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2-Irreducible and Strongly 2-Irreducible Submodules of a Module

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Abstract. Let R be a commutative ring with identity and M be an R-module. In this paper, we will introduce the concept of 2-irreducible (resp., strongly 2-irreducible) submodules of M as a generalization of irreducible (resp., strongly irreducible) submodules of M and investigated some properties of these classes of modules.

Keywords: Irreducible ideal, Strongly 2-irreducible ideal, 2-irreducible submodule, Strongly 2-irreducible submodule.

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1. Introduction

Throughout this paper, R will denote a commutative ring with identity and $\mathbb{Z}$ will denote the ring of integers.

An ideal I of R is said to be *irreducible* if $I = J_1 \cap J_2$ for ideals J_1 and J_2 of R implies that either $I = J_1$ or $I = J_2$. A proper ideal I of R is said to be *strongly irreducible* if for ideals J_1, J_2 of R, $J_1 \cap J_2 \subseteq I$ implies that $J_1 \subseteq I$ or $J_2 \subseteq I$ [12]. An ideal I of R is said to be *2-irreducible* if whenever $I = J_1 \cap J_2 \cap J_3$ for

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ideals J_1, J_1 and J_3 of R, then either $I = J_1 \cap J_2$ or $I = J_1 \cap J_3$ or $I = J_2 \cap J_3$. Clearly, any irreducible ideal is a 2-irreducible ideal [21].

A proper submodule N of an R-module M is said to be *irreducible* (resp., *strongly irreducible*) if for submodules H_1 and H_2 of M, $N = H_1 \cap H_2$ (resp., $H_1 \cap H_2 \subseteq N$) implies that $N = H_1$ or $N = H_2$.(resp., $H_1 \subseteq N$ or $H_2 \subseteq N$).

The main purpose of this paper is to introduce the concept of 2-irreducible and strongly 2-irreducible submodules of an R-module M as a generalization of irreducible and strongly irreducible submodules of M and obtain some related results.

A submodule N of an R-module M is said to be a *2-irreducible submodule* if whenever $N = H_1 \cap H_2 \cap H_3$ for submodules H_1, H_2 and H_3 of M, then either $N = H_1 \cap H_2$ or $N = H_2 \cap H_3$ or $N = H_1 \cap H_3$ (Definition 2.1).

A proper submodule N of an R-module M is said to be a *strongly 2-irreducible submodule* if whenever $H_1 \cap H_2 \cap H_3 \subseteq N$ for submodules H_1, H_2 and H_3 of M, then either $H_1 \cap H_2 \subseteq N$ or $H_2 \cap H_3 \subseteq N$ or $H_1 \cap H_3 \subseteq N$ (Definition 2.6).

In Section 2 of this paper, for an R-module M, among other results, we prove that if M is a Noetherian R-module and N is a 2-irreducible submodule of M, then either N is irreducible or N is an intersection of exactly two irreducible submodules of M (Theorem 2.22). In Theorem 2.9, we provide a characterization for strongly 2-irreducible submodules of M. Also, it is shown that if M is a strong comultiplication R-module, then every non-zero proper submodule of R is a strongly sum 2-irreducible R-module if and only if every non-zero proper submodule of M is a strongly 2-irreducible submodule of M (Theorem 2.11). Further, it is proved that if N is a submodule of a finitely generated multiplication R-module M, then N is a strongly 2-irreducible submodule of M if and only if $(N :_R M)$ is a strongly 2-irreducible ideal of R (Theorem 2.12). In Theorem 2.19 and 2.21, we provide some useful characterizations for strongly 2-irreducible submodules of some special classes of modules. Example 2.14 shows that the concepts of strongly irreducible submodules and strongly 2-irreducible submodules are different in general. Finally, let $R = R_1 \times R_2 \times \cdots \times R_n$ $(2 \leq n < \infty)$ be a decomposable ring and $M = M_1 \times M_2 \cdots \times M_n$ be an R-module, where for every $1 \leq i \leq n$, M_i is an R_i-module, respectively, it is proved that a proper submodule N of M is a strongly 2-irreducible submodule of M if and only if either $N = \times_{i=1}^n N_i$ such that for some $k \in \{1, 2, ..., n\}$, N_k is a strongly 2-irreducible submodule of M_k, and $N_i = M_i$ for every $i \in \{1, 2, ..., n\} \setminus \{k\}$ or $N = \times_{i=1}^n N_i$ such that for some $k, m \in \{1, 2, ..., n\}$, N_k is a strongly irreducible submodule of M_k, N_m is a strongly irreducible submodule of M_m, and $N_i = M_i$ for every $i \in \{1, 2, ..., n\} \setminus \{k, m\}$ (Theorem 2.28).

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2. Main results

Definition 2.1. We say that a submodule N of an R-module M is a *2-irreducible submodule* if whenever $N = H_1 \cap H_2 \cap H_3$ for submodules H_1, H_2 and H_3 of M, then either $N = H_1 \cap H_2$ or $N = H_2 \cap H_3$ or $N = H_1 \cap H_3$.

