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On Fejér Type Inequalities for (η_1, η_2)-Convex Functions

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Abstract. In this paper we find a characterization type result for (η_1, η_2)-convex functions. The Fejér integral inequality related to (η_1, η_2)-convex functions is obtained as a generalization of Fejér inequality related to the preinvex and η-convex functions. Also some Fejér trapezoid and mid-point type inequalities are given in the case that the absolute value of the derivative of considered function is (η_1, η_2)-convex.

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1. Introduction and Preliminaries

L. Fejér [4], proved the following integral inequality which is known in the literature as Fejér inequality.

$$f\Big(\frac{a+b}{2}\Big)\int_a^b g(x)dx \le \int_a^b f(x)g(x)dx \le \frac{f(a)+f(b)}{2}\int_a^b g(x)dx, \qquad (1.1)$$

where $f:[a;b]\to\mathbb{R}$ is convex and $g:[a,b]\to[0,\infty)=\mathbb{R}$ is integrable and symmetric to $x=\frac{a+b}{2}\Big(g(x)=g(a+b-x), \forall x\in[a,b]\Big)$.
Recently the concept of (η_1,η_2)-convex has been introduced in [15] as a generalization of preinvex functions [1, 7, 9, 12] and η-convex functions [2, 3, 21, 20]. In the following we can find the definition of (η_1,η_2)-convex function with some basic results. For the latest results about the Fejér's inequality obtained by the authors see [17, 18].

Definition 1.1. [1, 7] A set $I\subseteq\mathbb{R}$ is said to be *invex with respect to a real bifunction* $\eta: I\times I\to\mathbb{R}$, if

$$x,y\in I, \lambda\in[0,1]\Rightarrow y+\lambda\eta(x,y)\in I.$$

Definition 1.2. [15] Let $I\subseteq\mathbb{R}$ be an invex set with respect to $\eta_1: I\times I\to\mathbb{R}$. Consider $f: I\to\mathbb{R}$ and $\eta_2: f(I)\times f(I)\to\mathbb{R}$. The function f is said to be (η_1,η_2)-*convex* if

$$f\big(x+\lambda\eta_1(y,x)\big)\le f(x)+\lambda\eta_2\big(f(y),f(x)\big)$$

for all $x,y\in I$ and $\lambda\in[0,1]$.

Remark 1.3. An (η_1,η_2)-convex function reduces to

(i) an η-convex function if we consider $\eta_1(x,y)=x-y$ for all $x,y\in I$.
(ii) a preinvex function if we consider $\eta_2(x,y)=x-y$ for all $x,y\in f(I)$.
(iii) a convex function if satisfies (i) and (ii).

Example 1.4. [15] Consider the function $f:\mathbb{R}^+\to\mathbb{R}^+$ by

$$f(x)=\begin{cases} x, & 0\le x\le 1;\\ 1, & x>1.\end{cases}$$

Define two bifunction $\eta_1:\mathbb{R}^+\times\mathbb{R}^+\longrightarrow\mathbb{R}$ and $\eta_2:\mathbb{R}^+\times\mathbb{R}^+\longrightarrow\mathbb{R}^+$ by

$$\eta_1(x,y)=\begin{cases} -y, & 0\le y\le 1;\\ x+y, & y>1,\end{cases}$$

and

$$\eta_2(x,y)=\begin{cases} x+y, & x\le y;\\ 2(x+y), & x>y.\end{cases}$$

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Then f is an (η_1, η_2)-convex function. But f is not preinvex with respect to η_1 and it is not convex (consider $x = 0$, $y = 2$ and $\lambda > 0$).

Motivated by above works and references therein we use the concept of (η_1, η_2)-convex functions to obtain Fejér inequality related to this class of functions. Also we give trapezoid and mid-point type inequalities when the absolute value of derivative of considered function is (η_1, η_2)-convex.
The following characterization of (η_1, η_2)-convex functions is of interest which we point out it before the main results.

Theorem 1.5. *Suppose that I is an invex set with respect to η_1 such that for any $x, y \in I$ with $x < y$ we have $\eta_1(y, x) > 0$ and $x < x + \eta_1(y, x) \leq y$. A function $f : I \to \mathbb{R}$ is (η_1, η_2)-convex if and only if for any $x_1, x_2, x_3 \in I$ with $x_1 < x_2 < x_1 + \eta_1(x_3, x_1)$,*

$$\det \begin{pmatrix} \eta_1(x_3, x_1) & \eta_2\big(f(x_3), f(x_1)\big) \\ \\ x_2 - x_1 & f(x_2) - f(x_1) \end{pmatrix} \leq 0, \tag{1.2}$$

and

$$f\big(x_1 + \eta_1(x_3, x_1)\big) \leq f(x_1) + \eta_2\big(f(x_3), f(x_1)\big). \tag{1.3}$$

Proof. Suppose that f is an (η_1, η_2)-convex function. Consider $x_1, x_2, x_3 \in I$ with $x_1 < x_2 < x_1 + \eta_1(x_3, x_1)$. So from the assumption there is a $t \in (0, 1)$ such that $x_2 = x_1 + t\eta_1(x_3, x_1)$, namely $t = \frac{x_2 - x_1}{\eta_1(x_3, x_1)}$. From (η_1, η_2)-convexity of f we get

$$\eta_1(x_3, x_1)[f(x_2) - f(x_1)] - (x_2 - x_1)\eta_2\big(f(x_3), f(x_1)\big) \leq 0,$$

which is equivalent to above determinant being nonpositive. Also for $t = 1$,

$$f\big(x_1 + \eta_1(x_3, x_1)\big) \leq f(x_1) + \eta_2\big(f(x_3), f(x_1)\big) \tag{1.4}$$

and for $t = 0$,

$$f(x_1) \leq f(x_1).$$

For the inverse implications, consider $x, y \in I$ with $x < y$. Choosing any $t \in (0, 1)$ we have $x < x + t\eta_1(y, x) < x + \eta_1(y, x) < y$ and so

$$\det \begin{pmatrix} \eta_1(y, x) & \eta_2\big(f(y), f(x)\big) \\ \\ t\eta_1(y, x) & f\big(x + t\eta_1(y, x)\big) - f(x) \end{pmatrix} \leq 0.$$

By expanding this determinant we reach the inequality

$$f\big(x + t\eta_1(y, x)\big) - f(x) - t\eta_2\big(f(y), f(x)\big) \leq 0,$$

for any $t \in (0, 1)$. From the assumption we have

$$f\big(x + t\eta_1(y, x)\big) \leq f(x) + \eta_2\big(f(y), f(x)\big)$$

that gives (η_1, η_2)-convexity of f for $t = 1$. Also $f(x) \leq f(x)$ gives (η_1, η_2)-convexity of f for $t = 0$. $\square$

Remark 1.6. Theorem 1.5 is a generalization of theorem 1 in [21] and generally extends corresponding results related to the preinvex and convex functions.

