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Reduction of BL-general L-fuzzy Automata

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Abstract. In this paper, we show that for any BL-general L-fuzzy automaton (BL-GLFA) there exists a complete deterministic accessible reduced BL-general L-fuzzy automaton that recognizing the behavior of the BL-GLFA. Also, we prove that for any finite realization β, there exists a minimal complete deterministic BL-GLFA recognizing β. We prove any complete deterministic accessible reduced BL-GLFA is a minimal BL-GLFA. After that, we show that for any given finite realization β, the minimal complete deterministic BL-GLFA recognizing β is isomorphic to any complete accessible deterministic reduced BL-GLFA recognizing β. Moreover, we give some examples to clarify these notions. Finally, by using these notions, we give some theorems and algorithms and obtain some related results.

Keywords: BL-general fuzzy automata, Minimal automata, Reduction, Deterministic automata.

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1. Introduction

A general formulation of automata is given which is similar to that of sequential machines introduced in [17]. Study of fuzzy automata and languages was initiated in 1960s by Santos [16, 17, 18], Wee [29], Wee and Fu [27], and Lee and Zadeh [8].

Fuzzy finite automata have many applications in different branches of science, such as in the learning system, pattern recognition, neural networks, database theory, simulation theory [5, 6, 9, 11, 12, 19, 23, 24, 28].

Note that, state minimization is a fundamental problem in automata theory. There are many papers on the minimization trend of fuzzy automata, such as minimization of the mealy type of fuzzy finite automata, minimization of fuzzy finite automata with crisp final states without outputs, minimization of deterministic finite automaton with fuzzy (final) states, for more information see [2, 3, 10, 13, 14, 15, 20, 21, 23, 24, 26].

In 2004, M. Doostfatemeh and S.C. Kremer [4] extended the notion of fuzzy automata and gave the notion of general fuzzy automata. Their key motivation of introducing the notion general fuzzy automaton was the insufficiency of the current literature to handle the applications which rely on fuzzy automaton as a modeling tool, assigning membership values to active states of a fuzzy automaton, resolve the multi -membership. Another important insufficiency of the current literature is the lack of methodologies which enable us to define and analyze the continuous operation of fuzzy automaton.

Basic logic (BL) has been introduced by Hajek [7] in order to provide a general framework for formalizing statements of fuzzy nature. Formulas of propositional BL may be interpreted by means of BL-algebras. With respect to a semantics defined in this way, BL is complete: formulas proved by BL, exactly those valid in any BL-algebra.

In 2012, Kh. Abolpour and M. M. Zahedi [1] extended the notion of general fuzzy automata and gave the notion of BL-general fuzzy automata.

The rest of paper is organized as follows: In Section 2 we give some notions which will be necessary for Sections 3. In Section 3, we give the the notions of complete, deterministic, accessible and reduced for BL-general fuzzy automata. After that, for a BL-general L-fuzzy automata an algorithm to determine the complete BL-general fuzzy automata is given also we determine the time complexity of it. Moreover, we present an algorithm to determines deterministic BL-GLFA also, the time complexity of it is presented. After that, for a given realization β, we present the minimal complete deterministic BL-GLFA, where the given automaton recognizes β. Also, we present the notion of minimal complete deterministic BL-GLFA. Moreover, we prove that the minimal complete

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deterministic BL-GLFA recognizing β is isomorphic to any complete accessible deterministic reduced BL-GLFA recognizing β. Moreover, we give some examples to clarify these notions.

2. Preliminaries

First, we review some definitions which will be necessary for the next sections.

Definition 2.1. [4] A general fuzzy automaton (GFA) $\tilde{F}$ is an eight-tuple machine denoted by $\tilde{F} = (Q, X, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$, where

- Q is a finite set of states,
- X is a finite set of input symbols,
- $\tilde{R}$ is a set of fuzzy start states, $\tilde{R} \subseteq \tilde{P}(Q)$, where $\tilde{P}(Q)$ is the fuzzy power set of Q,
- Z is a finite set of output symbols,
- $\tilde{\delta} : (Q \times [0,1]) \times X \times Q \to [0,1]$ is the augmented transition function,
- $\omega : Q \to Z$ is the output function,
- $F_1 : [0,1] \times [0,1] \to [0,1]$ is called the membership assignment function.
- $F_2 : [0,1]^* \to [0,1]$ is called the multi-membership resolution function.

Let the set of all transitions of $\tilde{F}$ is denoted by Δ. Now, suppose that $Q_{act}(t_i)$ be the set of every active states at time t_i, for every $i \geq 0$. We have $Q_{act}(t_0) = \tilde{R}$ and $Q_{act}(t_i) = \{(q, \mu^{t_i}(q)) \big| \exists q' \in Q_{act}(t_{i-1}), \exists a \in X, \delta(q', a, q) \in \Delta\}$, for every $i \geq 1$, where $\mu^{t_i}(q)$ is the membership value of state q at time t_i.

Definition 2.2. [7] A BL-algebra is an algebra $(L, \wedge, \vee, *, \to, 0, 1)$ with four binary operations $\wedge, \vee, *, \to$ and two constants $0, 1$ in which: (i) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice, (ii) $(L, *, 1)$ is a commutative monoid, (iii) $*$ and $\to$ form an adjoint pair, i.e., $x \leq y \to z$ if and only if $x * y \leq z$, (iv) $x \wedge y = x * (x \to y)$, (v) $(x \to y) \vee (y \to x) = 1$, where $x, y, z \in L$.

From now on $L = (L, \vee, \wedge, 0, 1)$ is a bounded complete lattice.