Example 2.2. Let $R = K[X, Y]$ be a polynomial ring in variables X and Y over a field K. Let I be the ideal $\langle X^2, XY \rangle$. Then $\langle X^2, XY \rangle = \langle X \rangle \cap \langle X^2, Y \rangle$ implies that I is not an irreducible ideal of R. But since $\langle X \rangle \cap \langle X^2, Y \rangle$ is a primary decomplosition for I, one can see that I is a 2-irreducible ideal of R by using [17, 9.31].

Example 2.3. Let $R = K[X, Y]$ be a polynomial ring in variables X and Y over a field K and let $I = \langle X \rangle \cap \langle Y \rangle$. Then I is not an irreducible ideal of R. But since $\langle X \rangle$ and $\langle Y \rangle$ are prime and so strongly irreducible ideals of R, we have I is a 2-irreducible ideal of R by [21, Proposition 3].

Theorem 2.4. *Let M be a Noetherian R-module. If N is a 2-irreducible submodule of M, then either N is irreducible or N is an intersection of exactly two irreducible submodules of M.*

Proof. Let N be a 2-irreducible submodule of M. By [17, Exercise 9.31], N can be written as a finite irredundant irreducible decomposition $N = N_1 \cap N_2 \cap ... \cap N_k$. We show that either $k = 1$ or $k = 2$. If $k > 3$, then since N is 2-irreducible, $N = N_i \cap N_j$ for some $1 \leq i, j \leq k$, say $i = 1$ and $j = 2$. Therefore $N_1 \cap N_2 \subseteq N_3$, which is a contradiction. □

Corollary 2.5. *Let M be a Noetherian multiplication R-module. If N is a 2-irreducible submodule of M, then N a 2-absorbing primary submodule of M.*

Proof. Let N be a 2-irreducible submodule of M. By the fact that every irreducible submodule of a Noetherian R-module is primary and regarding Theorem 2.22, we have either N is a primary submodule or is a sum of two primary submodules. It is clear that every primary submodule is 2-absorbing primary, also the sum of two primary submodules is a 2-absorbing primary submodule, by [15, Theorem 2.20]. □

Definition 2.6. We say that a proper submodule N of an R-module M is a *strongly 2-irreducible submodule* if whenever $H_1 \cap H_2 \cap H_3 \subseteq N$ for submodules H_1, H_2 and H_3 of M, then either $H_1 \cap H_2 \subseteq N$ or $H_2 \cap H_3 \subseteq N$ or $H_1 \cap H_3 \subseteq N$.

Example 2.7. [21, Corollary 2] Consider the $\mathbb{Z}$-module $\mathbb{Z}$. Then $n\mathbb{Z}$ is a strongly 2-irreducible submodule of $\mathbb{Z}$ if $n = 0$, p^t or $p^r q^s$, where p, q are prime integers and t, r, s are natural numbers.

Proposition 2.8. *The strongly 2-irreducible submodules of a distributive R-module are precisely the 2-irreducible submodules.*

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Proof. This is straightforward. □

Theorem 2.9. *Let N be a proper submodule of an R-module M. Then the following conditions are equivalent:*

(a) *N is a strongly 2-irreducible submodule;*
(b) *For all elements x, y, z of M, we have $(Rx + Ry) \cap (Rx + Rz) \cap (Ry + Rz) \subseteq N$ implies that either $(Rx + Ry) \cap (Rx + Rz) \subseteq N$ or $(Rx + Ry) \cap (Ry + Rz) \subseteq N$ or $(Rx + Rz) \cap (Ry + Rz) \subseteq N$.*

Proof. $(a) \Rightarrow (b)$ This ia clear.

$(b) \Rightarrow (a)$ Let $H_1 \cap H_2 \cap H_3 \subseteq N$ for submodules H_1, H_2 and H_3 of M. If $H_1 \cap H_2 \not\subseteq N$, $H_1 \cap H_3 \not\subseteq N$, and $H_2 \cap H_3 \not\subseteq N$, then there exist elements x, y, z of M such that $x \in H_2 \cap H_3$, $y \in H_1 \cap H_3$, and $z \in H_1 \cap H_2$ but $x \notin N$, $y \notin N$, and $z \notin N$. Therefore,

$$(Ry + Rz) \cap (Rx + Rz) \cap (Rx + Ry) \subseteq H_1 \cap H_2 \cap H_3 \subseteq N.$$

Hence by the part (a), either $(Ry+Rz)\cap(Rx+Rz) \subseteq N$ or $(Ry+Rz)\cap(Rx+Ry) \subseteq N$ or $(Rx + Rz) \cap (Rx + Ry) \subseteq N$. Thus either $z \in N$ or $y \in N$ or $x \in N$. This contradiction completes the proof. □

Recall that a *waist submodule* of an R-module M is a submodule that is comparable to any other submodules of M.