2. Fejér inequality

In this section we give (η_1, η_2)-convex version of Fejér inequality. We separate this inequality to the left and right type respectively.

Theorem 2.1. *Let $I \subseteq \mathbb{R}$ be an invex set with respect $\eta_1 : I \times I \to \mathbb{R}$ such that*

$$\eta_1\big(x_2 + t_2\eta_1(x_1, x_2), x_2 + t_1\eta_1(x_1, x_2)\big) = (t_2 - t_1)\eta_1(x_1, x_2) \tag{2.1}$$

for all $x_1, x_2 \in I$ and $t_1, t_2 \in [0, 1]$ (compare (2.1) with Condition C in [16]*). Also let $f : I \to \mathbb{R}$ be a (η_1, η_2)-convex function where η_2 is an integrable bifunction on $f(I) \times f(I)$. For any $a, b \in I$ with $\eta_1(b, a) > 0$, suppose that the functions $g : [a, a + \eta_1(b, a)] \to \mathbb{R}^+$ and f are integrable on $[a, a + \eta_1(b, a)]$. Then*

$$f\left(\frac{2a + \eta_1(b, a)}{2}\right) \int_a^{a+\eta_1(b,a)} g(x)dx - \frac{1}{2} \int_a^{a+\eta_1(b,a)} \eta_2\Big(f(2a + \eta_1(b, a) - x), f(x)\Big) g(x)dx$$
$$\leq \int_a^{a+\eta_1(b,a)} f(x)g(x)dx.$$

Proof. Using (2.1) and (η_1, η_2)-convexity of f we obtain the following relations:

$$f\left(\frac{2a + \eta_1(b, a)}{2}\right) = f\left(\frac{2a + \eta_1(b, a) + t\eta_1(b, a)}{2} - \frac{t}{2}\eta_1(b, a)\right) \tag{2.2}$$
$$= f\left(\frac{2a + \eta_1(b, a) + t\eta_1(b, a)}{2} + \frac{1}{2}\eta_1\left(a + \frac{(1 - t)}{2}\eta_1(b, a), a + \frac{(1 + t)}{2}\eta_1(b, a)\right)\right)$$
$$= f\left(\frac{2a + \eta_1(b, a) + t\eta_1(b, a)}{2} + \frac{1}{2}\eta_1\left(\frac{2a + \eta_1(b, a) - t\eta_1(b, a)}{2}, \frac{2a + \eta_1(b, a) + t\eta_1(b, a)}{2}\right)\right)$$
$$\leq f\left(\frac{2a + \eta_1(b, a) + t\eta_1(b, a)}{2}\right)$$
$$+ \frac{1}{2}\eta_2\left(f\left(\frac{2a + \eta_1(b, a) - t\eta_1(b, a)}{2}\right), f\left(\frac{2a + \eta_1(b, a) + t\eta_1(b, a)}{2}\right)\right),$$

and with the same argument as above we have

$$f\left(\frac{2a + \eta_1(b, a)}{2}\right) \leq f\left(\frac{2a + \eta_1(b, a) - t\eta_1(b, a)}{2}\right) \tag{2.3}$$
$$+ \frac{1}{2}\eta_2\left(f\left(\frac{2a + \eta_1(b, a) + t\eta_1(b, a)}{2}\right), f\left(\frac{2a + \eta_1(b, a) - t\eta_1(b, a)}{2}\right)\right).$$

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Now consider two changes of variable

$$x = \frac{2a + \eta_1(b,a) - t\eta_1(b,a)}{2}, \tag{2.4}$$

and

$$x = \frac{2a + \eta_1(b,a) + t\eta_1(b,a)}{2}, \tag{2.5}$$

along with (2.2) and (2.3) to obtain the following inequalities:

$$\begin{aligned}
&\frac{1}{\eta_1(b,a)} \int_a^{a+\eta_1(b,a)} f(x)g(x)dx \qquad (2.6)\\
&= \frac{1}{\eta_1(b,a)} \left[\int_a^{a+\frac{\eta_1(b,a)}{2}} f(x)g(x)dx + \int_{a+\frac{\eta_1(b,a)}{2}}^{a+\eta_1(b,a)} f(x)g(x)dx \right]\\
&= \frac{1}{2}\int_0^1 f\left(\frac{2a+\eta_1(b,a)-t\eta_1(b,a)}{2}\right) g\left(\frac{2a+\eta_1(b,a)-t\eta_1(b,a)}{2}\right) dt\\
&+ \frac{1}{2}\int_0^1 f\left(\frac{2a+\eta_1(b,a)+t\eta_1(b,a)}{2}\right) g\left(\frac{2a+\eta_1(b,a)+t\eta_1(b,a)}{2}\right) dt\\
&\geq \frac{1}{2}\int_0^1 f\left(\frac{2a+\eta_1(b,a)}{2}\right) g\left(\frac{2a+\eta_1(b,a)-t\eta_1(b,a)}{2}\right) dt\\
&- \frac{1}{2}\int_0^1 \eta_2\left(f\left(\frac{2a+\eta_1(b,a)+t\eta_1(b,a)}{2}\right), f\left(\frac{2a+\eta_1(b,a)-t\eta_1(b,a)}{2}\right)\right)\\
&\times g\left(\frac{2a+\eta_1(b,a)-t\eta_1(b,a)}{2}\right) dt\\
&+ \frac{1}{2}\int_0^1 f\left(\frac{2a+\eta_1(b,a)}{2}\right) g\left(\frac{2a+\eta_1(b,a)+t\eta_1(b,a)}{2}\right) dt\\
&- \frac{1}{2}\int_0^1 \eta_2\left(f\left(\frac{2a+\eta_1(b,a)-t\eta_1(b,a)}{2}\right), f\left(\frac{2a+\eta_1(b,a)+t\eta_1(b,a)}{2}\right)\right)\\
&\times g\left(\frac{2a+\eta_1(b,a)+t\eta_1(b,a)}{2}\right) dt.
\end{aligned}$$