Definition 2.3. [19] Let $\tilde{F} = (Q, X, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$ be a general fuzzy automaton and $\bar{Q} = (P(Q), \subseteq, \cap, \cup, \emptyset, Q)$ be a BL-algebra as in Example 2 of [19]. Then the BL-general L-fuzzy automaton (BL-GLFA) as a ten-tuple machine denoted by $\tilde{F}_l = (\bar{Q}, X, \tilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2)$, where

(i) $\bar{Q} = P(Q)$, where Q is a finite set and $\bar{Q}$ is the power set of Q,
(ii) X is a finite set of input symbols,
(iii) $\tilde{R}$ is the set of fuzzy start states,
(iv) $\bar{Z}$ is a finite set of output symbols, where $\bar{Z}$ is the power set of Z,
(v) $\omega_l : \bar{Q} \to \bar{Z}$ is the output function defined by: $\omega_l(Q_i) = \{\omega(q) \big| q \in Q_i\}$,
(vi) $\delta_l : \bar{Q} \times X \times \bar{Q} \to L$ is the transition function defined by: $\delta_l(\{p\}, a, \{q\}) = \delta(p, a, q)$ and $\delta_l(Q_i, a, Q_j) = \vee_{q_i \in Q_i, q_j \in Q_j} \delta(q_i, a, q_j)$, for every $Q_i, Q_j \in P(Q)$ and $a \in X$,

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(vii) $f_l : \bar{Q} \times X \to \bar{Q}$ is the next state map defined by: $f_l(Q_i, a) = \cup_{q_i \in Q_i}\{q_j \big| \delta(q_i, a, q_j) \in \Delta\}$,

(viii) $\tilde{\delta}_l : (\bar{Q} \times L) \times X \times \bar{Q} \to L$ is the augmented transition function defined $\tilde{\delta}_l((Q_i, \mu^t(Q_i)), a, Q_j) = F_1(\mu^t(Q_i), \delta_l(Q_i, a, Q_j))$,

(ix) $F_1 : L \times L \to L$ is called membership assignment function,

(x) $F_2 : L^* \to L$ is called multi-membership resolution function.

Let $\tilde{F}_l = (\bar{Q}, X, \tilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2)$ be a BL-GLFA. Then the cardinality of $\tilde{F}_l$ is defined by $|\tilde{F}_l| = |\bar{Q}|$.

Definition 2.4. [24] Let $\tilde{F}_l = (\bar{Q}, X, \tilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2)$ be a BL-GLFA. The run map of the BL-GLFA $\tilde{F}_l$ is the map $\rho : X^* \to \bar{Q}$ defined by the following induction: $\rho(\Lambda) = \{q_0\}$ and $\rho(a_1a_2...a_n) = Q_{i_n}, \rho(a_1a_2...a_na_{n+1}) = f_l(Q_{i_n}, a_{n+1})$, where $(Q_{i_n}, \mu^{t_0+n}(Q_{i_n})) \in Q_{act}(a_1a_2...a_n)$, for every $a_1, ..., a_n \in X$.

The behavior of $\tilde{F}_l$ is the map $\beta = \omega_l \circ \rho : X^* \to \bar{Z}$.

Definition 2.5. [24] Given $(\bar{Q}, f_l, \delta_l)$ and $(\bar{Q}', f_l', \delta_l')$, we say that

$$g : (\bar{Q}, f_l, \delta_l) \to (\bar{Q}', f_l', \delta_l'),$$

is a homomorphism with threshold $\frac{\tau_1}{\tau_2}$ if there is a map of $\bar{Q}$ into $\bar{Q}'$ such that for every $Q_i, Q_j \in \bar{Q}$ the following hold:

(i) $g \circ f_l = f_l' \circ (g \times id_X)$,

(ii) $\tau_1 \leq \delta_l(f_l(Q_i, a_1), a_2, Q_j) \leq \tau_2$ if and only if $\tau_1 \leq \delta_l'(g(f_l(Q_i, a_1)), a_2, g(Q_j)) \leq \tau_2$,

where id_X is called the identity map on X.

We say that $g : (\bar{Q}, f_l, \delta_l) \to (\bar{Q}', f_l', \delta_l')$ is homomorphism if and only if $g : (\bar{Q}, f_l, \delta_l) \to (\bar{Q}', f_l', \delta_l')$ is homomorphism with threshold $\frac{0}{1}$.

Definition 2.6. [24] Let

$$\tilde{F}_{li} = (\bar{Q}_i, X, \tilde{R}_i = (\{q_{0i}\}, \mu^{t_0}(\{q_{0i}\})), \bar{Z}, \omega_{li}, \delta_{li}, f_{li}, \tilde{\delta}_{li}, F_1, F_2), i = 1, 2,$$

be two BL-GLFAs. We say that $(g, g_{out}) : \tilde{F}_l \to \tilde{F}_l'$ is a morphism with threshold $\frac{\tau_1}{\tau_2}$ if and only if the following hold:

(i) $g : (\bar{Q}, f_l, \delta_l) \to (\bar{Q}', f_l', \delta_l')$ is a homomorphism with threshold $\frac{\tau_1}{\tau_2}$.

(ii) $g_{out} \circ \omega_l = \omega_l' \circ g$,

(iii) $g(\{q_0\}) = \{q_0'\}$.

We say that $(g, g_{out}) : \tilde{F}_l \to \tilde{F}_l'$ is a morphism if and only if $(g, g_{out}) : \tilde{F}_l \to \tilde{F}_l'$ is morphism with threshold $\frac{0}{1}$.

Definition 2.7. [25] Let $\beta : X^* \to \bar{Z}$. Then we say that the behavior β has a finite realization if there exists a BL-GLFA $\tilde{F}_l$, where $\beta_{\tilde{F}_l} = \beta$.

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Definition 2.8. [25] Let $\tilde{F}_l = (\bar{Q}, X, \tilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2)$ be a BL-GLFA. Then we say that $\tilde{F}_l$ is a complete BL-GLFA if for any $\emptyset \neq Q' \in \bar{Q}$ and $a \in X$ there exists $\emptyset \neq Q'' \in \bar{Q}$ such that $f_l(Q', a) = Q''$.

3. Minimization and reduction of BL-general L-fuzzy automata

In this section, we present the definition of complete, deterministic, accessible and reduced for BL-general L-fuzzy automaton (BL-GLFA). After that, for a given realization β, we present the minimal complete deterministic BL-GLFA, where the given automaton recognizes β. Also, we prove that the minimal complete deterministic BL-GLFA recognizing β is isomorphic to any complete accessible deterministic reduced BL-GLFA recognizing β. Moreover, we present two algorithms to determine complete and deterministic BL-GLFA and also we obtain the complexity of them.

