituting $\big[\alpha^{-1}(1-e)xe, ye\big]$ into x in (2.12), we see that

$$a\Big(b\big[\alpha^{-1}(1-e)xe, ye\big] - \alpha\Big(\big[\alpha^{-1}(1-e)xe, ye\big]\Big)c\Big)^n \in Z(R) \tag{2.16}$$

for all $x, y \in R$. Since we assume $dim_C RC > 4$ and the rank of (2.16) is at most 3, then $a\Big(b\big[\alpha^{-1}(1-e)xe, ye\big] - \alpha\Big(\big[\alpha^{-1}(1-e)xe, ye\big]\Big)c\Big)^n = 0$ for all $x, y \in R$. Right multiplying by $(1-e)$ in the last equation, we have

$$a(1-e)\big(\alpha(x)\alpha(e)\alpha(y)\alpha(e)c(1-e)\big)^n = 0$$

for all $x, y \in R$. In view of [14], $a(1-e) = 0$ or $\alpha(e)R\alpha(e)c(1-e) = 0$. By the primeness of R, $a(1-e) = 0$ or $\alpha(e)c(1-e) = 0$. Analogously, we have $\alpha(1-e)ce = 0$. Thus $ce = \alpha(e)ce = \alpha(e)c$ for any idempotent e of rank 1. Let E be the additive subgroup of idempotents of R generated by all idempotents of rank 1 in R. Eventually $ce = \alpha(e)c$ for all $e \in E$. Since E is a noncommutative Lie ideal of R then $cx - \alpha(x)c = 0$ for all $x \in [R, R]$, by the proof of Lemma 1 in [8]. Hence $c[x, y] - \big[\alpha(x), \alpha(y)\big]c = 0$ for all $x, y \in R$. Since α is an X-outer automorphism of R then $c[x, y] - \big[r, s\big]c = 0$ for all $x, y, r, s \in R$ which means $c = 0$ or R is commutative, a contradiction. □

Now we give the proofs for Theorem 2.1 and Theorem 2.2 in the sequel.

Proof of Theorem 2.1. Assume $dim_C RC > 4$. The generalized α-derivation G is of the form $G(x) = sx + \delta(x)$ for all $x \in R$ and some $s \in Q$ in view of [9]. By assumption we have $a\big(sx + \delta(x)\big)^n \in Z(R)$ for all $x \in R$. If δ is an X-inner derivation of R then there exists $b \in R$ such that $\delta(x) = bx - \alpha(x)b$ for all $x \in R$. Thus $a\big((s+b)x - \alpha(x)b\big)^n \in Z(R)$ for all $x \in R$ and we are done by Lemma 2.5. Now suppose that δ is an X-outer derivation of R and $\big[a\big(sx + \delta(x)\big)^n, y\big] = 0$ for all $x, y \in R$. By Theorem 1 in [13]

$$\big[a\big(sx + w\big)^n, y\big] = 0 \tag{2.17}$$

for all $x, y, w \in R$. In particular, $\big[aw^n, y\big] = 0$ for all $w, y \in R$, that is, $aw^n \in Z(R)$ for all $w \in R$ and thereby $a = 0$ or R is commutative, a contradiction. □

Proof of Theorem 2.2. Assume $dim_C RC > 4$. Set $I = R\big[L, L\big]R$. Then $0 \neq \big[I, R\big] \subset L$ by Fact 1. There exists $s = f(1) \in Q$ such that $G(x) = sx + \delta(x)$ for all $x \in R$ where δ is an α-derivation of R in view of [9]. By the hypothesis

$$a\big(sx + \delta(x)\big)^n \in Z(R) \tag{2.18}$$

for all $x \in L$ and thus for all $x \in [I, R]$. In view of Theorem 2 in [13], I, R and Q satisfy the same GPI's with single skew derivation. So (2.18) holds for all $x \in [Q, Q]$. In turn we may assume that $I = R = Q$.

If δ is an X-inner α-derivation of R, then there exists $b \in R$ such that $\delta(x) = bx - \alpha(x)b$ for all $x \in R$. In this case (2.18) becomes $a\big((s+b)x - \alpha(x)b\big)^n \in Z(R)$ for all $x \in \big[R, R\big]$ and we are done by Lemma 2.7.

Now consider the case that δ is an X-outer derivation of R. Then

$$a\big(s[x, y] + \delta\big([x, y]\big)\big)^n \in Z(R)$$

for all $x, y \in R$. Thus

$$\big[a\big(s[x, y] + \delta(x)y + \alpha(x)\delta(y) - \delta(y)x - \alpha(y)\delta(x)\big)^n, z\big] = 0$$

for all $x, y, z \in R$. In view of Theorem 1 in [13]

$$\big[a\big(s[x, y] + wy + \alpha(x)u - ux - \alpha(y)w\big)^n, z\big] = 0 \tag{2.19}$$

for all $x, y, u, w, z \in R$. In particular, $\left[a\big(s[x,y]\big)^n, z\right] = 0$ which means $a\big(s[x,y]\big)^n \in Z(R)$ for all $x, y \in R$ and so we have $as = 0$ by Lemma 2.3 (ii). Hence

$$aG(x)^n = a(sx + \delta(x))^n = a\delta(x)^n \in C$$

for all $x \in R$. By virtue of Corollary 1 we obtain $a = 0$, a contradiction. $\square$

The condition of primeness can not be ommitted, as we see in the following example:

Example. Let $\mathbb{F}$ be a field of characteristic 2 and $q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ is an invertible element of the ring $R = \begin{bmatrix} \mathbb{F} & 0 & 0 \\ 0 & \mathbb{F} & \mathbb{F} \\ 0 & 0 & \mathbb{F} \end{bmatrix}$. Let $\alpha(x) = qxq^{-1} = \begin{bmatrix} u & 0 & 0 \\ 0 & v & v+w+z \\ 0 & 0 & z \end{bmatrix}$ for all $x = \begin{bmatrix} u & 0 & 0 \\ 0 & v & w \\ 0 & 0 & z \end{bmatrix} \in R$. For the elements $c = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, d = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in R$, it is easy to check that $G(x) = cx - \alpha(x)d = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & z \\ 0 & 0 & 0 \end{bmatrix}$. Hence for $a = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in R$ we have $aG(x)^n \in Z(R)$ where n is a fixed positive integer but $aG(x) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & z \\ 0 & 0 & 0 \end{bmatrix} \neq 0$ unless $z = 0$.

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