Proposition 2.10. *Let N be a proper submodule of an R-module M. Then we have the following.*

(a) *If N is a strongly 2-irreducible submodule of M, then it is also a 2-irreducible submodule of M.*
(b) *If N is a strongly 2-irreducible submodule of M, then N is a strongly 2-irreducible submodule of T and N/K is a strongly 2-irreducible submodule of M/K for any $K \subseteq N \subseteq T$.*
(c) *If for all elements x, y, z of M we have $Rx \cap Ry \cap Rz \subseteq N$ implies that either $Rx \cap Ry \subseteq N$ or $Rx \cap Rz \subseteq N$ or $Ry \cap Rz \subseteq N$, then N is a strongly 2-irreducible submodule of M.*
(d) *If N is a waist submodule of M, then N is strongly 2-irreducible submodule of M if and only if N is 2-irreducible module.*
(e) *If N satisfies $(N+T)\cap(N+K) = N+(T\cap K)$, whenever $T\cap K \subseteq N$, then N is strongly 2-irreducible submodule of M if and only if N is a 2-irreducible module.*

Proof. (a) Let N be a strongly 2-irreducible submodule of M and let $N = H_1 \cap H_2 \cap H_3$ for submodules H_1, H_2 and H_3 of M. Then by assumption, either $H_1 \cap H_2 \subseteq N$ or $H_1 \cap H_3 \subseteq N$ or $H_2 \cap H_3 \subseteq N$. Now the result follows from the fact that the reverse of inclusions are clear.

The parts (b), (d), and (e) are straightforward.

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(c) Let $H_1 \cap H_2 \cap H_3 \subseteq N$ for submodules H_1, H_2 and H_3 of M. If $H_1 \cap H_2 \not\subseteq N$, $H_1 \cap H_3 \not\subseteq N$, and $H_2 \cap H_3 \not\subseteq N$, then there exist elements x, y, z of M such that $x \in H_2 \cap H_3$, $y \in H_1 \cap H_3$, and $z \in H_1 \cap H_2$ but $x \notin N$, $y \notin N$, and $z \notin N$. Now the result follows by assumption. □

An R-module M is said to be a *comultiplication module* if for every submodule N of M there exists an ideal I of R such that $N = (0 :_M I)$, equivalently, for each submodule N of M, we have $N = (0 :_M Ann_R(N))$ [2].

An R-module M satisfies the *double annihilator conditions* (DAC for short) if for each ideal I of R we have $I = Ann_R(0 :_M I)$ [9].

An R-module M is said to be a *strong comultiplication module* if M is a comultiplication R-module and satisfies the DAC conditions [4].

A submodule N of an R-module M is said to be a *strongly sum 2-irreducible submodule* if whenever $N \subseteq H_1 + H_2 + H_3$ for submodules H_1, H_2 and H_3 of M, then either $N \subseteq H_1 + H_2$ or $N \subseteq H_2 + H_3$ or $N \subseteq H_1 + H_3$. Also, M is said to be a *strongly sum 2-irreducible module* if and only if M is a strongly sum 2-irreducible submodule of itself [10].

Theorem 2.11. *Let M be a strong comultiplication R-module. Then every non-zero proper submodule of R is a strongly sum 2-irreducible R-module if and only if every non-zero proper submodule of M is a strongly 2-irreducible submodule of M.*

Proof. " $\Rightarrow$ " Let N be a non-zero proper submodule of M and let $H_1 \cap H_2 \cap H_3 \subseteq N$ for submodules H_1, H_2 and H_3 of M. Then by using [11, 2.5],

$$Ann_R(N) \subseteq Ann_R(H_1) + Ann_R(H_2) + Ann_R(H_3).$$

This implies that either $Ann_R(N) \subseteq Ann_R(H_1) + Ann_R(H_2)$ or $Ann_R(N) \subseteq Ann_R(H_1) + Ann_R(H_3)$ or $Ann_R(N) \subseteq Ann_R(H_2) + Ann_R(H_3)$ since by assumption, $Ann_R(N)$ is a strongly sum 2-irreducible R-module. Therefore, either $H_1 \cap H_2 \subseteq N$ or $H_1 \cap H_3 \subseteq N$ or $H_2 \cap H_3 \subseteq N$ since M is a comultiplication R-module.

" $\Leftarrow$ " Let I be a non-zero proper submodule of R and let $I \subseteq I_1 + I_2 + I_3$. Then

$$(0 :_M I_1) \cap (0 :_M I_2) \cap (0 :_M I_3) \subseteq (0 :_M I).$$

Thus by assumption, either $(0 :_M I_1) \cap (0 :_M I_2) \subseteq (0 :_M I)$ or $(0 :_M I_1) \cap (0 :_M I_3) \subseteq (0 :_M I)$ or $(0 :_M I_2) \cap (0 :_M I_3) \subseteq (0 :_M I)$. This implies that either $(0 :_M I_1 + I_2) \subseteq (0 :_M I)$ or $(0 :_M I_1 + I_3) \subseteq (0 :_M I)$ or $(0 :_M I_2 + I_3) \subseteq (0 :_M I)$. Thus either $I \subseteq I_1 + I_2$ or $I \subseteq I_1 + I_3$ or $I \subseteq I_2 + I_3$ since M is a strong comultiplication R-module. □

An R-module M is said to be a *multiplication module* if for every submodule N of M there exists an ideal I of R such that $N = IM$ [6].