Theorem 3.1. *Let $\tilde{F}_l = (\bar{Q}, X, \tilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2)$ be a BL-GLFA. Then there exists a complete BL-GLFA $\tilde{F}_l^c$ such that $\beta_{\tilde{F}_l} = \beta_{\tilde{F}_l^c}$.*

Proof. Let $\tilde{F}_l = (\bar{Q}, X, \tilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2)$ does not be a complete BL-GLFA. Consider

$$\tilde{F}_l^c = (\bar{Q}^c, X, \tilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_l^c, \delta_l^c, f_l^c, \tilde{\delta}_l^c, F_1, F_2)$$

, where $\bar{Q}^c = P(Q \cup t)$, t is an element such that $t \notin Q$. If $f_l(Q', a) = \emptyset$, then $\delta_l^c(Q', a, P') = d$, for some fixed $d \in L$, where $\emptyset \neq Q' \in \bar{Q}, t \in P' \in \bar{Q}^c$. If $f_l(Q', a) \neq Q'$, then $\delta_l^c(Q', a, Q'') = \delta_l(Q', a, Q'')$, where $t \notin Q', Q'' \in \bar{Q}$. Also, let $\delta_l^c(\{t\}, a, Q') = d$, where $t \in Q'$, and consider $\delta_l^c(Q', a, Q'') = \delta_l(Q', a, P'')$, where $t \notin Q', Q'' = P'' \cup \{t\}$ and $P'' \neq \emptyset$. If $Q' = P' \cup \{t\}, P' \neq \emptyset$ and $t \notin Q''$, then consider $\delta_l^c(Q', a, Q'') = \delta_l(P', a, Q'')$. If $Q' = P' \cup \{t\}, Q'' = P'' \cup \{t\}$ and $P', P'' \neq \emptyset$, then consider $\delta_l^c(Q', a, Q'') = \delta_l(P', a, P'') \vee d$. Finally, If $Q' = P' \cup \{t\}$ and $P' \neq \emptyset$, then $\delta_l^c(Q', a, \{t\}) = d$. Also, let $\omega_l^c(Q') = \omega_l(Q')$, for every $Q' \in \bar{Q}$.

It is easy to see that the BL-GLFA

$$\tilde{F}_l^c = (\bar{Q}^c, X, \tilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_l^c, \delta_l^c, f_l^c, \tilde{\delta}_l^c, F_1, F_2),$$

is complete and $\beta_{\tilde{F}_l} = \beta_{\tilde{F}_l^c}$. □

1. Algorithm for computing the complete BL-general L-fuzzy automata

Step 1. **Input:** an incomplete BL-GLFA

$$\tilde{F}_l = (\bar{Q}, X, \tilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2),$$

and a bounded complete lattice $(L, \vee, \wedge, 0, 1)$.

Step 2. $\bar{Q}' = \emptyset$.

Step 3. If $P' \in \bar{Q}$, then $P' \cup \{t\} \in \bar{Q}'$.

Step 4. $\bar{Q}^c = \bar{Q} \cup \bar{Q}'$.

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Step 5. Let $d \in L$.
Step 6. If $\emptyset \neq Q' \in \bar{Q}$ and $f_l(Q', a) = \emptyset$, then $\delta_l^c(Q', a, P') = d$, where $P' \in \bar{Q}'$.
Step 7. If $Q' \in \bar{Q}$ and $f_l(Q', a) \neq \emptyset$, then $\delta_l^c(Q', a, Q'') = \delta_l(Q', a, Q'')$, where $Q'' \in \bar{Q}$.
Step 8. If $Q' \in \bar{Q}'$, then $\delta_l^c(\{t\}, a, Q') = d$.
Step 9. If $t \notin Q', Q'' = P'' \cup \{t\}, P'' \neq \emptyset$, then $\delta_l^c(Q', a, Q'') = \delta_l(Q', a, Q'')$.
Step 10. If $Q' = P' \cup \{t\}, P' \neq \emptyset$ and $Q'' \in \bar{Q}$, then $\delta_l^c(Q', a, Q'') = \delta_l(P', a, Q'')$.
Step 11. If $Q' = P' \cup \{t\}, Q'' = P'' \cup \{t\}$ and $P', P'' \neq \emptyset$, then $\delta_l^c(Q', a, Q'') = \delta_l(P', a, P'') \vee d$.
Step 12. If $Q' = P' \cup \{t\}$ and $P' \neq \emptyset$, then $\delta_l^c(Q', a, \{t\}) = d$.
Step 13. If $Q' \in \bar{Q}$, then $\omega_l^c(Q') = \omega_l(Q')$.
Step 14. **Output:** $\tilde{F}_l^c = (\bar{Q}^c, X, \tilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_l^c, \delta_l^c, f_l^c, \tilde{\delta}_l^c, F_1, F_2)$.

Steps 6, 7, 9, 10 and 11 of Algorithm 1, must be repeated at most $|\bar{Q}|^2$ also, Steps 3, 8, 12 and 13 must be repeated at most be repeated at most $|\bar{Q}|$. Then the order of time complexity of this algorithm is at most $O(|\bar{Q}|^2)$.

By considering Algorithm 1, we can obtain a complete BL-general L-fuzzy automata.

