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Inverse and Reverse 2-facility Location Problems with Equality Measures on a Network

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Abstract. In this paper we consider the inverse and reverse network facility location problems with considering the equity on servers. The inverse facility location with equality measure deals with modifying the weights of vertices with minimum cost, such that the difference between the maximum and minimum weights of clients allocated to the given facilities is minimized. On the other hand, the reverse case of facility location problem with equality measure considers modifying the weights of vertices with a given budget constraint, such that the difference between the maximum and minimum weights of vertices allocated to the given facilities is reduced as much as possible. Two algorithms with time complexity $O(nlogn)$ are presented for the inverse and reverse 2-facility location problems with equality measures. Computational results show their superiority with respect to the linear programming models.

Keywords: Inverse facility location, Reverse facility location, Balanced allocation, Equality measure.

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1. Introduction

The equity location models have been interested in recent years. These facility location problems deal with to locate the facilities such that the equality in serving to the demand points is maximized. This subject has been considered by many authors. Among them Gavalec and Hudec [18] studied the balancing function model which its objective function is the maximum difference in the distance from a demand point to its farthest and nearest facility. Berman et al. [5] considered the problem of finding the location of p facilities such that the maximum weight assigned to each facility is minimized. Marin [22] considered the balanced location problem in which the difference between the maximum and minimum weights allocated to different facilities is minimized. Fathali and Zaferanieh [16] presented polynomial algorithms for balanced location models on tree networks. A trade off between effectiveness and equity has been considered by Lejeune and Prasad [21]. They presented a bi-criteria model for this problem. Landete and Marin [20] considered the problem of minimizing the differences among the weights that allocated to the facilities. Some properties to describe the behavior of the equality measures in facility location models have been presented by Barbati and Piccolo [3]. The interested reader is referred to [23, 14], two reviews of the literature on equity measurement in location theory.

The p-median and p-center problems are two important classic facility location models. These problems ask to find the location of p facilities such that respectively the sum and maximum weighted distances from clients to the closest facility is minimized. The classical location models deal with to find the optimal locations of the facilities. However, in some cases the facilities may already exist and the problem is to improve the given locations by changing some parameters. If we want to change the parameters with minimum cost such that the given locations are optimal then the problem is called inverse location problem. On the other hand, if we should change the parameters to improve the given locations as much as possible within a given budget constraint, then the problem is called reverse location problem.

Many authors have been considered the inverse and reverse location models. Cai et al. [13] showed that the inverse center problem is NP-hard. Burkard et al. [12] investigated the inverse p-median problems and presented an $O(nlogn)$ algorithm for the inverse 1-median problem on a tree and in the plane. Then Galavii [17] improved the time complexity of the inverse 1-median problem on trees to linear time. Burkard et al. [11] developed an $O(n^2)$ algorithm for the inverse 1-median problem on a cycle. The inverse 1-median problem on tree networks with variable weights and edge lengths have been considered by Guan and Zhang [19] and Wu et al. [28], respectively. Baroughi Bonab et al. [8] showed that the inverse p-median problem with variable edge lengths

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is NP-hard on general graphs. Alizadeh et al. [1] considered the inverse 1-center location problem with edge length augmentation on trees and presented an $O(nlogn)$ time algorithm. Later, Alizadeh and Burkard [2] proved that the inverse absolute and vertex 1-center problems can be solved in $O(n^2)$ time. Nazari et al. [25] considered the inverse backup 2-median problem on a tree. Recently, Omidi et al. [27] proposed an $O(n \log n)$ algorithm for solving the inverse balanced facility location problem with variable edge lengths. Fathali [15] developed an algorithm for solving the general case of inverse continuous location problems with variable weights.

The reverse 1-median and 1-center problems are known to be NP-hard [7, 9]. Berman et al. [6] considered the reverse 1-median problem on a tree and Burkard et al. [9] developed a linear time algorithm for the reverse 1-median problem on a cycle. Berman et al. [7] and Zhang et al. [29] presented polynomial time algorithms for the reverse 1-center problem. Then Nguyen [26] developed an $O(n^2)$ time algorithm for the 1-center problem on trees. Burkard et al. [10] developed polynomial time algorithms for reverse 2-median problem on trees and paths. Recently, Nazari and Fathali [24] considered the reverse backup 2-median problem on the plane.