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Theorem 2.12. *Let N be a submodule of a finitely generated multiplication R-module M. Then N is a strongly 2-irreducible submodule of M if and only if $(N :_R M)$ is a strongly 2-irreducible ideal of R.*

Proof. " $\Rightarrow$ " Let N be a strongly 2-irreducible submodule of M and let $J_1 \cap J_2 \cap J_3 \subseteq (N :_R M)$ for some ideals J_1, J_2, and J_3 of R. Then

$$J_1M \cap J_2M \cap J_3M \subseteq (N :_R M)M = N$$

by [8, Corollary 1.7]. Thus by assumption, either $J_1M \cap J_2M \subseteq N$ or $J_1M \cap J_3M \subseteq N$ or $J_2M \cap J_3M \subseteq N$. Hence, either $(J_1 \cap J_2)M \subseteq (N :_R M)M$ or $(J_1 \cap J_3)M \subseteq (N :_R M)M$ or $(J_2 \cap J_3)M \subseteq (N :_R M)M$. Therefore, either $J_1 \cap J_2 \subseteq (N :_R M)$ or $J_1 \cap J_3 \subseteq (N :_R M)$ or $J_2 \cap J_3 \subseteq (N :_R M)$ by [18, Corollary of Theorem 9].

" $\Leftarrow$ " Let $(N :_R M)$ is a strongly 2-irreducible ideal of R and let $H_1 \cap H_2 \cap H_3 \subseteq N$ for some submodules H_1, H_2 and H_3 of M. Then we have

$$(H_1 \cap H_2 \cap H_3 :_R M)M = ((H_1 :_R M) \cap (H_2 :_R M) \cap (H_3 :_R M))M \subseteq (N :_R M)M.$$

Thus $(H_1 :_R M) \cap (H_2 :_R M) \cap (H_3 :_R M) \subseteq (N :_R M)$ by [18, Corollary of Theorem 9]. Hence, either $(H_1 :_R M) \cap (H_2 :_R M) \subseteq (N :_R M)$ or $(H_1 :_R M) \cap (H_3 :_R M)) \subseteq (N :_R M)$ or $(H_2 :_R M) \cap (H_3 :_R M) \subseteq (N :_R M)$ since $(N :_R M)$ is a strongly 2-irreducible ideal of R. Therefore, either $H_1 \cap H_2 \subseteq N$ or $H_1 \cap H_3 \subseteq N$ or $H_2 \cap H_3 \subseteq N$ by [8, Corollary 1.7]. □

Example 2.13. Consider the $\mathbb{Z}$-module $\mathbb{Z}_{p^t q^n r^m}$, where p, q, r are prime integers and t, n, m are natural numbers.

(a) By using Theorem 2.12 and Example 2.7, one can see that $\bar{p^t}\mathbb{Z}_{p^t q^n r^m}$ and $q^n \bar{r}^m \mathbb{Z}_{p^t q^n r^m}$ are strongly 2-irreducible submodules of $\mathbb{Z}_{p^t q^n r^m}$.

(b) $p\bar{q}r\mathbb{Z}_{p^3qr} = \bar{pq}Z_{p^3qr} \cap \bar{pr}\mathbb{Z}_{p^3qr} \cap \bar{qr}\mathbb{Z}_{p^3qr}$ implies that $p\bar{q}r\mathbb{Z}_{p^3qr}$ is not a 2-irreducible submodule of $\mathbb{Z}_{p^3qr}$.

The following example shows that the concepts of strongly irreducible submodules and strongly 2-irreducible submodules are different in general.

Example 2.14. Consider the $\mathbb{Z}$-module $\mathbb{Z}_6$. Then $0 = \bar{3}\mathbb{Z}_6 \cap \bar{2}\mathbb{Z}_6$ implies that the 0 submodule of $\mathbb{Z}_6$ is not strongly irreducible. But $(0 :_{\mathbb{Z}} \mathbb{Z}_6) = 6\mathbb{Z}$ is a strongly 2-irreducible ideal of $\mathbb{Z}$ by Example 2.7. Since the $\mathbb{Z}$-module $\mathbb{Z}_6$ is a finitely generated multiplication $\mathbb{Z}$-module, 0 is a strongly 2-irreducible submodule of $\mathbb{Z}_6$ by Theorem 2.12.

Lemma 2.15. *Let M be an R-module. If N_1 and N_2 are strongly irreducible submodules of M, then $N_1 \cap N_2$ is a strongly 2-irreducible submodule of M.*

Proof. This is straightforward. □

A proper submodule P of an R-module M is said to be *prime* if for any $r \in R$ and $m \in M$ with $rm \in P$, we have $m \in P$ or $r \in (P :_R M)$ [7].

Proposition 2.16. *Let M be a multiplication R-module and let N_1, N_2, and N_3 be prime submodules of M such that $N_1 + N_2 = N_1 + N_3 = N_2 + N_3 = M$. Then $N_1 \cap N_2 \cap N_3$ is not a strongly 2-irreducible submodule of M.*

Proof. Assume on the contrary that $N_1 \cap N_2 \cap N_3$ is a strongly 2-irreducible submodule of M. Then $N_1 \cap N_2 \cap N_3 \subseteq N_1 \cap N_2 \cap N_3$ implies that either $N_1 \cap N_2 \subseteq N_1 \cap N_2 \cap N_3$ or $N_1 \cap N_3 \subseteq N_1 \cap N_2 \cap N_3$ or $N_2 \cap N_3 \subseteq N_1 \cap N_2 \cap N_3$. We can assume without loss of generality that $N_1 \cap N_2 \subseteq N_1 \cap N_2 \cap N_3$. Then $N_1 \cap N_2 \subseteq N_3$. It follows that $(N_1 :_R M)N_2 \subseteq N_3$. As N_3 is a prime submodule of M, we have $N_2 \subseteq N_3$ or $(N_2 :_R M) \subseteq (N_3 :_R M)$. Thus $N_2 \subseteq N_3$ or $N_1 \subseteq N_3$ since M is a multiplication R-module. Therefore, $N_3 = M$, which is a contradiction. □