Example 3.2. Let $(L, \wedge, \vee, 0, 1)$ be a complete lattice as in Figure 1. Now,

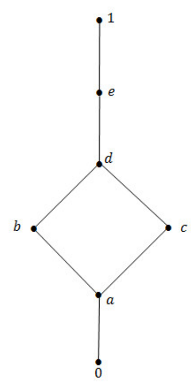

Figure 1. The complete lattice L of Example 3.2.

consider the general L-fuzzy automaton $\widetilde{F} = (Q, X, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$, where $Q = \{q_0, q_1\}$, $\tilde{R} = \{(q_0, 1)\}, X = \{\sigma\}, Z = \{z_1, z_2\}, \omega(q_0) = z_1, \omega(q_1) = z_2$ and

$$\delta(q_0, \sigma, q_0) = a, \ \delta(q_0, \sigma, q_1) = b.$$

Now, we have BL-general L-fuzzy automaton $\widetilde{F}_l$ as follow: $\tilde{F}_l = (\bar{Q}, X, \tilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2)$, where

$$\bar{Q} = \{\{q_0\}, \{q_1\}, \{q_0, q_1\}, \emptyset\},$$

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$\omega_l(\{q_0\}) = \{z_1\}, \omega_l(\{q_1\}) = \{z_2\}, \omega_l(\{q_0, q_1\}) = \{z_1, z_2\}$and

$$\begin{aligned}
&\delta(\{q_0\}, \sigma, \{q_0\}) = a,\ \delta(\{q_0\}, \sigma, \{q_1\}) = b,\\
&\delta(\{q_0\}, \sigma, \{q_0, q_1\}) = b,\ \delta(\{q_0, q_1\}, \sigma, \{q_0\}) = a,\\
&\delta(\{q_0, q_1\}, \sigma, \{q_1\}) = b,\ \delta(\{q_0, q_1\}, \sigma, \{q_0, q_1\}) = b.
\end{aligned}$$

It is obvious that $\tilde{F}_l$ is not complete, because $f_l(\{q_1\}, \sigma) = \emptyset$. Then by Theorem 3.1, we have $\bar{Q}^c = P(Q \cup t) = \{\{q_0\}, \{q_1\}, \{t\}, \{q_0, q_1\}, \{q_0, t\}, \{q_1, t\}, \{q_0, q_1, t\}, \emptyset\}$ and

$$\begin{aligned}
&\delta_l^c(\{q_0\}, \sigma, \{q_0\}) = a, \quad \delta_l^c(\{q_0\}, \sigma, \{q_1\}) = b,\\
&\delta_l^c(\{q_0\}, \sigma, \{q_0, q_1\}) = b, \quad \delta_l^c(\{q_0\}, \sigma, \{q_0, t\}) = a,\\
&\delta_l^c(\{q_0\}, \sigma, \{q_1, t\}) = b, \quad \delta_l^c(\{q_0\}, \sigma, \{q_0, q_1, t\}) = b,\\
&\delta_l^c(\{q_1\}, \sigma, \{t\}) = d, \quad \delta_l^c(\{q_1\}, \sigma, \{q_0, t\}) = d,\\
&\delta_l^c(\{q_1\}, \sigma, \{q_1, t\}) = d, \quad \delta_l^c(\{q_1\}, \sigma, \{q_0, q_1, t\}) = d,\\
&\delta_l^c(\{q_0, q_1\}, \sigma, \{q_0\}) = a, \quad \delta_l^c(\{q_0, q_1\}, \sigma, \{q_1\}) = b,\\
&\delta_l^c(\{q_0, q_1\}, \sigma, \{q_0, q_1\}) = b, \quad \delta_l^c(\{q_0, q_1\}, \sigma, \{q_0, t\}) = a,\\
&\delta_l^c(\{q_0, q_1\}, \sigma, \{q_1, t\}) = b, \quad \delta_l^c(\{q_0, q_1\}, \sigma, \{q_0, q_1, t\}) = b,\\
&\delta_l^c(\{t\}, \sigma, \{t\}) = d, \quad \delta_l^c(\{t\}, \sigma, \{q_0, t\}) = d,\\
&\delta_l^c(\{t\}, \sigma, \{q_1, t\}) = d,\ \delta_l^c(\{t\}, \sigma, \{q_0, q_1, t\}) = d,\\
&\delta_l^c(\{q_0, t\}, \sigma, \{q_0\}) = a, \quad \delta_l^c(\{q_0, t\}, \sigma, \{q_1\}) = b,\\
&\delta_l^c(\{q_0, t\}, \sigma, \{t\}) = d, \quad \delta_l^c(\{q_0, t\}, \sigma, \{q_0, q_1\}) = b,\\
&\delta_l^c(\{q_0, t\}, \sigma, \{q_0, t\}) = d, \quad \delta_l^c(\{q_0, t\}, \sigma, \{q_1, t\}) = d,\\
&\delta_l^c(\{q_0, t\}, \sigma, \{q_0, q_1, t\}) = d,\ \delta_l^c(\{q_1, t\}, \sigma, \{t\}) = d,\\
&\delta_l^c(\{q_1, t\}, \sigma, \{q_0, t\}) = d, \quad \delta_l^c(\{q_1, t\}, \sigma, \{q_1, t\}) = d,\\
&\delta_l^c(\{q_1, t\}, \sigma, \{q_0, q_1, t\}) = d,\ \delta_l^c(\{q_0, q_1, t\}, \sigma, \{q_0\}) = a,\\
&\delta_l^c(\{q_0, q_1, t\}, \sigma, \{q_1\}) = b, \quad \delta_l^c(\{q_0, q_1, t\}, \sigma, \{t\}) = d,\\
&\delta_l^c(\{q_0, q_1, t\}, \sigma, \{q_0, q_1\}) = b,\ \delta_l^c(\{q_0, q_1, t\}, \sigma, \{q_0, t\}) = d,\\
&\delta_l^c(\{q_0, q_1, t\}, \sigma, \{q_1, t\}) = d,\ \delta_l^c(\{q_0, q_1, t\}, \sigma, \{q_0, q_1, t\}) = d,
\end{aligned}$$

It is clear that $\tilde{F}_l^c = (\bar{Q}^c, X, \tilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_l^c, \delta_l^c, f_l^c, \tilde{\delta}_l^c, F_1, F_2)$ is a complete BL-GLFA.

Definition 3.3. Let $\tilde{F}_l = (\bar{Q}, X, \tilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2)$ be a max-min BL-GLFA.Then we say that $\tilde{F}_l$ is deterministic if for any $\emptyset \neq Q' \in \bar{Q}$ and $a \in X$ there exists at most one $Q'' \in \bar{Q}$ such that $\delta_l(Q', a, Q'') > 0$.