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Again using the changes of variable (2.4) and (2.5) in the last relations obtained in (2.6) respectively we have

$$\begin{aligned}
&\frac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)} f(x)g(x)dx\\
&\geq \frac{1}{\eta_1(b,a)} f\left(\frac{2a+\eta_1(b,a)}{2}\right)\int_a^{a+\frac{\eta_1(b,a)}{2}} g(x)dx\\
&-\frac{1}{2\eta_1(b,a)}\int_a^{a+\frac{\eta_1(b,a)}{2}} \eta_2\Big(f\big(2a+\eta_1(b,a)-x\big), f(x)\Big)g(x)dx\\
&+\frac{1}{\eta_1(b,a)} f\left(\frac{2a+\eta_1(b,a)}{2}\right)\int_{a+\frac{\eta_1(b,a)}{2}}^{a+\eta_1(b,a)} g(x)dx\\
&-\frac{1}{2\eta_1(b,a)}\int_{a+\frac{\eta_1(b,a)}{2}}^{a+\eta_1(b,a)} \eta_2\Big(f\big(2a+\eta_1(b,a)-x\big), f(x)\Big)g(x)dx\\
&=\frac{1}{\eta_1(b,a)} f\left(\frac{2a+\eta_1(b,a)}{2}\right)\int_a^{a+\eta_1(b,a)} g(x)dx\\
&-\frac{1}{2\eta_1(b,a)}\int_a^{a+\eta_1(b,a)} \eta_2\Big(f\big(2a+\eta_1(b,a)-x\big), f(x)\Big)g(x)dx.
\end{aligned}$$

Thus we arrive at the desired result. □

If in Theorem 2.1 we consider $\eta_2(x,y)=x-y$ for all $x,y\in f(I)$, then we have the following result (see [14] and references therein):

Corollary 2.2. *Let $I\subseteq\mathbb{R}$ be an invex set with respect to $\eta_1: I\times I\to\mathbb{R}$ satisfying (2.1) and $f: I\to\mathbb{R}$ be a preinvex function. For any $a,b\in I$ with $\eta_1(b,a)>0$, suppose that $g:[a,a+\eta_1(b,a)]\to\mathbb{R}^+$ is integrable and symmetric to $a+\frac{1}{2}\eta_1(b,a)$ and $f\in L^1[a,a+\eta_1(b,a)]$. Then*

$$f\left(\frac{2a+\eta_1(b,a)}{2}\right)\int_a^{a+\eta_1(b,a)} g(x)dx \leq \int_a^{a+\eta_1(b,a)} f(x)g(x)dx.$$

Also Theorem 2.1 gives a generalization of Theorem 7 in [21]. Furthermore if we consider $\eta_1(x,y)=x-y$ for all $x,y\in I$, then we recapture the left side of (1.1).

Theorem 2.3. *Let $I\subseteq\mathbb{R}$ be an invex set with respect to $\eta_1: I\times I\to\mathbb{R}$ and $f: I\to\mathbb{R}$ be a (η_1,η_2)-convex function where η_2 is an integrable bifunction on $f(I)\times f(I)$. For any $a,b\in I$ with $\eta_1(b,a)>0$ suppose that $f\in L^1[a,a+\eta_1(b,a)]$ and $g:[a,a+\eta_1(b,a)]\to\mathbb{R}^+$ is integrable and symmetric with respect to $a+\frac{1}{2}\eta_1(b,a)$. Then*

$$\int_a^{a+\eta_1(b,a)} f(x)g(x)dx \leq \left[f(a)+\frac{\eta_2\big(f(b),f(a)\big)}{2}\right]\int_a^{a+\eta_1(b,a)} g(x)dx. \quad (2.7)$$

Proof. From (η_1, η_2)-convexity of f, using the changes of variable $x = a + t\eta_1(b,a)$ and $x = a + (1-t)\eta_1(a,b)$ we can obtain two following inequalities.

$$\int_a^{a+\eta_1(b,a)} f(x)g(x)dx \le \eta_1(b,a) \int_0^1 \left[f(a) + t\eta_2\big(f(b), f(a)\big)\right] g\big(a + t\eta_1(b,a)\big)dt \tag{2.8}$$

$$= \eta_1(b,a) \left[\int_0^1 f(a) g\big(a + t\eta_1(b,a)\big)dt + \eta_2\big(f(b), f(a)\big) \int_0^1 tg\big(a + t\eta_1(b,a)\big)dt\right],$$

and

$$\begin{aligned} &\int_a^{a+\eta_1(b,a)} f(x)g(x)dx \qquad (2.9)\\ &\le \eta_1(b,a) \int_0^1 \left[f(a) + (1-t)\eta_2\big(f(b), f(a)\big)\right] g\big(a + (1-t)\eta_1(b,a)\big)dt \\ &= \eta_1(b,a) \Big[\int_0^1 f(a) g\big(a + (1-t)\eta_1(b,a)\big)dt \\ &+ \eta_2\big(f(b), f(a)\big) \int_0^1 (1-t) g\big(a + (1-t)\eta_1(b,a)\big)dt \Big]. \end{aligned}$$

Now adding (2.8) to (2.9) along with the fact that g is symmetric with respect to $a + \frac{1}{2}\eta_1(b,a)$ imply that

$$\begin{aligned} &2\int_a^{a+\eta_1(b,a)} f(x)g(x)dx \\ &\le \eta_1(b,a) \left[2f(a) \int_0^1 g\big(a + t\eta_1(b,a)\big)dt + \eta_2\big(f(b), f(a)\big) \int_0^1 g\big(a + t\eta_1(b,a)\big)dt\right]. \end{aligned}$$

So again by the use of the change of variable $x = a+t\eta_1(b,a)$ in above inequality we deduce the respected result.

$$\int_a^{a+\eta_1(b,a)} f(x)g(x)dx \le \left[f(a) + \frac{\eta_2\big(f(b), f(a)\big)}{2}\right] \int_a^{a+\eta_1(b,a)} g(x)dx.$$

□

If in Theorem 2.3 we consider $\eta_2(x,y) = x - y$ for all $x, y \in f(I)$, then we have the following result (see [14] and references therein).

Corollary 2.4. *Let $I \subseteq \mathbb{R}$ be an invex set with respect to $\eta_1 : I \times I \to \mathbb{R}$ and let $f : I \to \mathbb{R}$ be a preinvex function. For any $a, b \in I$ with $\eta_1(b,a) > 0$ suppose that $f \in L^1[a, a + \eta_1(b,a)]$ and $g : [a, a + \eta_1(b,a)] \to \mathbb{R}^+$ is integrable and symmetric with respect to $a + \frac{1}{2}\eta_1(b,a)$. Then*

$$\int_a^{a+\eta_1(b,a)} f(x)g(x)dx \le \frac{f(a) + f(b)}{2} \int_a^{a+\eta_1(b,a)} g(x)dx. \tag{2.10}$$

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Also Theorem 2.3 gives a generalization of Theorem 6 in [21]. Furthermore if we consider $\eta_1(x, y) = x - y$ for all $x, y \in I$, then we recapture the right side of (1.1). For some recent works about Fejér's inequality see [6, 10, 11, 12] and references therein.