Corollary 2.17. *Let M be a multiplication R-module such that every proper submodule of M is strongly 2-irreducible. Then M has at most two maximal submodules.*

Proof. This follows from Proposition 2.16 □

Let N be a submodule of an R-module M. The intersection of all prime submodules of M containing N is said to be the (*prime*) *radical* of N and denote by $rad_M N$ (or simply by $rad(N)$). In case N does not contained in any prime submodule, the radical of N is defined to be M. Also, $N \neq M$ is said to be a *radical submodule* of M if $rad(N) = N$ [14]

Lemma 2.18. *Let I be an ideal of R and N be a submodule of an R-module M. Then $rad(IN) = rad(N) \cap rad(IM)$.*

Proof. By [13, Corollary of Theorem 6], we have $rad(N \cap IM)) = rad(N) \cap rad(IM)$. Since $IN \subseteq IM \cap N$, $rad(IN) \subseteq rad(IM \cap N)$. Thus $rad(IN) \subseteq rad(N) \cap rad(IM)$. Now let P be a prime submodule of M such that $IN \subseteq P$. As P is prime, $N \subseteq P$ or $I \subseteq (P :_R M)$. Hence $N \cap IM \subseteq P$. This in tourn implies that $rad(N) \cap rad(IM) \subseteq rad(IN)$, as desired. □

A proper ideal I of R is said to be a *2-absorbing ideal* of R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$ [5].

A proper submodule N of an R-module M is said to be a *2-absorbing primary submodule* of M if whenever $a, b \in R$, $m \in M$, and $abm \in N$, then $am \in rad(N)$ or $bm \in rad(N)$ or $ab \in (N :_R M)$ [15].

A proper submodule N of an R-module M is called a *2-absorbing submodule* of M if whenever $abm \in N$ for some $a, b \in R$ and $m \in M$, then $am \in N$ or $bm \in N$ or $ab \in (N :_R M)$ [19] and [16].

Theorem 2.19. *Let M be a finitely generated multiplication R-module and N be a radical submodule of M. Then the following conditions are equivalent:*

(a) *N is a strongly 2-irreducible submodule of M;*

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(b) *N is a 2-absorbing submodule of M;*
(c) *N is a 2-absorbing primary submodule of M;*
(d) *N is either a prime submodule of M or is an intersection of exactly two prime submodules of M.*

Proof. $(a) \Rightarrow (b)$ Let I, J be ideals of R and K be a submodule of M such that $IJK \subseteq N$. Then by using Lemma 2.18,

$$K \cap IM \cap JM \subseteq rad(K) \cap rad(IM) \cap rad(JM) = rad(IJK) \subseteq rad(N) = N$$

Hence by part (a), either $K \cap IM \subseteq N$ or $K \cap JM \subseteq N$ or $IM \cap JM \subseteq N$. Thus either $IK \subseteq N$ or $JK \subseteq N$ or $IJM \subseteq N$ as needed.

$(b) \Rightarrow (c)$ This is clear.

$(c) \Rightarrow (b)$ This is clear by using [15, Theorem 2.6].

$(b) \Rightarrow (d)$ Since N is a 2-absorbing submodule of M, $(N :_R M)$ is a 2-absorbing ideal of R by [20, Proposition 1]. Hence $\sqrt{(N :_R M)} = P$ is a prime ideal of R or $\sqrt{(N :_R M)} = P \cap Q$, where P and Q are distinct prime ideals of R that are minimal over $(N :_R M)$ by [5, Theorem 2.4]. We have $\sqrt{(N :_R M)}M = rad(N)$ by [14, Theorem 4]. If $\sqrt{(N :_R M)} = P$, then $PM = rad(N)$. Since M is a multiplication R-module, PM is a prime submodule of M by [8, Corollary 2.11]. Now let $\sqrt{Ann_R(N)} = P \cap Q$, where P and Q are distinct prime ideals of R. Then $(P \cap Q)M = rad(N)$. By [8, Corollary 1.7], $(P \cap Q)M = PM \cap QM$. Thus by [8, Corollary 2.11], $rad(N)$ is an intersection of two prime submodules of M. Now, we prove the claim by assumption that N is a radical submodule of M.

$(d) \Rightarrow (a)$ This follows from Lemma 2.15. □

The following example shows that parts (a) and (b) of Theorem 2.19 are not equivalent in general.

Example 2.20. Consider the submodule $G_t = \langle 1/p^t + \mathbb{Z} \rangle$ of the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$. Then G_t is a strongly 2-irreducible submodule of $\mathbb{Z}_{p^\infty}$. But G_t is not a 2-absorbing submodule of $\mathbb{Z}_{p^\infty}$. It should be note that the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ is not a finitely genrated multiplication $\mathbb{Z}$-module.