Theorem 3.4. *Let $\tilde{F}$ be a BL-GLFA. Then there exists a deterministic BL-GLFA $\tilde{F}_d$ such that $\beta(\tilde{F}) = \beta(\tilde{F}_d)$.*

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Proof. Let

$$D_x = \{Q' \in \bar{Q} \big| \exists Q_1, ..., Q_{n-1} \in \bar{Q}, \text{ where } \delta_l(\{q_0\}, a_1, Q_1) > 0, \delta_l(Q_1, a_2, Q_2) > 0, ..., \delta_l(Q_{n-1}, a_n, Q') > 0\}, \quad (3.1)$$

for every $x = a_1a_2...a_n \in X^*$. Then $D_\Lambda = \{Q' \in \bar{Q} \big| Q' \in \tilde{R}\} = \{\{q_0\}\}$. Let $\bar{Q}_d = \{D_x \big| x \in X^*\} \subseteq \bar{Q}$. Define $\delta_{ld} : \bar{Q}_d \times X \times \bar{Q}_d \to L$, where

$$\delta_{ld}(D_y, a, D_x) = \begin{cases} 1 & if\ D_x = D_{ya} \\ 0 & otherwise \end{cases},$$

and $\omega_{ld} : \bar{Q}_d \to \bar{Z}_d$, by $\omega_{ld}(D_x) = \cup_{Q' \in D_x} \omega_l(Q')$, where $\bar{Z}_d = \bar{Z}$. Consider

$$\tilde{F}_d = (\bar{Q}_d, X, \tilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_{ld}, \delta_{ld}, f_{ld}, \tilde{\delta}_{ld}, F_1, F_2).$$

Now, we show that δ_{ld} is well defined. Let $D_y = D_u, D_x = D_v$. If $D_x = D_{ya}$ or $D_{ua} = D_v$, then we have $D_{ua} = D_{ya} = D_x = D_v$. So, $\delta_{ld}(D_y, a, D_x) = \delta_{ld}(D_u, a, D_v)$. It is clear that ω_{ld} is well-defined. Now, we show that $\beta_{ld} = \beta_l$. By considering (3.1), clear that $f_{ld} = f_l$. Therefore, $\rho_{ld} = \rho_l$. Hence, $\beta_{ld}(x) = \omega_l(\rho_{ld}(x)) = \omega_l(\rho_l(x)) = \beta_l(x)$. □

2. Algorithm for computing deterministic BL-general L-fuzzy automata

Step 1. **Input:** a nondeterministic BL-GLFA

$$\tilde{F}_l = (\bar{Q}, X, \tilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2),$$

and a bounded complete lattice $(L, \vee, \wedge, 0, 1)$.

Step 2. Let $D_\Lambda = \{\{q_0\}\}$.

Step 3. For $x \in X^*$ and $a \in X$, let $D_x.a = \{Q'' | \delta_l(Q', a, Q'') > 0, Q' \in D_x\}$.

Step 4. Let $D_{xa} := D_x.a$.

Step 5. Let $\bar{Q}_d = \{D_x | x \in X^*\}$.

Step 6. If $D_x = D_{ya}$, then $\delta_{ld}(D_y, a, D_x) = 1$,
else $\delta_{ld}(D_y, a, D_x) = 0$.

Step 7. Consider $\omega_{ld}(D_x) = \cup_{Q' \in D_x} \omega_l(Q')$.

Step 8. $\bar{Z}_d = \bar{Z}$.

Step 9. **Output:** $\tilde{F}_d = (\bar{Q}_d, X, \tilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_{ld}, \delta_{ld}, f_{ld}, \tilde{\delta}_{ld}, F_1, F_2)$.

Step 3 of Algorithm 2, must be repeated at most $2^{|\bar{Q}|}$ also, time complexity of Step 7 is at most $2^{|\bar{Q}|} \times |\bar{Q}|$. Then the order of time complexity of this algorithm is at most $O(2^{|\bar{Q}|} \times |\bar{Q}|)$.

By using Algorithms 1 and 2, we can obtain a complete and deterministic BL-general L-fuzzy automata.

Example 3.5. Let $\tilde{F}_l$ be as defined in Example 3.2. Clearly, $\tilde{F}_l$ is deterministic. Also, $\tilde{F}_l^c$ defined in Example 3.2, is deterministic, too.

Definition 3.6. Let $\tilde{F}_l$ be a BL-GLFA. Then we say that

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1. $Q' \in \bar{Q}$ is accessible if there exist $x \in X^*$ such that $\tilde{\delta}_l^*(\{q_0\}, \mu^{t_0}(\{q_0\}), x, Q') > 0$.
2. $\tilde{F}_l$ is accessible if for every $\emptyset \neq Q' \in \bar{Q}$, Q' is an accessible state.

Theorem 3.7. *Let $\tilde{F}$ be a BL-GLFA. Then there exists a BL-GLFA $\tilde{F}_a$ such that $\beta(\tilde{F}) = \beta(\tilde{F}_a)$.*

Proof. By Theorem 3.4 and without loss of generality we assume that $\tilde{F}$ be deterministic. Let $S = \{Q' \in \bar{Q} | Q' \text{ be an accessible state}\}$, $Z_a = Z$, $\delta_a = \delta|_{S\times X\times S}$, $\omega_a = \omega|_S$ and $f_a = f|_{S\times X}$, i.e., δ_a is the restriction of δ to $S \times X \times S$ and ω_a is the restriction of ω to S. Then the BL-GLFA

$$\tilde{F}_a = (S, X, \tilde{R}, \bar{Z}, \omega_a, \delta_a, f_a, \tilde{\delta}_a, F_1, F_2),$$

is an accessible BL-GLFA. Now, we show that $\beta_a(\tilde{F}_a) = \beta(\tilde{F})$. We have $\rho_a(x) = \rho_a(a_1...a_{n+1}) = f_{ld}(Q_{in}, a_{n+1}) = f_l(Q_{in}, a_{n+1}) = \rho(x)$, where $\rho(a_1...a_n) = Q_{in}$. Hence, $\beta_a(x) = \omega_{la}(\rho_a(x)) = \omega_{la}(\rho(x)) = \omega_l(\rho(x)) = \beta(x)$.

□

Example 3.8. Let $\tilde{F}_l$ be as defined in Example 3.2. States $\{q_0\}, \{q_1\}$ and $\{q_0, q_1\}$ are accessible. Then $\tilde{F}_l$ is accessible.