In this paper we develop two $O(nlogn)$ algorithms for inverse and reverse 2-facility location problems with equality measures on general networks. In the inverse model we should change the weights of vertices with minimum cost such that the difference number of clients that allocated to the two given facilities is minimized. However, the reverse model investigated the modifying the weights of vertices to reduce the difference number of clients that allocated to the two given facilities such that the changing cost of vertices does not exceed a given budget.

In what follows we define the inverse and reverse equity location problems in Section 2. The models of these problems and two algorithms with $O(nlogn)$ time complexity are presented in Sections 3 and 4, respectively. Section 5 contains the computaional results of presented algorithms on some test problems.

2. Problems definition

Let $G = (V, E)$ be a graph with $|V| = n$ and $|E| = m$. The vertex $v_i \in V$ has a nonnegative weight w_i, which is the demands of clients on vertex v_i. For any pair of vertices, v_i and v_j, let $d_{ij} = d(v_i, v_j)$ be the length of a shortest path between vertices v_i and v_j in G. For any $S \subset V$, let $W(S) = \sum_{v_i \in S} w_i$. Let m_1 and m_2 be two given vertices in G which are assumed the location of facilities in the network. Let $V_1 = \{v_i \in V | d(v_i, m_1) \leq d(v_i, m_2)\}$ and $V_2 = V \setminus V_1$ be the sets of vertices that assigned to facilities in m_1 and m_2, respectively. In the inverse equity model of 2-facility location problem we want to modify the weights of vertices at minimum cost such that the difference of total weights of vertices in V_1 and V_2 is minimized. For any vertex v_i, suppose that the cost

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of increasing per unit of w_i is c_i^+ and the cost of decreasing per unit of w_i is c_i^-. Let q_i^+ and q_i^- be the amounts by which the weight w_i is increased and decreased, respectively. We suppose that q_i^+ obey the upper bounds u_i. Let

$$D^+ = \{q_1^+, q_2^+, ..., q_n^+\}, \ D^- = \{q_1^-, q_2^-, ..., q_n^-\},$$

and for $i = 1, ..., n$, let $\hat{w}_i = w_i + q_i^+ - q_i^-$. Therefore, we consider the minimizing of the following objective functions:

$$f_1(D^+, D^-) = \sum_{i=1}^{n} (c_i^+ q_i^+ + c_i^- q_i^-). \tag{2.1}$$

$$f_2(D^+, D^-) = |\sum_{v_i \in V_1} \hat{w}_i - \sum_{v_i \in V_2} \hat{w}_i|. \tag{2.2}$$

Note that the optimal value of the objective function f_2 is zero, which is that $W(V_1) = W(V_2)$. However, sometimes the limitation on budgeting we can not adjust the weights of vertices to satisfy this optimal condition. In the following we consider the problems with limited and unlimited budgeting which called reverse and inverse models, respectively.

3. The inverse model

In this section we suppose that the budget is unlimited, i.e. we consider the inverse case model. In the inverse model we want to change the weights of vertices with minimum cost such that the wights of vertices which assigned to m_1 and m_2 are balanced. Therefore, the model can be stated as follows,

$$\mathbf{P_1}: \qquad \min f_1 = \sum_{i=1}^{n} (c_i^+ q_i^+ + c_i^- q_i^-) \tag{3.1}$$

$$s.t.$$

$$|\sum_{v_i \in V_1} \hat{w}_i - \sum_{v_i \in V_2} \hat{w}_i| = 0, \tag{3.2}$$

$$\hat{w}_i = w_i + q_i^+ - q_i^-, \quad i = 1, 2, ..., n. \tag{3.3}$$

$$0 \leq q_i^+ \leq u_i, \quad i = 1, 2, ..., n. \tag{3.4}$$

$$0 \leq q_i^- \leq w_i, \quad i = 1, 2, ..., n \tag{3.5}$$

$$0 \leq \hat{w}_i, \quad i = 1, 2, ..., n. \tag{3.6}$$

By substitute constraints (3.3) to (3.2) the following model will be obtained.

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$$\mathbf{P_2}: \qquad \min f_1 = \sum_{i=1}^{n} (c_i^+ q_i^+ + c_i^- q_i^-) \tag{3.7}$$

$s.t.$

$$\sum_{v_i \in V_1} (q_i^+ - q_i^-) - \sum_{v_i \in V_2} (q_i^+ - q_i^-) = W(V_2) - W(V_1) \tag{3.8}$$

$$0 \leq q_i^+ \leq u_i, \quad i = 1, 2, ..., n. \tag{3.9}$$

$$0 \leq q_i^- \leq w_i, \quad i = 1, 2, ..., n \tag{3.10}$$

$$\tag{3.11}$$

Which is a bounded variable linear programming model with one constraint. In the following, we present an $O(nlogn)$ algorithm for this problem.