A submodule N of an R-module M is said to be *pure* if $IN = IM \cap N$ for every ideal I of R [1]. Also, an R-module M is said to be *fully pure* if every submodule of M is pure [3].

Theorem 2.21. *Let M be a fully pure multiplication R-module and N be a submodule of M. Then the following conditions are equivalent:*

(a) *N is a strongly 2-irreducible submodule of M;*
(b) *N is a 2-absorbing submodule of M;*
(c) *N is a 2-irreducible submodule of M.*

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Proof. $(a) \Rightarrow (b)$ Let I, J be ideals of R and K be a submodule of M such that $IJK \subseteq N$. Then since M is fully pure,

$$K \cap IM \cap JM = IJK \subseteq N.$$

Hence by part (a), either $K \cap IM \subseteq N$ or $K \cap JM \subseteq N$ or $IM \cap JM \subseteq N$. Thus either $IK \subseteq N$ or $JK \subseteq N$ or $IJM \subseteq N$.

$(b) \Rightarrow (a)$ Let $H_1 \cap H_2 \cap H_3 \subseteq N$ for submodules H_1, H_2 and H_3 of M. Then

$$(H_1 :_R M) \cap (H_2 :_R M) \cap (H_3 :_R M) = (H_1 \cap H_2 \cap H_3 :_R M) \subseteq (N :_R M).$$

Thus either $(H_1 :_R M)(H_2 :_R M) \subseteq (N :_R M)$ or $(H_1 :_R M)(H_3 :_R M) \subseteq (N :_R M)$ or $(H_2 :_R M)(H_3 :_R M) \subseteq (N :_R M)$ since $(N :_R M)$ is a 2-absorbing ideal of R by [20, Proposition 1]. We can assume without loss of generality that $(H_1 :_R M)(H_2 :_R M) \subseteq (N :_R M)$. Thus as M is fully pure, we have

$$(H_1 :_R M)M \cap (H_2 :_R M)M \subseteq (N :_R M)M \subseteq N.$$

Therefore, $H_1 \cap H_2 \subseteq N$ since M is a multiplication R-module.

$(a) \Leftrightarrow (c)$ By [3, proof of Theorem 2.19], M is a distributive R-module. Now the result follows from Proposition 2.8. $\square$

Lemma 2.22. *Let M be an R-module, S a multiplicatively closed subset of R, and N be a finitely generated submodule of M. If $S^{-1}N \subseteq S^{-1}K$ for a submodule K of M, then there exists $s \in S$ such that $sN \subseteq K$.*

Proof. This is straightforward. $\square$

Proposition 2.23. *Let M be an R-module, S be a multiplicatively closed subset of R and N be a finitely generated prime strongly 2-irreducible submodule of M such that $(N :_R M) \cap S = \emptyset$. Then $S^{-1}N$ is a strongly 2-irreducible submodule of $S^{-1}M$ if $S^{-1}N \neq S^{-1}M$.*

Proof. Let $S^{-1}H_1 \cap S^{-1}H_2 \cap S^{-1}H_3 \subseteq S^{-1}N$ for submodules $S^{-1}H_1$, $S^{-1}H_2$ and $S^{-1}H_3$ of $S^{-1}M$. Then $S^{-1}(H_1 \cap H_2 \cap H_3) \subseteq S^{-1}N$. By Lemma 2.22, there exists $s \in S$ such that $s(H_1 \cap H_2 \cap H_3) \subseteq N$. This implies that $H_1 \cap H_2 \cap H_3 \subseteq N$ since N is prime and $(N :_R M) \cap S = \emptyset$. Now as N is a strongly 2-irreducible submodule of M, we have either $H_1 \cap H_2 \subseteq N$ or $H_1 \cap H_3 \subseteq N$ or $H_2 \cap H_3 \subseteq N$. Therefore, either $S^{-1}H_1 \cap S^{-1}H_2 \subseteq S^{-1}N$ or $S^{-1}H_1 \cap S^{-1}H_3 \subseteq S^{-1}N$ or $S^{-1}H_2 \cap S^{-1}H_3 \subseteq S^{-1}N$, as needed. $\square$

Proposition 2.24. *Let M be an R-module and $\{K_i\}_{i \in I}$ be a chain of strongly 2-irreducible submodules of M. Then $\cap_{i \in I} K_i$ is a strongly 2-irreducible submodule of M.*

Proof. Let $H_1 \cap H_2 \cap H_3 \subseteq \cap_{i \in I} K_i$ for submodules H_1, H_2 and H_3 of M. Assume that $H_1 + H_2 \not\subseteq \cap_{i \in I} K_i$, $H_1 + H_3 \not\subseteq \cap_{i \in I} K_i$, and $H_2 + H_3 \not\subseteq \cap_{i \in I} K_i$.