Definition 3.9. Let $\tilde{F}$ be a deterministic, accessible BL-GLFA. We define a relation on $\bar{Q}$ by $Q'\varphi Q''$, if and only if $\omega_l(f(Q', x)) = \omega_l(f(Q'', x))$, for every $x \in X^*$.

Example 3.10. Let $\tilde{F}_l$ be as defined in Example 3.2. By considering Examples 3.5 and 3.8 and Definition 3.9, $\tilde{F}_l$ is complete and deterministic. Then $\varphi = \{\{q_0\}, \{q_1\}, \{q_0, q_1\}\}$.

Lemma 3.11. *Let $\tilde{F}$ be an accessible complete deterministic BL-GLFA. Then φ is an equivalence relation on $\bar{Q}$.*

Proof. It is clear that $Q'\varphi Q'$ and if $Q'\varphi Q''$, then $Q''\varphi Q'$. Now, let $Q'\varphi Q''$ and $Q''\varphi Q'''$. Then for every $x \in X^*$ we have $\omega_l(f(Q', x)) = \omega_l(f(Q'', x)) = \omega_l(f(Q'', x))$. Hence, φ is an equivalence relation. □

Definition 3.12. An accessible complete deterministic max-min BL-GLFA $\tilde{F}$ is called reduced if for every $Q'\varphi Q''$ implies that $Q' = Q''$, for any $Q', Q'' \in \bar{Q}$.

Example 3.13. Let $\tilde{F}_l$ be as defined in Example 3.2. By Examples 3.5, 3.8, 3.10 and Definition 3.12, $\tilde{F}_l$ is reduced.

Definition 3.14. Let $\tilde{F}_l = (\bar{Q}, X, \tilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2)$ be an accessible complete deterministic BL-GLFA and φ be the equivalence relation defined in Definition 3.9. Suppose that $\bar{Q}/\varphi = \{Q'\varphi | Q' \in \bar{Q}\}$, and

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$\tilde{R}/\varphi = \{q_0\}\varphi, \mu^{t_0}(\{q_0\}\varphi) = \mu^{t_0}(\{q_0\})$. Now, define $\delta_{l\varphi} : \bar{Q}/\varphi \times X \times \bar{Q}/\varphi \to L$ by:

$$\delta_{l\varphi}(Q'\varphi, a, Q''\varphi) = \begin{cases} \gamma & if\, f_l(Q', a) = Q''', \text{ where } Q'''\varphi Q'' \\ 0 & otherwise \end{cases},$$

where $\gamma \in L$. Also, consider $\omega_{l\varphi} : \bar{Q}/\varphi \to Z$, where $\omega_{l\varphi}(Q'\varphi) = \omega_l(Q')$.

Lemma 3.15. *$\delta_{l\varphi}$ and $\omega_{l\varphi}$ are well-defined.*

Proof. Let $Q'\varphi, Q''\varphi, P'\varphi, P''\varphi \in \bar{Q}/\varphi$, $Q'\varphi = P'\varphi$ and $Q''\varphi = P''\varphi$. If $\delta_{l\varphi}(Q'\varphi, a, Q''\varphi) = \gamma$, then there exists $Q''' \in \bar{Q}$ such that $Q'''\varphi Q''$ and $f_l(Q', a)$ $= Q'''$. So, $Q'''\varphi P''\varphi Q''$. By considering Definition 3.14, $\omega_l(f_l(Q', ax)) =$ $\omega_l(f_l(P', ax))$, for every $x \in X^*$. Therefore, $\omega_l(f_l(Q'', x)) = \omega_l(f_l(P, x))$, where $f_l(P', a) = P$ and $P\varphi Q''$. So, $P\varphi Q''\varphi P''$ and $\delta_{l\varphi}(P'\varphi, a, P''\varphi) = \gamma$. Hence, $\delta_{l\varphi}$ is well-defined.

Now, we show that ω_{φ} is well-defined. Let $Q'\varphi, Q''\varphi \in \bar{Q}\ \varphi$ and $Q'\varphi = Q''\varphi$. Then $Q'\varphi Q''$. So, $\omega_l(Q'\varphi) = \omega_l(Q') = \omega_l(Q'') = \omega_l(Q''\varphi)$. Hence, the claim holds. □

Theorem 3.16. *Let $\tilde{F}_l = (\bar{Q}, X, \tilde{R} = (\{q_0\}, \mu^{t_0}(\{q_0\})), \bar{Z}, \omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2)$ be an accessible complete deterministic BL-GLFA. Then*

$$\frac{\tilde{F}_l}{\varphi} = (\bar{Q}\varphi, X, \tilde{R}/\varphi, \bar{Z}, \omega_{l\varphi}, \delta_{l\varphi}, f_{l\varphi}, \tilde{\delta\varphi}_l, F_1, F_2),$$

is a reduced BL-GLFA.

Proof. Let $Q'\varphi\varphi Q''\varphi$. We have to prove that $Q'\varphi = Q''\varphi$. It is suffices to show that $Q'\varphi Q''$. For every $x \in X^*$, $\omega_{l\varphi}(f_l(Q'\varphi, x)) = \omega_{l\varphi}(f_l(Q''\varphi, x))$. Also, $\omega_l(f_l(Q', x)) = \omega_{l\varphi}(f_l(Q'\varphi, x)) = \omega_{l\varphi}(f_l(Q''\varphi, x)) = \omega_l(f_l(Q'', x))$, for every $x \in X^*$. Hence, the claim holds. □

Theorem 3.17. *Let $\tilde{F}_l$ be a BL-GLFA. Then $\beta_{\frac{\tilde{F}}{\varphi}} = \beta_{\tilde{F}}$.*

Proof. By considering Theorem 3.16, $\beta_{\frac{\tilde{F}}{\varphi}} = \omega_{l\varphi}(f_l(\{q_0\})\varphi, x) = \omega_l(f_l(\{q_0\}), x) =$ $\beta_{\tilde{F}}$, for every $x \in X^*$.

□

Definition 3.18. Let β be a finite realization. A relation R_β on X^* is defined by:
For every two strings x and y in X^*, $xR_\beta y$ if for every $z \in X^*$ we have $\beta(xz) = \beta(yz)$.