3. Mid-Point Type Inequalities

In this section we consider the problem of estimating the difference between the middle and left terms of (1.1), when the absolute value of derivative of considered function is (η_1, η_2)-convex.

The following lemma is needed to achieve Fejér mid-point type inequalities. By the Fejér mid-point type inequalities we mean the estimation of the difference between the middle and left terms of (1.1).

Lemma 3.1. [13] Let $I^\circ \subseteq \mathbb{R}$ be an invex set with respect to $\eta_1 : I^\circ \times I^\circ \to \mathbb{R}$ and $a, b \in I^\circ$ with $a < a + \eta_1(b, a)$. Suppose that $f : I^\circ \to \mathbb{R}$ is a differentiable mapping on I° such that $f' \in L^1[a, a + \eta_1(b, a)]$. If $g : [a, a + \eta_1(b, a)] \to \mathbb{R}^+$ is an integrable mapping, then

$$\frac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)} f(x)g(x)dx - \frac{1}{\eta_1(b,a)} f\big(a + \frac{1}{2}\eta_1(b,a)\big)\int_a^{a+\eta_1(b,a)} g(x)dx$$
$$= \eta_1(b,a)\int_0^1 k(t) f'\big(a + t\eta_1(b,a)\big)dt$$

where

$$k(t) = \begin{cases} \displaystyle\int_0^t g\big(a + s\eta_1(b,a)\big)ds & t \in [0, \frac{1}{2}); \\[2ex] \displaystyle -\int_t^1 g\big(a + s\eta_1(b,a)\big)ds & t \in [\frac{1}{2}, 1]. \end{cases}$$

By the use of Lemma 3.1, we obtain the following result which is a Fejér mid-point type inequality.

Theorem 3.2. *Let $I^\circ \subseteq \mathbb{R}$ be an invex set with respect to $\eta_1 : I^\circ \times I^\circ \to \mathbb{R}$ and $a, b \in I^\circ$ with $a < a + \eta_1(b, a)$. Suppose that $f : I^\circ \to \mathbb{R}$ is a differentiable mapping on I° such that $f' \in L^1[a, a + \eta_1(b, a)]$, the function $g : [a, a + \eta_1(b, a)] \to \mathbb{R}^+$ is integrable and symmetric with respect to $a + \frac{1}{2}\eta_1(b, a)$. If $|f'|$ is (η_1, η_2)-convex on I°, then:*

$$\left|\frac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)} f(x)g(x)dx - \frac{1}{\eta_1(b,a)} f\big(a + \frac{1}{2}\eta_1(b,a)\big)\int_a^{a+\eta_1(b,a)} g(x)dx\right|$$
$$\leq \frac{2|f'(a)| + \eta_2(|f'(b)|, |f'(a)|)}{2\eta_1(b,a)}\int_a^{a+\frac{1}{2}\eta_1(b,a)} \big[\eta_1(b,a) - 2(x-a)\big]g(x)dx.$$

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Proof. From lemma (3.1) and (η_1, η_2)-convexity of $|f'|$ on I°, we have

$$J_1 = \left| \frac{1}{\eta_1(b,a)} \int_a^{a+\eta_1(b,a)} f(x)g(x)dx - \frac{1}{\eta_1(b,a)} f\big(a + \frac{1}{2}\eta_1(b,a)\big) \int_a^{a+\eta_1(b,a)} g(x)dx \right|$$
$$= \Bigg| \eta_1(b,a) \Big[\int_0^{\frac{1}{2}} \left(\int_0^t g\big(a + s\eta_1(b,a)\big)ds \right) f'\big(a + t\eta_1(b,a)\big)dt$$
$$+ \int_{\frac{1}{2}}^1 \left(- \int_t^1 g\big(a + s\eta_1(b,a)\big)ds \right) f'\big(a + t\eta_1(b,a)\big)dt \Big] \Bigg|$$
$$\leq \eta_1(b,a) \int_0^{\frac{1}{2}} \left(\int_0^t g\big(a + s\eta_1(b,a)\big)ds \right) [|f'(a)| + t\eta_2(|f'(b)|, |f'(a)|)]\, dt \quad (3.1)$$
$$+ \eta_1(b,a) \int_{\frac{1}{2}}^1 \left(\int_t^1 g\big(a + s\eta_1(b,a)\big)ds \right) [|f'(a)| + t\eta_2(|f'(b)|, |f'(a)|)]\, dt. \quad (3.2)$$

Changing the order of integration in (3.1) we get

$$\int_0^{\frac{1}{2}} \int_0^t g\big(a + s\eta_1(b,a)\big) \big[|f'(a)| + t\eta_2(|f'(b)|, |f'(a)|)\big] dsdt \quad (3.3)$$
$$= \int_0^{\frac{1}{2}} \int_s^{\frac{1}{2}} g\big(a + s\eta_1(b,a)\big) \big[|f'(a)| + t\eta_2(|f'(b)|, |f'(a)|)\big] dtds$$
$$= \int_0^{\frac{1}{2}} g\big(a + s\eta_1(b,a)\big) \Big[(\frac{1}{2} - s)|f'(a)| + (\frac{1}{8} - \frac{s^2}{2})\eta_2(|f'(b)|, |f'(a)|) \Big] ds.$$

Using the change of variable $x = a + s\eta_1(b,a)$ for $s \in [0,1]$ in (3.3) we obtain

$$\int_0^{\frac{1}{2}} \int_0^t g\big(a + s\eta_1(b,a)\big) \big[|f'(a)| + t\eta_2(|f'(b)|, |f'(a)|)\big] dsdt \quad (3.4)$$
$$= \frac{|f'(a)|}{\eta_1(b,a)} \int_a^{a+\frac{1}{2}\eta_1(b,a)} \Big(\frac{1}{2} - \frac{x-a}{\eta_1(b,a)} \Big) g(x)dx$$
$$+ \frac{\eta_2(|f'(b)|, |f'(a)|)}{\eta_1(b,a)} \int_a^{a+\frac{1}{2}\eta_1(b,a)} \left(\frac{1}{8} - \frac{1}{2}\big(\frac{x-a}{\eta_1(b,a)}\big)^2 \right) g(x)dx.$$