If $W(V_1) = W(V_2)$ then the servers are balanced and the weights of vertices remain unchanged. Otherwise, without loss of generality, let $W(V_1) > W(V_2)$. Note that the vertices with the same distances to m_1 and m_2 are assigned to the set with smaller weight. Then the following property can be stated.

Lemma 3.1. *To obtain a feasible solution, either the weights of vertices in V_1 should be reduced or the weights of vertices in V_2 should be augmented.*

Let

$$C = \{r_1, r_2, ..., r_{2n}\}$$

where r_i is either c_i^+ or c_i^- such that

$$r_1 \leq r_2 \leq r_3 \leq ... \leq r_{2n}.$$

To find a feasible solution with minimum cost, we start with r_1. Then r_1 may be either c_k^+ or c_k^-. Firstly, consider the case that $r_1 = c_k^+$, if $v_k \in V_2$ then we set $q_k^+ = min\{W(V_1) - W(V_2), u_k\}$. However, if $v_k \in V_1$ then we should consider r_2. In the case that $r_1 = c_k^-$, if $v_k \in V_1$ then we set $q_k^- = min\{W(V_1) - W(V_2), w_k\}$, and if $v_k \in V_2$ then we should consider r_2. With continue this method for r_2, r_3 and at most r_{2n}, we will find the optimal solution.

These ideas lead us the following algorithm.

Algorithm [IE2FLP].
Input: The weighted graph G, two vertices m_1 and m_2 of G as location of facilities and the cost of increasing and decreasing of vertices weights.
Output: The new weights of vertices $\hat{w}_i$ for balancing the weights of vertices which assigned to the facilities in m_1 and m_2.
Initialization:
Set $V_1 = \{v_i \in V | d(v_i, m_1) \leq d(v_i, m_2)\}$ and $V_2 = V \setminus V_1$.
If $W(V_1) = W(V_2)$ then **Stop**, the current weights are optimal.
If $W(V_1) > W(V_2)$ then set $\hat{V}_1 = V_1$ and $\hat{V}_2 = V_2$,

Else set $\hat{V}_1 = V_2$ and $\hat{V}_2 = V_1$.
For each vertex $v_i \in \hat{V}_1$ that $d(v_i, m_1) = d(v_i, m_2)$, move it from $\hat{V}_1$ to $\hat{V}_2$ and update $W(\hat{V}_1)$ and $W(\hat{V}_2)$.
Sort the cost of changing vertices weights, i.e. $c_1^+, ..., c_n^+$ and $c_1^-, ..., c_n^-$, in an increasing order and call them $r_1, ..., r_{2n}$.
Iteration counter $i := 0$.
(For any vertex v_i in $\hat{V}_1$ or $\hat{V}_2$, let $\hat{w}_i$ be the weight of v_i in the current iteration. Let also $\hat{W}_1$ and $\hat{W}_2$ be the sum of weights of vertices in $\hat{V}_1$ and $\hat{V}_2$, respectively.)
Set $f_1 := 0$ and for $i = 1, ..., n,\ \hat{w}_i = w_i$.
Iteration step:
While $\hat{W}_1 \neq \hat{W}_2$ **do** the following:

(1) **If** $r_i = c_k^+$ and $v_i \in \hat{V}_2$ **then** set
  (a) $q_k^+ := min\{\hat{W}_1 - \hat{W}_2, u_k\}$,
  (b) $\hat{w}_k := w_k + q_k^+$,
  (c) $\hat{W}_2 = \hat{W}_2 + q_k^-$
  (d) $f_1 := f_1 + r_i q_k^+$.
  End if
(2) **If** $r_i = c_k^-$ and $v_i \in \hat{V}_1$ **then** set
  (a) $q_k^- := min\{\hat{W}_1 - \hat{W}_2, w_k\}$,
  (b) $\hat{w}_k := w_k - q_k^-$,
  (c) $\hat{W}_1 = \hat{W}_1 - q_k^-$
  (d) $f_1 := f_1 + r_i q_k^-$.
  End if
(3) **Set** $i := i + 1$.