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Then there are $m, n, t \in I$, where $H_1 \cap H_2 \not\subseteq K_m$, $H_1 \cap H_3 \not\subseteq K_n$, and $H_2 \cap H_3 \not\subseteq K_t$. Since $\{K_i\}_{i\in I}$ is a chain we can assume that $K_m \subseteq K_n \subseteq K_t$. But as $H_1 \cap H_2 \cap H_3 \subseteq K_m$ and K_m is a strongly sum 2-irreducible submodule of M, we have either $H_1 \cap H_2 \subseteq K_m$ or $H_1 \cap H_3 \subseteq K_m$ or $H_2 \cap H_3 \subseteq K_m$. In any case, we get a contradiction. □

Theorem 2.25. *Let $f : M \to \acute{M}$ be a epimorphism of R-modules. Then we have the following.*

(a) *If N is a strongly 2-irreducible submodule of M such that $ker(f) \subseteq N$, then $f(N)$ is a strongly 2-irreducible submodule of $\acute{M}$.*

(b) *If $\acute{N}$ is a strongly 2-irreducible submodule of $\acute{M}$, then $f^{-1}(\acute{N})$ is a strongly 2-irreducible submodule of M.*

Proof. (a) Let N be a strongly 2-irreducible submodule of M. If $f(N) = \acute{M}$, then we have $N + Ker(f) = f^{-1}(f(N)) = f^{-1}(\acute{M}) = f^{-1}(f(M)) = M$. Now as $ker(f) \subseteq N$, we get that $N = M$, which is a contradiction. Therefore, $f(N) \neq \acute{M}$. Suppose that $\acute{H_1} \cap \acute{H_2} \cap \acute{H_3} \subseteq f(N)$ for submodules $\acute{H_1}$, $\acute{H_2}$ and $\acute{H_3}$ of $\acute{M}$. Then $f^{-1}(\acute{H_1}) \cap f^{-1}(\acute{H_2}) \cap f^{-1}(\acute{H_3}) \subseteq f^{-1}(f(N)) = N$ since $ker(f) \subseteq N$. Thus by assumption, either $f^{-1}(\acute{H_1}) \cap f^{-1}(\acute{H_2}) \subseteq N$ or $f^{-1}(\acute{H_1}) \cap f^{-1}(\acute{H_3}) \subseteq N$ or $f^{-1}(\acute{H_2}) \cap f^{-1}(\acute{H_3}) \subseteq N$. Now as f is epimorphism, we have either $\acute{H_1} \cap \acute{H_2} \subseteq f(N)$ or $\acute{H_1} \cap \acute{H_3} \subseteq f(N)$ or $\acute{H_2} \cap \acute{H_3} \subseteq f(N)$, as needed.

(b) Let $\acute{N}$ be a strongly 2-irreducible submodule of $\acute{M}$. Since $\acute{N} \neq \acute{M}$ and f is a epimorphism, we have $f^{-1}(\acute{N}) \neq M$. Now let $H_1 \cap H_2 \cap H_3 \subseteq f^{-1}(\acute{N})$ for submodules H_1, H_2 and H_3 of M. Then $f(H_1) \cap f(H_2) \cap f(H_3) \subseteq f(f^{-1}(\acute{N})) = \acute{N}$. Thus by assumption, either $f(H_1) \cap f(H_2) \subseteq \acute{N}$ or $f(H_1) \cap f(H_3) \subseteq \acute{N}$ or $f(H_2) \cap f(H_3) \subseteq \acute{N}$. Now we have either $H_1 \cap H_2 \subseteq f^{-1}(\acute{N})$ or $H_1 \cap H_3 \subseteq f^{-1}(\acute{N})$ or $H_2 \cap H_3 \subseteq f^{-1}(\acute{N})$, as required. □

Theorem 2.26. *Let M be a finitely generated multiplication distributive R-module and let N be a non-zero proper submodule of M. Then the following statements are equivalent:*

(a) *N is a strongly 2-irreducible submodule of M;*

(b) *$(N :_R M)$ is a strongly 2-irreducible ideal of R;*

(c) *$(N :_R M)$ is a 2-irreducible ideal of R.*

Proof. $(a) \Rightarrow (b)$ This follows from Theorem 2.12.

$(b) \Rightarrow (c)$ This follows from [21, Proposition 1].

$(c) \Rightarrow (a)$ Let $H_1 \cap H_2 \cap H_3 \subseteq N$ for submodules H_1, H_2 and H_3 of M. Then as M is a distributive R-module, we have

$$N = N + (H_1 \cap H_2 \cap H_3) = (N + H_1) \cap (N \cap H_2) \cap (N \cap H_3).$$

This implies that $(N :_R M) = (N+H_1 :_R M) \cap (N+H_2 :_R M) \cap (N+H_3 :_R M)$. Thus by assumption, either$(N :_R M) = (N + H_1 :_R M) \cap (N + H_2 :_R M)$ or $(N :_R M) = (N + H_1 :_R M) \cap (N + H_3 :_R M)$ or $(N :_R M) = (N + H_2 :_R$

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$M)\cap(N+H_3:_R M)$. Therefore, by [8, Corollary 1.7], either $N = N+(H_1\cap H_2)$ or $N = N + (H_1 \cap H_3)$ or $N = N + (H_2 \cap H_3)$, since M is a finitely generated multiplication R-module. Thus either, $H_1 \cap H_2 \subseteq N$ or $H_1 \cap H_3 \subseteq N$ or $H_2 \cap H_3 \subseteq N$ as needed. □

Let R_i be a commutative ring with identity and M_i be an R_i-module, for $i = 1, 2$. Let $R = R_1 \times R_2$. Then $M = M_1 \times M_2$ is an R-module and each submodule of M is in the form of $N = N_1 \times N_2$ for some submodules N_1 of M_1 and N_2 of M_2.