Similarly by the change of order of integration in (3.2) and using the fact that g is symmetric to $a + \frac{1}{2}\eta_1(b,a)$ we have

$$\int_{\frac{1}{2}}^1 \int_t^1 g\big(a + s\eta_1(b,a)\big) \big[|f'(a)| + t\eta_2(|f'(b)|, |f'(a)|)\big] dsdt \quad (3.5)$$
$$= \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^s g\big(a + s\eta_1(b,a)\big) \big[|f'(a)| + t\eta_2(|f'(b)|, |f'(a)|)\big] dtds$$
$$= \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^s g\big(a + (1-s)\eta_1(b,a)\big) \big[|f'(a)| + t\eta_2(|f'(b)|, |f'(a)|)\big] dtds$$
$$= \int_{\frac{1}{2}}^1 g\big(a + (1-s)\eta_1(b,a)\big) \big[(s - \frac{1}{2})|f'(a)| + (\frac{s^2}{2} - \frac{1}{8})\eta_2(|f'(b)|, |f'(a)|) \big] ds.$$

By the change of variable $x = a + (1-s)\eta_1(b,a)$ it follows from (3.5) that

$$\begin{aligned}
&\int_{\frac{1}{2}}^{1}\int_{t}^{1} g\big(a + s\eta_1(b,a)\big)\big[|f'(a)| + t\eta_2(|f'(b)|,|f'(a)|)\big]dsdt \qquad (3.6)\\
&= \frac{|f'(a)|}{\eta_1(b,a)}\int_a^{a+\frac{1}{2}\eta_1(b,a)}\left(1 - \frac{x-a}{\eta_1(b,a)} - \frac{1}{2}\right)g(x)dx\\
&+ \frac{\eta_2(|f'(b)|,|f'(a)|)}{\eta_1(b,a)}\int_a^{a+\frac{1}{2}\eta_1(b,a)}\Big[\frac{1}{2}\Big(1 - \frac{x-a}{\eta_1(b,a)}\Big)^2 - \frac{1}{8}\Big]g(x)dx.
\end{aligned}$$

Substituting (3.4) and (3.6) in (3.1) and (3.2) respectively and then simplifying, we get

$$J_1 \le \frac{2|f'(a)| + \eta_2(|f'(b)|,|f'(a)|)}{2\eta_1(b,a)}\int_a^{a+\frac{1}{2}\eta_1(b,a)}\big[\eta_1(b,a) - 2(x-a)\big]g(x)dx.$$

□

Corollary 3.3 (Theorem 2.3 in [22]). *Let $f : I^\circ \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$ and $g : [a,b] \to \mathbb{R}^+$ be a differentiable mapping and symmetric with respect to $\frac{a+b}{2}$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}f\Big(\frac{a+b}{2}\Big)\int_a^b g(x)dx\right|\\
&\le \frac{|f'(a)| + |f'(b)|}{2(b-a)}\int_a^{\frac{a+b}{2}}\big[a + b - 2x\big]g(x)dx.
\end{aligned}$$

Also Theorem 3.2 is a generalization of Theorem 3.3 in [19] and Theorem 2 in [13].

By the use of Hölder's inequality we can obtain another form of Fejér mid-point type inequality.

Theorem 3.4. *Let $I^\circ \subseteq \mathbb{R}$ be an invex set with respect to $\eta_1 : I^\circ \times I^\circ \to \mathbb{R}$ and $a, b \in I^\circ$ with $a < a + \eta_1(b,a)$. Suppose that $f : I^\circ \to \mathbb{R}$ is a differentiable mapping on I° such that $f' \in L^1([a, a + \eta_1(b,a)])$, the function $g : [a, a + \eta_1(b,a)] \to \mathbb{R}^+$ is integrable and symmetric with respect to $a + \frac{1}{2}\eta_1(b,a)$. If $|f'|^q$, $q > 1$ is (η_1, η_2)-convex on I°, then we have the following inequality.*

$$\left|\frac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)} f(x)g(x)dx - \frac{1}{\eta_1(b,a)}f\Big(a + \frac{1}{2}\eta_1(b,a)\Big)\int_a^{a+\eta_1(b,a)} g(x)dx\right|$$

$$\le \big(\eta_1(b,a)\big)^{\frac{p-2}{p}}\Big(\frac{1}{24}\Big)^{\frac{1}{q}}M\left(\int_a^{a+\frac{1}{2}\eta_1(b,a)}\Big[\frac{\eta_1(b,a)}{2} - (x-a)\Big]g^p(x)dx\right)^{\frac{1}{p}},$$

where $M = \Big(3|f'(a)|^q + 2\eta_2\big(|f'(b)|^q,|f'(a)|^q\big)\Big)^{\frac{1}{q}} + \Big(3|f'(a)|^q + 2\eta_2\big(|f'(b)|^q,|f'(a)|^q\big)\Big)^{\frac{1}{q}}$ *and* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. According to the proof of Theorem 3.2, by the use of lemma (3.1) and changing the order of integration we get

$$\left|\frac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)} f(x)g(x)dx - \frac{1}{\eta_1(b,a)} f\big(a+\frac{1}{2}\eta_1(b,a)\big)\int_a^{a+\eta_1(b,a)} g(x)dx\right|$$
$$\leq \eta_1(b,a)\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}} g\big(a+s\eta_1(b,a)\big)\Big|f'\big(a+t\eta_1(b,a)\big)\Big|dtds \tag{3.7}$$
$$+\,\eta_1(b,a)\int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^s g\big(a+s\eta_1(b,a)\big)\Big|f'\big(a+t\eta_1(b,a)\big)\Big|dtds. \tag{3.8}$$

If we apply Hölder's inequality in (3.7), then

$$\eta_1(b,a)\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}} g\big(a+s\eta_1(b,a)\big)\Big|f'\big(a+t\eta_1(b,a)\big)\Big|dtds$$
$$\leq \eta_1(b,a)\left(\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}} g^p\big(a+s\eta_1(b,a)\big)dtds\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}}\Big|f'\big(a+t\eta_1(b,a)\big)\Big|^q dtds\right)^{\frac{1}{q}}.$$