end while

Theorem 3.2. *The IE2FLP algorithm find an optimal solution of the inverse 2-facility location problem with equality measure.*

Proof. Since the algorithm terminates when $W(\hat{V}_1) = W(\hat{V}_2)$, then obviously the weights that obtained by the algorithm is a feasible solution for model P_1. Moreover, the algorithm starts with an infeasible solution and change the weights of vertices with minimum cost to improve the feasibility. Therefore, after finding a feasible solution the minimum costs are used. $\square$

Since the iteration step needs an $O(n)$ time and $r_1, ..., r_{2n}$ can be sorted in $O(nlogn)$ time, therefore the time complexity of the algorithm is $O(nlogn)$.

Theorem 3.3. *The inverse 2-facility location problem with equality measure can be solved in $O(nlogn)$ time.*

To illustrate the presented algorithm consider the following example.

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EXAMPLE 3.4. Consider the tree T depicted in Fig. 2, which is presented by Berman et al. [5]. The numbers next to the nodes and the links are demand weights and links lengths, respectively. The costs of increasing and decreasing the weights of vertices are given in Table 1.

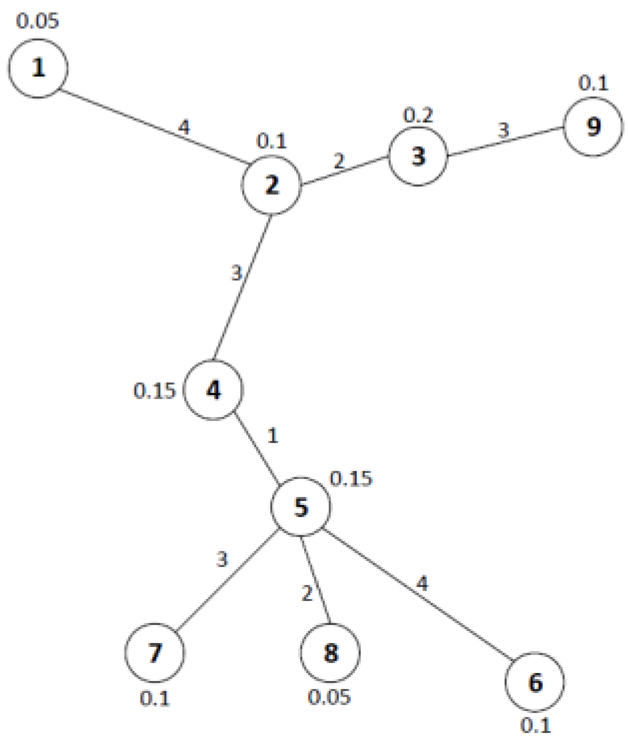

FIGURE 1. The tree T with 9 verteices.

v_i	w_i	c_i^+	c_i^-	u_i
v_1	0.05	1	0.2	0.5
v_2	0.1	0.25	0.1	0.4
v_3	0.2	0.3	1	0.2
v_4	0.15	0.7	0.6	0.3
v_5	0.15	2	0.5	0.2
v_6	0.1	1.5	0.7	0.1
v_7	0.1	0.4	1.5	0.1
v_8	0.05	1.5	1	0.2
v_9	0.1	2	2	0.1

TABLE 1. The costs of changing weights of vertices in tree T.

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Let $m_1 = v_3$ and $m_2 = v_6$. Then

$$V_1 = \{v_1, v_2, v_3, v_4, v_9\},$$
$$V_2 = \{v_5, v_6, v_7, v_8\},$$

where $W(V_1) = 0.6$ and $W(V_2) = 0.4$. By sorting the costs of changing weights of vertices, we will see,

$$\begin{aligned} r_1 = c_2^- \le r_2 = c_1^- \le r_3 = c_2^+ \le r_4 = c_3^+ \le r_5 = c_7^+ \le r_6 = c_5^- \le r_7 = c_4^- \\ \le r_8 = c_4^+ \le r_9 = c_6^- \le r_{10} = c_8^- \le r_{11} = c_3^- \le r_{12} = c_1^+ \le r_{13} = c_6^+ \\ \le r_{14} = c_7^- \le r_{15} = c_8^+ \le r_{16} = c_5^+ \le r_{17} = c_9^- \le r_{18} = c_9^+. \end{aligned}$$

Therefore, we start with $r_1 = c_2^-$ and set $q_2^- = 0.1$, $\hat{w}_2 = 0$ and $f_1 = 0.01$. After 3 iterations we will obtained $q_1^- = 0.05$, $\hat{w}_1 = 0$, $q_7^+ = 0.05$, $\hat{w}_7 = 0.15$ and $f_1 = 0.04$.