Theorem 2.27. *Let $R = R_1 \times R_2$ be a decomposable ring and $M = M_1 \times M_2$ be an R-module, where M_1 is an R_1-module and M_2 is an R_2-module. Suppose that $N = N_1 \times N_2$ is a proper submodule of M. Then the following conditions are equivalent:*

(a) *N is a strongly 2-irreducible submodule of M;*
(b) *Either $N_1 = M_1$ and N_2 strongly 2-irreducible submodule of M_2 or $N_2 = M_2$ and N_1 is a strongly 2-irreducible submodule of M_1 or N_1, N_2 are strongly irreducible submodules of M_1, M_2, respectively.*

Proof. $(a) \Rightarrow (b)$. Let $N = N_1 \times N_2$ be a strongly 2-irreducible submodule of M such that $N_2 = M_2$. From our hypothesis, N is proper, so $N_1 \neq M_1$. Set $\acute{M} = M/(0 \times M_2)$. One can see that $\acute{N} = N/(0 \times M_2)$ is a strongly 2-irreducible submodule of $\acute{M}$. Also, observe that $\acute{M} \cong M_1$ and $\acute{N} \cong N_1$. Thus N_1 is a strongly 2-irreducible submodule of M_1. By a similar argument as in the previous case, N_2 is a strongly 2-irreducible submodule of M_2, where, $N_1 = M_1$. Now suppose that $N_1 \neq M_1$ and $N_2 \neq M_2$. We show that N_1 is a irreducible submodule of M_1. Suppose that $H_1\cap K_1 \subseteq N_1$ for some submodules H_1 and K_1 of M_1. Then

$$(H_1 \times M_2) \cap (M_1 \times 0) \cap (K_1 \times M_2) \subseteq (H_1 \cap K_1) \times 0 \subseteq N_1 \times N_2.$$

Thus by assumption, either $(H_1 \times M_2) \cap (M_1 \times 0) \subseteq N_1 \times N_2$ or $(H_1 \times M_2) \cap (K_1\times M_2) \subseteq N_1\times N_2$ or $(M_1\times 0)\cap(K_1\times M_2) \subseteq N_1\times N_2$. Therefore, $H_1 \subseteq N_1$ or $K_1 \subseteq N_1$ since $N_2 \neq M_2$. Thus N_1 is a strongly irreducible submodule of M_1. Similarly, we can show that N_2 is strongly irreducible submodule of M_2.

$(b) \Rightarrow (a)$. Suppose that $N = N_1 \times M_2$, where N_1 is a strongly 2-irreducible submodule of M_1. Then it is clear that N is a strongly 2-irreducible submodule of M. Now, assume that $N = N_1 \times N_2$, where N_1 and N_2 are strongly irreducible submodules of M_1 and M_2, respectively. Hence $(N_1\times M_2)\cap(M_1\times N_2) = N_1 \times N_2 = N$ is a strongly 2-irreducible submodule of M, by Lemma 2.15. □

Theorem 2.28. *Let $R = R_1 \times R_2 \times \cdots \times R_n$ $(2 \leq n < \infty)$ be a decomposable ring and $M = M_1 \times M_2 \cdots \times M_n$ be an R-module, where for every $1 \leq i \leq n$, M_i is an R_i-module, respectively. Then for a proper submodule N of M the following conditions are equivalent:*

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(a) *N is a strongly 2-irreducible submodule of M;*

(b) *Either $N = \times_{i=1}^{n} N_i$ such that for some $k \in \{1, 2, ..., n\}$, N_k is a strongly 2-irreducible submodule of M_k, and $N_i = M_i$ for every $i \in \{1, 2, ..., n\} \setminus \{k\}$ or $N = \times_{i=1}^{n} N_i$ such that for some $k, m \in \{1, 2, ..., n\}$, N_k is a strongly irreducible submodule of M_k, N_m is a strongly irreducible submodule of M_m, and $N_i = M_i$ for every $i \in \{1, 2, ..., n\} \setminus \{k, m\}$.*

Proof. We use induction on n. For $n = 2$ the result holds by Theorem 2.27. Now let $3 \leq n < \infty$ and suppose that the result is valid when $K = M_1 \times \cdots \times M_{n-1}$. We show that the result holds when $M = K \times M_n$. By Theorem 2.27, N is a strongly 2-irreducible submodule of M if and only if either $N = L \times M_n$ for some strongly 2-irreducible submodule L of K or $N = K \times L_n$ for some strongly 2-irreducible submodule L_n of M_n or $N = L \times L_n$ for some strongly irreducible submodule L of K and some strongly irreducible submodule L_n of M_n. Note that a proper submodule L of K is a strongly irreducible submodule of K if and only if $L = \times_{i=1}^{n-1} N_i$ such that for some $k \in \{1, 2, ..., n-1\}$, N_k is a strongly irreducible submodule of M_k, and $N_i = M_i$ for every $i \in \{1, 2, ..., n-1\} \setminus \{k\}$. Consequently the claim is now verified. $\square$

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