Since $|f'|^q$ is (η_1,η_2)-convex on I°, for any $a, b \in I^\circ$ and $t \in [0,1]$ we have

$$\Big|f'\big(a+t\eta_1(b,a)\big)\Big|^q \leq |f'(a)|^q + t\eta_2\big(|f'(b)|^q, |f'(a)|^q\big),$$

which by the use of substitution $x = a + s\eta_1(b,a)$, we deduce that

$$\eta_1(b,a)\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}} g\big(a+s\eta_1(b,a)\big)\Big|f'\big(a+t\eta_1(b,a)\big)\Big|dtds \tag{3.9}$$
$$\leq \eta_1(b,a)\left(\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}} g^p\big(a+s\eta_1(b,a)\big)dtds\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}} |f'(a)|^q + t\eta_2\big(|f'(b)|^q, |f'(a)|^q\big)dtds\right)^{\frac{1}{q}}$$
$$= \eta_1(b,a)\left(\frac{1}{(\eta_1(b,a))^2}\int_a^{a+\frac{1}{2}\eta_1(b,a)}\Big[\frac{\eta_1(b,a)}{2} - (x-a)\Big]g^p(x)dx\right)^{\frac{1}{p}}$$
$$\times\left(\frac{3|f'(a)|^q + 2\eta_2\big(|f'(b)|^q, |f'(a)|^q\big)}{24}\right)^{\frac{1}{q}}.$$

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Also from (3.8) and the fact that g is symmetric with respect to $a+\frac{1}{2}\eta_1(b,a)$ we obtain

$$\eta_1(b,a)\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{s} g\big(a+s\eta_1(b,a)\big)\Big|f'\big(a+t\eta_1(b,a)\big)\Big|dtds \tag{3.10}$$
$$\leq \eta_1(b,a)\left(\frac{1}{(\eta_1(b,a))^2}\int_a^{a+\frac{1}{2}\eta_1(b,a)}\Big[\frac{\eta_1(b,a)}{2}-(x-a)\Big]g^p(x)dx\right)^{\frac{1}{p}}$$
$$\times\left(\frac{3|f'(a)|^q+2\eta_2\big(|f'(b)|^q,|f'(a)|^q\big)}{24}\right)^{\frac{1}{q}}.$$

Using (3.9) and (3.10) in (3.7) and (3.8) respectively we get the required inequality. □

Corollary 3.5 (Theorem 2.5 in [22])**.** *Let $f: I^\circ\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on $I^\circ, a, b\in I^\circ$ with $a<b$, and $g:[a,b]\to\mathbb{R}^+$ be a differentiable mapping and symmetric with respect to $\frac{a+b}{2}$. If $|f'|^q$ is convex on $[a,b], q>1$, then the following inequality holds:*

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx-\frac{1}{b-a}f\Big(\frac{a+b}{2}\Big)\int_a^b g(x)dx\right|$$
$$\leq (b-a)\left(\frac{1}{(b-a)^2}\int_a^{\frac{a+b}{2}}\Big[\frac{a+b}{2}-x\Big]g^p(x)dx\right)^{\frac{1}{p}}$$
$$\times\left(\frac{|f'(a)|^q+2|f'(b)|^q}{24}\right)^{\frac{1}{q}}+\left(\frac{2|f'(a)|^q+|f'(b)|^q}{24}\right)^{\frac{1}{q}},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Also Theorem 3.4 gives a generalized form of Theorem 3 in [13].

4. Trapezoid type Inequalities

In this section we consider the problem of estimating the difference between the middle and right terms of (1.1), when the absolute value of derivative of considered function is (η_1,η_2)-convex. First we prove the following lemma.

Lemma 4.1. *Suppose that $I^\circ\subseteq\mathbb{R}$ is an invex set with respect to $\eta_1: I^\circ\times I^\circ\to\mathbb{R}$ and consider $a,b\in I^\circ$ with $\eta_1(b,a)>0$. Suppose that $f: I^\circ\to\mathbb{R}$ is a differentiable mapping on I° such that $f'\in L^1([a,a+\eta_1(b,a)])$. If $g: [a,a+\eta_1(b,a)]\to\mathbb{R}^+$ is an integrable mapping, then*

$$\frac{f(a)+f\big(a+\eta_1(b,a)\big)}{2}\int_a^{a+\eta_1(b,a)}g(x)dx-\int_a^{a+\eta_1(b,a)}f(x)g(x)dx$$
$$=\frac{1}{2}\int_a^{a+\eta_1(b,a)}\int_a^{x}g(u)f'(x)dudx-\frac{1}{2}\int_a^{a+\eta_1(b,a)}\int_x^{a+\eta_1(b,a)}g(u)f'(x)dudx.$$

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Proof. By Leibniz integral rule and integration by parts we have

$$\int_a^{a+\eta_1(b,a)} f(x)g(x)dx = \int_a^{a+\eta_1(b,a)} f(x)\left(\int_a^x g(u)du\right)' dx \tag{4.1}$$
$$= f\big(a+\eta_1(b,a)\big)\int_a^{a+\eta_1(b,a)} g(u)du - \int_a^{a+\eta_1(b,a)}\int_a^x g(u)f'(x)dudx.$$

with the same argument

$$\int_a^{a+\eta_1(b,a)} f(x)g(x)dx = \int_a^{a+\eta_1(b,a)} f(x)\left(-\int_x^{a+\eta_1(b,a)} g(u)du\right)' dx \tag{4.2}$$
$$= f(a)\int_a^{a+\eta_1(b,a)} g(u)du + \int_a^{a+\eta_1(b,a)}\int_x^{a+\eta_1(b,a)} g(u)f'(x)dudx.$$

Now it is enough to add relation (4.1) to (4.2). □

The following lemma is needed to obtain Fejér trapezoid type inequalities. By the Fejér trapezoidal type inequality we mean the estimation of the difference between the middle and right terms of (1.1).

Lemma 4.2. *Suppose that $I^\circ \subseteq \mathbb{R}$ is an invex set with respect to $\eta_1 : I^\circ \times I^\circ \to \mathbb{R}$ and consider $a, b \in I^\circ$ with $\eta_1(b,a) > 0$. Suppose that $f : I^\circ \to \mathbb{R}$ is a differentiable mapping on I° such that $f' \in L^1([a, a+\eta_1(b,a)])$. If $g : [a, a+\eta_1(b,a)] \to \mathbb{R}^+$ is an integrable mapping and symmetric with respect to $a + \frac{1}{2}\eta_1(b,a)$, then*

$$\frac{f(a)+f\big(a+\eta_1(b,a)\big)}{2}\int_a^{a+\eta_1(b,a)} g(x)dx - \int_a^{a+\eta_1(b,a)} f(x)g(x)dx$$
$$= \frac{\eta_1(b,a)}{4}\Bigg\{\int_0^1\left(\int_{a+(\frac{1-t}{2})\eta_1(b,a)}^{a+(\frac{1+t}{2})\eta_1(b,a)} g(u)du\right)f'\Big(a+(\frac{1-t}{2})\eta_1(b,a)\Big)dt$$
$$+\int_0^1\left(\int_{a+(\frac{1-t}{2})\eta_1(b,a)}^{a+(\frac{1+t}{2})\eta_1(b,a)} g(u)du\right)f'\Big(a+(\frac{1+t}{2})\eta_1(b,a)\Big)dt\Bigg\}.$$