4. The reverse model

Let the budget be limited and equal to $B > 0$. In this section we consider the case of using the budget in order to change the weights of vertices such that the difference of total weights of vertices which assigned to m_1 and m_2 becomes as small as possible, i.e we consider the reverse model. The model of this problem can be written as follows.

$$\mathbf{P_3}: \qquad \min\ f_2 = |\sum_{v_i\in V_1}(w_i + q_i^+ - q_i^-) - \sum_{v_i\in V_2}(w_i + q_i^+ - q_i^-)| \tag{4.1}$$

$$s.t.$$

$$\sum_{i=1}^{n}(c_i^+ q_i^+ + c_i^- q_i^-) \le B, \tag{4.2}$$

$$0 \le q_i^+ \le u_i, \quad i = 1, 2, ..., n. \tag{4.3}$$

$$0 \le q_i^- \le w_i, \quad i = 1, 2, ..., n. \tag{4.4}$$

Let

$$y = |\sum_{v_i\in V_1}(w_i + q_i^+ - q_i^-) - \sum_{v_i\in V_2}(w_i + q_i^+ - q_i^-)|.$$

Then the model can be converted to the following linear programming problem.

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$$\mathbf{P_4}: \quad \min \quad y \tag{4.5}$$

$$s.t. \tag{4.6}$$

$$y \geq \sum_{v_i \in V_1} (w_i + q_i^+ - q_i^-) - \sum_{v_i \in V_2} (w_i + q_i^+ - q_i^-) \tag{4.7}$$

$$y \geq -\sum_{v_i \in V_1} (w_i + q_i^+ - q_i^-) + \sum_{v_i \in V_2} (w_i + q_i^+ - q_i^-) \tag{4.8}$$

$$\sum_{i=1}^{n} (c_i^+ q_i^+ + c_i^- q_i^-) \leq B, \tag{4.9}$$

$$0 \leq q_i^+ \leq u_i, \quad i = 1, 2, ..., n. \tag{4.10}$$

$$0 \leq q_i^- \leq w_i, \quad i = 1, 2, ..., n. \tag{4.11}$$

In the same as the inverse problem, we can solve the problem by an $O(nlogn)$ algorithm. The details are given in the following.

If $W(V_1) = W(V_2)$ then the solution $q_i^+ = 0$, $q_i^- = 0$ for $i = 1, ..., n$, is optimal and therefor the weights of vertices remain unchanged. Otherwise, without loss of generality, let $W(V_1) > W(V_2)$. Then to find the optimal solution, either the weights of vertices in V_1 should be decreased or the weights of vertices in V_2 should be increased. With the same notation as the Section 3, let $C = \{r_1, r_2, ..., r_{2n}\}$ be the sorted set of $c_1^+, ..., c_n^+$ and $c_1^-, ..., c_n^-$. Then, we start with r_1 which may either be c_k^+ or c_k^-. In the case $r_1 = c_k^+$, if $v_k \in V_2$ then we set

$$q_k^+ = min\{\frac{B}{c_k^+},\ W(V_1) - W(V_2),\ u_k, \}.$$

However, in this case if $v_k \in V_1$ then we consider r_2.

In the case that $r_1 = c_k^-$, if $v_k \in V_1$ then we set

$$q_k^- = min\{\frac{B}{c_k^-},\ W(V_1) - W(V_2),\ w_k\}.$$

and if $v_k \in V_2$ then we consider r_2. With continue this method at most in $2n$ iterations we will find the optimal solution.

The following algorithm can be applied to find the optimal solution of reverse 2-facility location problem with equality measure.

Algorithm [RE2FLP].

Input: The weighted graph G, two vertices m_1 and m_2 of G as location of facilities, the total budget B, and the cost of increasing and decreasing of vertices weights.

Output: The new weights of vertices $\hat{w}_i$ for minimizing the difference weights of vertices which assigned to facilities in m_1 and m_2.