Definition 3.19. Let $\tilde{F}$ be a deterministic BL-GLFA. Then for every string $x, y \in X^*$, $xR_F y$ if and only if there exists $Q' \in \bar{Q}$ such that $\tilde{\delta}^*_l((\{q_0\}, \mu^{t_0}(\{q_0\}))$, $x, Q') > 0$ if and only if $\tilde{\delta}^*_l((\{q_0\}, \mu^{t_0}(\{q_0\})), y, Q') > 0$.

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Lemma 3.20. *Let $\tilde{F}$ be a deterministic BL-GLFA. Then R_β is an equivalence relation, where $\beta_{\tilde{F}} = \beta$.*

Proof. It is clear that $xR_\beta x$ and if $xR_\beta y$, then $yR_\beta x$, for every $x, y \in X^*$. Now, let $xR_\beta y$ and $yR_\beta z$, for every $x, y, z \in X^*$. Then $\beta(xw) = \beta(yw) = \beta(zw)$, for every $w \in X^*$. Hence, R_β is an equivalence relation. □

Lemma 3.21. *R_F is an equivalence relation.*

Proof. It is clear that $xR_F x$ and if $xR_F y$, then $yR_F x$, for every $x, y \in X^*$. Now, let $xR_F y$ and $yR_F z$, for every $x, y, z \in X^*$. If there exists $Q' \in \bar{Q}$ such that $\tilde{\delta}^*_l((\{q_0\}, \mu^{t_0}(\{q_0\})), x, Q') > 0$, then $\tilde{\delta}^*_l((\{q_0\}, \mu^{t_0}(\{q_0\})), y, Q') > 0$. Also, by considering $yR_F z$, we have $\tilde{\delta}^*_l((\{q_0\}, \mu^{t_0}(\{q_0\})), Z, Q') > 0$. Therefore, $xR_F z$. Similarity, we can obtain the converse. Hence, the claim holds. □

Corollary 3.22. *By considering Definitions 3.19, for any complete deterministic BL-GLFA $\tilde{F}$, the number of classes of equivalence relation R_F is not more than the number of states of $\tilde{F}$.*

Theorem 3.23. *Let $\tilde{F}$ be a complete deterministic BL-GLFA. Then for a given equivalence class $[w]_{R_F}$ of R_F, there exists an equivalence class $[w]_{R_\beta}$ of R_β in which $[w]_{R_F} \subseteq [w]_{R_\beta}$. Every equivalence class $[w]_{R_\beta}$ of the relation R_β is a finite union of equivalence classes of R_F.*

Proof. Let $[w]_{R_F}$ be an equivalence class of R_F and $x \in [w]_{R_F}$. Since, $\tilde{F}$ is complete, then there is $Q' \in \bar{Q}$ such that $\tilde{\delta}_l((\{q_0\}, \mu^{t_0}(q_0)), x, Q') > 0$. Then by considering Definition 3.19, $\tilde{\delta}^*_l((\{q_0\}, \mu^{t_0}(\{q_0\})), w, Q') > 0$. Since, $\tilde{F}$ is a complete BL-GLFA, then there exists $Q'' \in \bar{Q}$ such that $\tilde{\delta}^*_l((\{q_0\}, \mu^{t_0}(q_0)), xz, Q'') > 0$ and $\tilde{\delta}^*_l((\{q_0\}, \mu^{t_0}(q_0)), wz, Q'') > 0$, for every $z \in X^*$. Therefore, $\beta(xz) = \beta(wz)$. So, $xR_\beta w$. Then $x \in [w]_{R_\beta}$ and $[w]_{R_F} \subseteq [w]_{R_\beta}$. Clearly, $\beta(x) = \beta(w)$, for every $x \in [w]_{R_\beta}$. Consider $S = \{Q' \in \bar{Q} | \tilde{\delta}^*_l((\{q_0\}, \mu^{t_0}(q_0)), x, Q') > 0, x \in [w]_{R_\beta}\}$. Hence, the equivalence class $[w]_{R_\beta}$ of R_β is a finite union of the equivalence classes $[w]_{R_F}$ of R_F. □

Theorem 3.24. *Let β be a finite realization. Then there exists a complete deterministic BL-GLFA $\tilde{F}_m$ such that $\beta(\tilde{F}_m) = \beta$ and $\tilde{F}_m$ is a minimal automaton.*

Proof. Let $\tilde{F}$ be a complete deterministic BL-GLFA such that $\beta(\tilde{F}) = \beta$. By considering Theorem 3.23, the number of equivalence classes of R_β is finite. Let $\bar{Q}_m$ be the set of equivalence classes of R_β i.e.,

$$\bar{Q}_m = \{[w] \big| [w] \text{ is an equivalence class of } R_\beta\}.$$

Consider $\tilde{R}_m = \{([\Lambda], 1)\}$. Define $\delta_m : \bar{Q}_m \times X \times \bar{Q}_m \to L$ by

$$\delta_{lm}([z], a, [x]) = \begin{cases} \alpha & if\ [za] = [x] \\ 0 & otherwise \end{cases}, \tag{3.2}$$

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Also, define $\omega_{lm} : \bar{Q}_m \to Z$ by $\omega_{lm}([z]) = \omega_l(\rho(z))$. It is obvious that, δ_m is well-defined. Now, we show that ω_{lm} is well-defined. Let $[z] = [w]$. Then $zR_\beta w$. Therefore, $\omega_{lm}([z]) = \omega_l(\rho(z)) = \omega_l(\rho(w)) = \omega_{lm}([w])$. Clearly, $\tilde{F}_m = (\bar{Q}_m, X, \tilde{R}_m, \bar{Z}, \omega_{lm}, \delta_{lm}, f_{lm}, \tilde{\delta}_{lm}, F_1, F_2)$ is a complete deterministic BL-GLFA. Now, we show that $\beta(\tilde{F}) = \beta(\tilde{F}_m)$. Then $\beta(\tilde{F}_m)(x) = \omega_{lm}(\rho_m(x)) = \omega_{lm}([x]) = \omega_l(\rho(x)) = \beta(\tilde{F})(x)$. □