Proof. From Lemma (4.1) we obtain that

$$J_2 = \frac{f(a)+f\big(a+\eta_1(b,a)\big)}{2}\int_a^{a+\eta_1(b,a)} g(x)dx - \int_a^{a+\eta_1(b,a)} f(x)g(x)dx \tag{4.3}$$
$$= \frac{1}{2}\Bigg\{\int_a^{a+\frac{1}{2}\eta_1(b,a)}\int_a^x g(u)f'(x)dudx + \int_{a+\frac{1}{2}\eta_1(b,a)}^{a+\eta_1(b,a)}\int_a^x g(u)f'(x)dudx$$
$$-\int_a^{a+\frac{1}{2}\eta_1(b,a)}\int_x^{a+\eta_1(b,a)} g(u)f'(x)dudx - \int_{a+\frac{1}{2}\eta_1(b,a)}^{a+\eta_1(b,a)}\int_x^{a+\eta_1(b,a)} g(u)f'(x)dudx\Bigg\}.$$

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By applying the changes of variable $x = a + (\frac{1-t}{2})\eta_1(b,a)$ and $x = a + (\frac{1+t}{2})\eta_1(b,a)$ in (4.3) respectively we have

$$J_2 = \frac{\eta_1(b,a)}{4}\Bigg\{ \int_0^1 \int_a^{a+(\frac{1-t}{2})\eta_1(b,a)} g(u) f'\Big(a + (\frac{1-t}{2})\eta_1(b,a)\Big) du dt \tag{4.4}$$

$$+ \int_0^1 \int_a^{a+(\frac{1+t}{2})\eta_1(b,a)} g(u) f'\Big(a + (\frac{1+t}{2})\eta_1(b,a)\Big) du dt \tag{4.5}$$

$$- \int_0^1 \int_{a+(\frac{1-t}{2})\eta_1(b,a)}^{a+\eta_1(b,a)} g(u) f'\Big(a + (\frac{1-t}{2})\eta_1(b,a)\Big) du dt \tag{4.6}$$

$$- \int_0^1 \int_{a+(\frac{1+t}{2})\eta_1(b,a)}^{a+\eta_1(b,a)} g(u) f'\Big(a + (\frac{1+t}{2})\eta_1(b,a)\Big) du dt \Bigg\}. \tag{4.7}$$

Now if we consider (4.4) with (4.6) and consider (4.5) with (4.7) together, then

$$J_2 = \frac{\eta_1(b,a)}{4}\Bigg\{ \int_0^1 \Big[2\int_a^{a+(\frac{1-t}{2})\eta_1(b,a)} g(u)du - \int_a^{a+\eta_1(b,a)} g(u)du\Big] f'\Big(a + (\frac{1-t}{2})\eta_1(b,a)\Big) dt \tag{4.8}$$

$$+ \int_0^1 \Big[2\int_a^{a+(\frac{1+t}{2})\eta_1(b,a)} g(u)du - \int_a^{a+\eta_1(b,a)} g(u)du\Big] f'\Big(a + (\frac{1+t}{2})\eta_1(b,a)\Big) dt \Bigg\}.$$

Since g is symmetric to $a + \frac{1}{2}\eta_1(b,a)$ then

$$2\int_a^{a+(\frac{1-t}{2})\eta_1(b,a)} g(u)du - \int_a^{a+\eta_1(b,a)} g(u)du = \int_{a+(\frac{1-t}{2})\eta_1(b,a)}^{a+(\frac{1+t}{2})\eta_1(b,a)} g(u)du, \tag{4.9}$$

and

$$2\int_a^{a+(\frac{1+t}{2})\eta_1(b,a)} g(u)du - \int_a^{a+\eta_1(b,a)} g(u)du = \int_{a+(\frac{1-t}{2})\eta_1(b,a)}^{a+(\frac{1+t}{2})\eta_1(b,a)} g(u)du. \tag{4.10}$$

Implying (4.9) and (4.10) in (4.8) respectively we have

$$J_2 = \frac{\eta_1(b,a)}{4}\Bigg\{ \int_0^1 \Big(\int_{a+(\frac{1-t}{2})\eta_1(b,a)}^{a+(\frac{1+t}{2})\eta_1(b,a)} g(u)du\Big) f'\Big(a + (\frac{1-t}{2})\eta_1(b,a)\Big) dt$$

$$+ \int_0^1 \Big(\int_{a+(\frac{1-t}{2})\eta_1(b,a)}^{a+(\frac{1+t}{2})\eta_1(b,a)} g(u)du\Big) f'\Big(a + (\frac{1+t}{2})\eta_1(b,a)\Big) dt \Bigg\}.$$

$\square$

Using Lemma 4.2, the following Fejér trapezoid type inequality holds.

Theorem 4.3. *Suppose that $I^\circ \subseteq \mathbb{R}$ is an invex set with respect to $\eta_1 : I^\circ \times I^\circ \to \mathbb{R}$ and consider $a, b \in I^\circ$ with $\eta_1(b,a) > 0$. Suppose that $f : I^\circ \to \mathbb{R}$ is a differentiable mapping on I° such that $f' \in L^1([a, a + \eta_1(b,a)])$. If g :*

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$[a, a+\eta_1(b,a)] \to \mathbb{R}^+$ is an integrable mapping and symmetric with respect to $a + \frac{1}{2}\eta_1(b,a)$ and if $|f'|$ is (η_1, η_2)-convex on I°, then

$$\left|\frac{f(a)+f\big(a+\eta_1(b,a)\big)}{2}\int_a^{a+\eta_1(b,a)} g(x)dx - \int_a^{a+\eta_1(b,a)} f(x)g(x)dx\right|$$
$$\leq \frac{\eta_1(b,a)}{4}\Big[2|f'(a)| + \eta_2(|f'(b)|, |f'(a)|)\Big]\int_0^1 \int_{a+(\frac{1-t}{2})\eta_1(b,a)}^{a+(\frac{1+t}{2})\eta_1(b,a)} g(u)dudt.$$