Initialization:

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Set $V_1 = \{v_i \in V | d(v_i, m_1) \leq d(v_i, m_2)\}$ and $V_2 = V \setminus V_1$.
If $W(V_1) = W(V_2)$ or $B = 0$ then **Stop**, the current weights are optimal.
If $W(V_1) > W(V_2)$ then set $\hat{V}_1 = V_1$ and $\hat{V}_2 = V_2$,
Else set $\hat{V}_1 = V_2$ and $\hat{V}_2 = V_1$.
For each vertex $v_i \in \hat{V}_1$ that $d(v_i, m_1) = d(v_i, m_2)$, move it from $\hat{V}_1$ to $\hat{V}_2$ and update $W(\hat{V}_1)$ and $W(\hat{V}_2)$.
Sort the cost of changing vertices weights, i.e. $c_1^+, ..., c_n^+$ and $c_1^-, ..., c_n^-$, in an increasing order and called them $r_1, ..., r_{2n}$.
Iteration counter $i := 0$.
(For any vertex v_i in $\hat{V}_1$ or $\hat{V}_2$, let $\hat{w}_i$ be the weight of v_i in the current iteration. Let also $\hat{W}_1$ and $\hat{W}_2$ be the sum of weights of vertices in $\hat{V}_1$ and $\hat{V}_2$, respectively.)
Set $f_2 := \hat{W}_1 - \hat{W}_2$ and for $i = 1, ..., n,\ \hat{w}_i = w_i$.
Iteration step:
While the $\hat{W}_1 \neq \hat{W}_2$ and $B \neq 0$, **do** the following:

(1) **If** $r_i = c_k^+$ and $v_i \in \hat{V}_2$ **then** set
  (a) $q_k^+ := min\{\frac{B}{c_k^+},\ \hat{W}_1 - \hat{W}_2,\ u_k\}$,
  (b) $\hat{w}_k := w_k + q_k^+$,
  (c) $B = B - c_k^+ q_k^+$,
  (d) $\hat{W}_2 = \hat{W}_2 + q_k^+$,
  (e) $f_2 := f_2 - q_k^+$.
  End if
(2) **If** $r_i = c_k^-$ and $v_i \in \hat{V}_1$ **then** set
  (a) $q_k^- := min\{\frac{B}{c_k^-},\ \hat{W}_1 - \hat{W}_2,\ w_k\}$,
  (b) $\hat{w}_k := w_k - q_k^-$,
  (c) $\hat{W}_1 = \hat{W}_1 - q_k^-$,
  (d) $B = B - c_k^- q_k^-$,
  (e) $f_2 := f_2 - q_k^-$.
  End if
(3) **Set** $i := i + 1$.

end while

Theorem 4.1. *The RE2FLP algorithm find an optimal solution of the reverse 2-facility location problem with equality measure.*

Proof. In all iterations of the algorithm feasibility holds. The algorithm tries to improve the value of objective function by using minimum cost of changing the weight of vertices. It terminates in the cases that either $W(\hat{V}_1) = W(\hat{V}_2)$ or $B = 0$. If sufficient budgeting exist the algorithm reach a solution with $f_2 = 0$ (the ideal case), otherwise it reach a feasible solution with minimum value of objective function. $\square$

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The time complexity of this algorithm is the same as IE2FLP, and we can stat the following theorem.

Theorem 4.2. *The reverse 2-facility location problem with equality measure can be solved in $O(nlogn)$ time.*

EXAMPLE 4.3. Consider the network G depicted in Fig. 2. The costs of changing weights of vertices and upper bounds are given in Table 2.

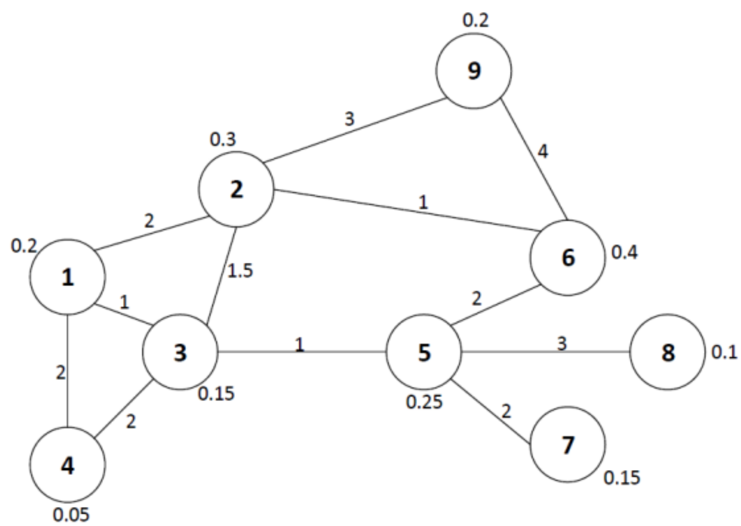

FIGURE 2. The network G with 9 vertices.