Definition 3.25. Let $\tilde{F}_1 = (\bar{Q}_1, X, \tilde{R}_1, \bar{Z}_1, \omega_{l1}, \delta_{l1}, f_{l1}, \tilde{\delta}_{l1}, F_1, F_2)$ and $\tilde{F}_2 = (\bar{Q}_2, X, \tilde{R}_2, \bar{Z}_2, \omega_{l2}, \delta_{l2}, f_{l2}, \tilde{\delta}_{l2}, F_1, F_2)$ be two BL-GLFA. A homomorphism from $\tilde{F}_1$ onto $\tilde{F}_2$ is a function ξ from $\bar{Q}_1$ onto $\bar{Q}_2$ such that for every $Q', Q'' \in \bar{Q}_1$ and $u \in X$ the following conditions hold:

(1) $\delta_{l1}(Q', a, Q'') > 0$ if and only if $\delta_{l2}(\xi(Q'), a, \xi(Q''))$,
(2) $\omega_{l1}(Q') = B$ implies that $\omega_{l2}(\xi(Q')) = B$.

We say that ξ is isomorphism if and only if ξ is homomorphism, one-one and $\omega_{l1}(Q') = B$ if and only if $\omega_{l2}(\xi(Q')) = B$.

Theorem 3.26. *Let β be a finite realization and $\tilde{F}_m$ be the BL-GLFA defined in the proof of Theorem 3.24, and $\tilde{F}$ be a complete accessible deterministic reduced BL-GLFA. Then $\tilde{F}_m$ and $\tilde{F}$ are isomorphic.*

Proof. Let $\tilde{F} = (\bar{Q}, X, \tilde{R}, \bar{Z}, \omega_l, \delta_l, f_l, \tilde{\delta}_l, F_1, F_2)$ be a complete accessible deterministic reduced BL-GLFA and $\tilde{F}_m = (\bar{Q}_m, X, \tilde{R}_m, \bar{Z}, \omega_{lm}, \delta_{lm}, f_{lm}, \tilde{\delta}_{lm}, F_1, F_2)$ be the BL-GLFA defined in the proof of Theorem 3.24. Define $\xi : \bar{Q} \to \bar{Q}_m$ by $\xi(Q') = [u]$, where $f_l(\{q_0\}, u) = Q'$. First, let $Q' = Q''$, where $Q', Q'' \in \bar{Q}$. Then $Q' \varphi Q''$. Therefore, $\omega_l(f(Q', x)) = \omega_l(f(Q'', x))$, for every $x \in X^*$. Since $\tilde{F}$ is accessible, then there exists $u, v \in X^*$ such that $\tilde{\delta}_1((\{q_0\}, \mu^{t_0}(\{q_0\})), u, Q') > 0$. Since, $\tilde{F}$ is complete and deterministic, so $f_l(\{q_0\}, u) = Q'$. Also, $f_l(\{q_0\}, v) = Q''$. Therefore,

$$\begin{aligned}\beta(ux) &= \omega_l(f_l(\{q_0\}, ux))\\ &= \omega_l(f_l(f_l(\{q_0\}, u), x))\\ &= \omega_l(f_l(f_l(\{q_0\}, v), x)) = \beta(vx).\end{aligned}$$

Then $[u] = [v]$. Hence, $\xi(Q') = \xi(Q'')$ and ξ is well-defined.

Now, let $[u] \in \bar{Q}_m$. Since $\tilde{F}$ is complete, then there is $Q' \in \bar{Q}$ such that $f_l(\{q_0\}, u) = Q'$. So, $\xi(Q') = [u]$. Hence, ξ is onto.

Let $\tilde{\delta}^*((\{q_0\}, \mu^{t_0}(\{q_0\})), u, Q') > 0$. Then $\tilde{\delta}^*((\{q_0\}, \mu^{t_0}(\{q_0\})), ua, Q'') > 0$. Therefore, $\xi(Q') = [u]$, $\xi(Q'') = [ua]$ and $\delta_{lm}([u], a, [ua]) > 0$. So, $\delta_{lm}(\xi(Q'), a, \xi(Q'')) > 0$.

Now, let $\delta_{lm}(\xi(Q'), a, \xi(Q'')) > 0$, where $\xi(Q') = [u]$ and $\xi(Q'') = [v]$. Then $[ua] = [v]$. So, $f_l(\{q_0\}, u) = Q'$ and $f_l(\{q_0\}, ua) = Q''$. Hence, $\delta_l(Q', u, Q'') > 0$.

Let $\xi(Q') = [u]$. Then

$$\omega_l(\xi(Q')) = \omega_l([u]) = \omega_l(\rho(u)) = \omega_l(f_l(\{q_0\}, u)) = \omega_l(Q').$$

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Let $Q', Q'' \in \bar{Q}$ and $\xi(Q') = \xi(Q'')$. Then there exist $u, v \in X^*$ such that $[u] = \xi(Q') = \xi(Q'') = [v]$. So, $\beta(uz) = \beta(vz)$, for every $z \in X^*$. Therefore, $Q'\varphi Q''$. Since, $\tilde{F}$ is reduced, then $Q' = Q''$. So, ξ is one-to-one. Hence, $\tilde{F}_m$ and $\tilde{F}$ are isomorphic.

□

4. Conclusion

In this note, the notions of complete, deterministic, accessible and reduced for a BL-general L-fuzzy automaton is presented. After that, an algorithm for computing the complete BL-general L-fuzzy automata is given. Also, it is proved that for any finite realization β, there exists a minimal quotient complete deterministic BL-GLFA, where recognize β. After that, it is shown that any complete deterministic accessible reduced BL-GLFA is a minimal BL-GLFA. Moreover, it is proved that for any given finite realization β, the minimal quotient complete deterministic BL-GLFA recognizing β is isomorphic to any complete accessible deterministic reduced BL-GLFA recognizing β. For an accessible complete deterministic BL-GLFA $\tilde{F}$, an equivalence relation on states of $\tilde{F}$ is presented. Also, an algorithm for computing deterministic BL-general L-fuzzy automata is presented.

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