Proof. From Lemma (4.2) and (η_1, η_2)-convexity of $|f'|$ on I°, we have

$$\left|\frac{f(a)+f\big(a+\eta_1(b,a)\big)}{2}\int_a^{a+\eta_1(b,a)} g(x)dx - \int_a^{a+\eta_1(b,a)} f(x)g(x)dx\right|$$
$$\leq \frac{\eta_1(b,a)}{4}\int_0^1 \int_{a+(\frac{1-t}{2})\eta_1(b,a)}^{a+(\frac{1+t}{2})\eta_1(b,a)} g(u)\Big[\Big|f'\Big(a+(\frac{1-t}{2})\eta_1(b,a)\Big)\Big|$$
$$+\Big|f'\Big(a+(\frac{1+t}{2})\eta_1(b,a)\Big)\Big|\Big]dudt$$
$$\leq \frac{\eta_1(b,a)}{4}\int_0^1 \int_{a+(\frac{1-t}{2})\eta_1(b,a)}^{a+(\frac{1+t}{2})\eta_1(b,a)} g(u)\Big[|f'(a)| + (\frac{1-t}{2})\eta_2(|f'(b)|, |f'(a)|)$$
$$+|f'(a)| + (\frac{1+t}{2})\eta_2(|f'(b)|, |f'(a)|)\Big)\Big]dudt$$
$$= \frac{\eta_1(b,a)}{4}\int_0^1 \int_{a+(\frac{1-t}{2})\eta_1(b,a)}^{a+(\frac{1+t}{2})\eta_1(b,a)} g(u)\Big[2|f'(a)| + \eta_2(|f'(b)|, |f'(a)|)\Big]dudt.$$

□

Remark 4.4. (i) In Theorem 4.3 if we consider $\eta_1(x,y) = x-y$ and $\eta_2(x,y) = x-y$, then we recapture inequality (1.11) in [8].

$$\left|\frac{f(a)+f(b)}{2}\int_a^b g(x)dx - \int_a^b f(x)g(x)dx\right|$$
$$\leq \frac{b-a}{4}\Big[|f'(a)| + |f'(b)|\Big]\int_0^1 \int_{a+(\frac{1-t}{2})(b-a)}^{a+(\frac{1+t}{2})(b-a)} g(u)dudt.$$

(ii) In [15], with all assumptions of Theorem 4.3, we can find another presentation of Fejér trapezoid type inequality as the following:

$$\left|\frac{f(a)+f\big(a+\eta_1(b,a)\big)}{2}\int_a^{a+\eta_1(b,a)} g(x)dx - \int_a^{a+\eta_1(b,a)} f(x)g(x)dx\right| \leq$$
$$\Big[2|f'(a)| + \eta_2\big(|f'(b)|, |f'(a)|\big)\Big]\int_{a+\frac{1}{2}\eta_1(b,a)}^{a+\eta_1(b,a)} g(x)\big(a+\eta_1(b,a)-x\big)dx.$$

By the use of Hölder's inequality we can obtain another form of Fejér trapezoidal type inequality.

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Theorem 4.5. *Suppose that $I^\circ \subseteq \mathbb{R}$ is an invex set with respect to $\eta_1 : I^\circ \times I^\circ \to \mathbb{R}$ and consider $a, b \in I^\circ$ with $\eta_1(b,a) > 0$. Suppose that $f : I^\circ \to \mathbb{R}$ is a differentiable mapping on I° such that $f' \in L^1([a, a+\eta_1(b,a)])$. If $g : [a, a+\eta_1(b,a)] \to \mathbb{R}^+$ is an integrable mapping and symmetric with respect to $a + \frac{1}{2}\eta_1(b,a)$ and if $|f'|^q$ is (η_1, η_2)-convex on I° for $q > 1$, then we have following inequality.*

$$\left|\frac{f(a)+f(a+\eta_1(b,a))}{2}\int_a^{a+\eta_1(b,a)} g(x)dx - \int_a^{a+\eta_1(b,a)} f(x)g(x)dx\right|$$
$$\leq \frac{\eta_1(b,a)}{2}\left[\frac{2|f'(a)|^q + \eta_2(|f'(b)|^q, |f'(a)|^q)}{2}\right]^{\frac{1}{q}} \left(\int_0^1 \Big[\int_{a+(\frac{1-t}{2})\eta_1(b,a)}^{a+(\frac{1+t}{2})\eta_1(b,a)} g(x)dx\Big]^p dt\right)^{\frac{1}{p}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. By lemma (4.2) and using Hölder's inequality we have

$$\left|\frac{f(a)+f(a+\eta_1(b,a))}{2}\int_a^{a+\eta_1(b,a)} g(x)dx - \int_a^{a+\eta_1(b,a)} f(x)g(x)dx\right| \tag{4.11}$$
$$\leq \frac{\eta_1(b,a)}{4}\left(\int_0^1 \Big[\int_{a+(\frac{1-t}{2})\eta_1(b,a)}^{a+(\frac{1+t}{2})\eta_1(b,a)} g(u)du\Big]^p dt\right)^{\frac{1}{p}}$$
$$\times\left[\left(\int_0^1 \Big|f'\Big(a+(\frac{1-t}{2})\eta_1(b,a)\Big)\Big|^q dt\right)^{\frac{1}{q}} + \left(\int_0^1 \Big|f'\Big(a+(\frac{1+t}{2})\eta_1(b,a)\Big)\Big|^q dt\right)^{\frac{1}{q}}\right].$$

By (η_1, η_2)-convexity of $|f'|^q$ and power-mean inequality $c^s + d^s < 2^{1-s}(c+d)^s$ for $c > 0$, $d > 0$ and $s < 1$, we have the following inequality:

$$\left(\int_0^1 \Big|f'\Big(a+(\frac{1-t}{2})\eta_1(b,a)\Big)\Big|^q dt\right)^{\frac{1}{q}} + \left(\int_0^1 \Big|f'\Big(a+(\frac{1+t}{2})\eta_1(b,a)\Big)\Big|^q dt\right)^{\frac{1}{q}} \tag{4.12}$$
$$\leq 2^{1-\frac{1}{q}}\left[\int_0^1 \Big|f'\Big(a+(\frac{1-t}{2})\eta_1(b,a)\Big)\Big|^q dt + \int_0^1 \Big|f'\Big(a+(\frac{1+t}{2})\eta_1(b,a)\Big)\Big|^q dt\right]^{\frac{1}{q}}$$
$$\leq 2^{1-\frac{1}{q}}\left[\int_0^1 \Big\{|f'(a)|^q + (\frac{1-t}{2})\eta_2\big(|f'(b)|^q, |f'(a)|^q\big) + |f'(a)|^q + (\frac{1+t}{2})\eta_2\big(|f'(b)|^q, |f'(a)|^q\big)\Big\}dt\right]^{\frac{1}{q}}$$
$$= 2^{1-\frac{1}{q}}\Big[2|f'(a)|^q + \eta_2\big(|f'(b)|^q, |f'(a)|^q\big)\Big]^{\frac{1}{q}}.$$

Now it is enough to apply (4.11) in (4.12). This completes the proof. □

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