v_i	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9
c_i^+	1	1.5	1.2	1	1.3	1.1	1.2	1.3	1.2
c_i^-	1	1.5	1.2	1	1.3	1.1	1.2	1.3	1.2
u_i	0.2	0.1	0.2	0.1	0.2	0.15	0.2	0.3	0.1

TABLE 2. The upper bounds and costs of changing weights of vertices of network G.

The sorted costs are as follows.

$$\begin{aligned} r_1 = c_1^+ \le r_2 = c_1^- \le r_3 = c_4^+ \le r_4 = c_4^- \le r_5 = c_6^+ \le r_6 = c_6^- \le r_7 = c_7^+ \\ \le r_8 = c_7^- \le r_9 = c_3^+ \le r_{10} = c_3^- \le r_{11} = c_9^+ \le r_{12} = c_9^- \le r_{13} = c_5^+ \\ \le r_{14} = c_5^- \le r_{15} = c_8^+ \le r_{16} = c_8^- \le r_{17} = c_2^+ \le r_{18} = c_2^-. \end{aligned}$$

Let $B = 0.3$, $m_1 = v_2$ and $m_2 = v_5$. Then

$$V_1 = \{v_1, v_2, v_6, v_9\},$$
$$V_2 = \{v_3, v_4, v_5, v_7, v_8\},$$

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Instance	n	(m_1, m_2)	W_1	W_2	$\hat{W}_1 = \hat{W}_2$	f_1^*	Linprog CPU (in sec)	IE2FLP Iter	IE2FLP CPU (in sec)
pmed1	100	(75, 20)	407	164	232	635	0.02657	85	0.00064
		(40, 60)	519	52	90	1992	0.02020	167	0.00193
pmed2	100	(60, 80)	353	218	270	247	0.04726	45	0.00062
		(15, 75)	471	100	152	1349	0.02121	122	0.00093
pmed3	100	(5,95)	346	225	256	192	0.01972	43	0.00037
		(70,30)	332	239	267	145	0.01556	39	0.00036
pmed4	100	(70,30)	390	181	230	552	0.02355	82	0.00059
		(20,80)	371	200	238	334	0.01750	55	0.00084
pmed5	100	(10,60)	308	263	274	49	0.02026	20	0.00062
		(45,55)	377	194	240	383	0.01829	61	0.00046
pmed6	200	(50,150)	995	177	283	3324	0.02437	258	0.00464
		(70,180)	879	293	471	1824	0.02210	182	0.00387
pmed7	200	(10,190)	630	542	573	88	0.02093	20	0.00184
		(80,120)	899	273	435	2037	0.02196	195	0.00329
pmed8	200	(130,170)	1012	160	294	3608	0.03042	283	0.00439
		(50,110)	723	449	526	511	0.02209	91	0.00238
pmed9	200	(30,90)	1025	147	264	3776	0.02996	289	0.00450
		(60,160)	1129	43	94	5555	0.02400	363	0.00458
pmed10	200	(65,180)	731	441	554	616	0.02119	111	0.00248
		(30,120)	760	412	529	743	0.02232	103	0.00241

TABLE 3. The results of IE2FLP algorithm and linear programming model of P_2.

where $W(V_1) = 1.1$ and $W(V_2) = 0.7$. After 2 iterations all budget will be spends and we obtain $q_1^- = 0.2$, $q_4^+ = 0.1$, $\hat{W}_1 = 0.9$, $\hat{W}_2 = 0.8$ and $f_2 = 0.1$.

5. Computational results

In this section we examine our proposed algorithms on some test problems from ORLIB (see Beasley [4]) which were presented for the traditional p-median problem. The algorithms were written in MATLAB 2014 and run on a PC with Intel Core i7 processor, 8 GB of RAM and CPU 2.4 GHz.

The proposed algorithms were tested on 10 test problems with varying given points and the results are compared with those obtained by the linear programming models. All the costs, weights, and upper bounds are randomly generated in the interval $[1, 10]$.

Tables 3 and 4 contain the results of solving the instances using IE2FLP and RE2FLP algorithms, respectively and the linear programming models. In these tables the columns with the heading "Iter" show the number of last iteration of the algorithms for finding the optimal solution. The results indicate that both IE2FLP and RE2FLP algorithms could find the optimal solution for all instances. The obtained value of objective functions with these methods are the same as linear programming models. However, IE2FLP and RE2FLP algorithms are faster than linear programming methods.

6. Summary and conclusion

In this paper we investigated the inverse and reverse facility location problems with equality measures. The balancing on the weights of clients allocated

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									Linprog	IE2FLP	
Instance	n	B	(m_1, m_2)	W_1	W_2	$\hat{W}_1$	$\hat{W}_2$	f_1^*	CPU (in sec)	Iter	CPU (in sec)
pmed1	100	700	(75, 20)	407	164	232.00	232.00	0.00	0.01392	85	0.00480
		500	(75, 20)	407	164	239.75	206.00	147.00	0.07242	68	0.00256
		1000	(40, 60)	519	52	214.00	67.00	147.00	0.01567	136	0.00337
pmed2	100	250	(60, 80)	353	218	270.00	270.00	0.00	0.02859	45	0.00301
		200	(60, 80)	353	218	285.66	270.00	15.66	0.01458	43	0.00301
		1000	(15, 75)	471	100	197.00	136.20	60.80	0.01683	109	0.00301
pmed3	100	200	(5,95)	346	225	256.00	256.00	0.00	0.02189	43	0.00237
		100	(5,95)	346	225	286.00	244.00	42.00	0.01405	21	0.00221
		100	(70,30)	332	239	267.00	244.50	22.50	0.01408	27	0.00229
pmed4	100	600	(70,30)	390	181	230.00	230.00	0.00	0.01523	82	0.00301
		400	(70,30)	390	181	247.00	209.00	38.00	0.01629	68	0.00343
		200	(20,80)	371	200	278.66	234.00	44.66	0.01425	48	0.00238
pmed5	100	100	(10,60)	308	263	274.00	274.00	0.00	0.01569	20	0.00221
		40	(10,60)	308	263	278.00	273.00	5.00	0.01333	18	0.00219
		200	(45,55)	377	194	284.00	223.00	61.00	0.01527	47	0.00289
pmed6	200	3500	(50,150)	995	177	283.00	283.00	0.00	0.02452	258	0.00826
		2000	(50,150)	995	177	479.16	277.00	202.16	0.01696	202	0.00736
		1500	(70,180)	879	293	514.80	450.00	64.80	0.01816	167	0.00331
pmed7	200	100	(10,190)	630	542	573.00	573.00	0.00	0.01804	20	0.00222
		50	(10,190)	630	542	580.00	542.00	38.00	0.01752	13	0.00179
		1500	(80,120)	899	273	499.20	393.00	106.20	0.02270	163	0.00332
pmed8	200	4000	(130,170)	1012	160	294.00	294.00	0.00	0.01961	283	0.00435
		2000	(130,170)	1012	160	507.33	266.00	241.33	0.01912	201	0.00609
		200	(50,110)	723	449	618.00	496.00	122.00	0.01634	44	0.00239
pmed9	200	4000	(30,90)	1025	147	264.00	264.00	0.00	0.01716	289	0.00421
		2000	(30,90)	1025	147	504.00	235.00	269.00	0.04569	197	0.00392
		4000	(60,160)	1129	43	262.75	86.00	176.75	0.01614	291	0.00415
pmed10	200	700	(65,180)	731	441	554.00	554.00	0.00	0.01780	111	0.00289
		500	(65,180)	731	441	563.66	525.00	38.66	0.01819	93	0.00277
		600	(30,120)	760	412	549.66	502.00	47.66	0.01732	91	0.00272

Table 4. The results of RE2FLP algorithm and linear programming model of P_4.

to the facilities are considered as the measure of equality. The models for problems with 2 facilities are presented and $O(nlogn)$ algorithms are developed for solving these models. The results were compared with those obtained by the linear programming models. It was shown that for almost all problems the ant presented algorithms outperforms the linear programming approaches.

Other measuring functions such as maximizing the difference of distances from a client to nearest and farthest facilities, can be considered as the future works. Also, presenting polynomial time algorithms on inverse and reverse p facilities with equality measures are interesting developments of the considered models in this paper.

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