Iranian Journal of Mathematical Sciences and Informatics
Vol. 17, No. 2 (2022), pp 87-95
DOI: 10.52547/ijmsi.17.2.87

On a New Reverse Hilbert's Type Inequality

Gaowen Xi

Department of Mathematics, Chongqing University of Science and Technology, Chongqing, 401331, P. R. China

E-mail: xigaowen@163.com

Abstract. In this paper, by using the Euler-Maclaurin expansion for the Riemann-ζ function, we establish an inequality of a weight coefficient. Using this inequality, we derive a new reverse Hilbert's type inequality. As an applications, an equivalent form is obtained.

Keywords: Hilbert's type inequality, Weight coefficien, Hölder inequality, Riemann-ζ function, Reverse.

2000 Mathematics subject classification: 26D15.

1. Introduction

If $p, q > 1$, $\frac{1}{p}+\frac{1}{q}=1$, $a_n \geq 0$, $b_n \geq 0$, for $n \geq 1, n \in N$ and $0 < \sum\limits_{n=1}^{\infty} a_n^p < \infty$, $0 < \sum\limits_{n=1}^{\infty} b_n^q < \infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\left\{\sum_{n=1}^{\infty} a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} b_n^q\right\}^{\frac{1}{q}}, \tag{1.1}$$

and

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m,n\}} < pq\left\{\sum_{n=1}^{\infty} a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} b_n^q\right\}^{\frac{1}{q}}, \tag{1.2}$$

Received 26 September 2018; Accepted 22 May 2020

where the constant $\frac{\pi}{\sin\frac{\pi}{p}}$ and pq is best possible for each inequality respectively. Inequality (1.1) is Hardy-Hilbert's inequality. Inequality (1.2) is a Hilbert's type inequality [1].

In [5], [10] and [9], Krnic, Pecaric and Yang gave some generalization and reinforcement of inequality (1.1). In [3], Kuang and Debnath gave a reinforcement of inequality (1.2):

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m,n\}}<\left\{\sum_{n=1}^{\infty}[pq-G(p,n)]a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}[pq-G(q,n)]b_n^q\right\}^{\frac{1}{q}} \quad (1.3)$$

where $G(r,n)=\frac{r+\frac{1}{3r}-\frac{4}{3}}{(2n+1)^{\frac{1}{r}}}>0\ (r=p,q)$.

In [6] and [7], Xi gave a generalization and reinforcement of inequalities (1.2) and (1.3):

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^\lambda,n^\lambda\}} &< \left\{\sum_{n=1}^{\infty}\left[\kappa\left(\lambda\right)-\frac{1}{3qn^{\frac{q+\lambda-2}{q}}}\right]n^{1-\lambda}a_n^p\right\}^{\frac{1}{p}}\\ &\times\left\{\sum_{n=1}^{\infty}\left[\kappa\left(\lambda\right)-\frac{1}{3pn^{\frac{p+\lambda-2}{p}}}\right]n^{1-\lambda}b_n^q\right\}^{\frac{1}{q}},(1.4)\end{aligned}$$

where $\kappa(\lambda)=\frac{pq\lambda}{(p+\lambda-2)(q+\lambda-2)}>0$, $2-\min\{p,q\}<\lambda\le 2$.

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^\lambda+A,n^\lambda+B\}} &< \left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda)-\frac{1}{n^{\frac{q+\lambda-2}{q}}}\left(\frac{1}{3q}-\frac{B}{1+B}\right)\right]\right.\\ &\times n^{1-\lambda}a_n^p\Big\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda)-\frac{1}{n^{\frac{p+\lambda-2}{p}}}\left(\frac{1}{3p}-\frac{A}{1+A}\right)\right]n^{1-\lambda}b_n^q\right\}^{\frac{1}{q}}, \quad (1.5)\end{aligned}$$

For the reverse Hilbert's type inequality, In [8] , Xi and Wang gave a reverse Hilbert's type inequality:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^2,n^2\}} > 2\left[\sum_{n=1}^{\infty}\left(1-\frac{1}{2n}\right)\frac{1}{n}a_n^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}\frac{1}{n}b_n^q\right]^{\frac{1}{q}}. \quad (1.6)$$

In this paper, by introducing a parameter λ and using the Euler-Maclaurin expansion for the Riemann-ζ function, we establish an inequality of a weight coefficient. Using this inequality, we derive a reverse of the Hilbert's type inequality (1.4).

2. A Lemma

First, we need the following formula of the Riemann-ζ function (see [4], [12] and [11]):

[DOI: 10.52547/ijmsi.17.2.87]

$$\zeta(\sigma) = \sum_{k=1}^{n} \frac{1}{k^{\sigma}} - \frac{n^{1-\sigma}}{1-\sigma} - \frac{1}{2n^{\sigma}} - \sum_{k=1}^{l-1} \frac{B_{2k}}{2k} \begin{pmatrix} -\sigma \\ 2k-1 \end{pmatrix} \frac{1}{n^{\sigma+2k-1}}$$
$$- \frac{B_{2l}}{2l} \begin{pmatrix} -\sigma \\ 2l-1 \end{pmatrix} \frac{\varepsilon}{n^{\sigma+2l-1}}, \tag{2.1}$$

where $\sigma > 0$, $\sigma \neq 1$, $n,\ l \geq 1$, $n,\ l \in N$, $0 < \varepsilon = \varepsilon(\sigma, l, n) < 1$. The numbers $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0$, $B_4 = -1/30$, $\cdots$ are Bernoulli numbers. In particular, $\zeta(\sigma) = \sum_{k=1}^{\infty} \frac{1}{k^{\sigma}}$ $(\sigma > 1)$.

Since $\zeta(0) = -1/2$, then the formula of the Riemann-ζ function (2.1) is also true for $\sigma = 0$.

Lemma 2.1. *If* $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $2 - p < \lambda \leq 2$, $n \geq 1$ *and* $n \in N$, *then*

$$\omega(n, \lambda, q) = \sum_{k=1}^{\infty} \frac{1}{\max\{k^{\lambda}, n^{\lambda}\}} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{q}} > \frac{qn^{1-\lambda}}{q+\lambda-2}, \tag{2.2}$$

and

$$\omega(n, \lambda, p) = \sum_{k=1}^{\infty} \frac{1}{\max\{k^{\lambda}, n^{\lambda}\}} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}}$$
$$< n^{1-\lambda} \left[\kappa(\lambda) - \frac{p+2}{2(p+\lambda-2)n^{\frac{p+\lambda-2}{p}}}\right], \tag{2.3}$$

where $\kappa(\lambda) = \frac{pq\lambda}{(p+\lambda-2)(q+\lambda-2)}$.

Proof. Equalities (2.2) and (2.3) define the weight coefficient. When $2 - p < \lambda \leq 2$, taking $\sigma = \frac{2-\lambda}{p} \geq 0$, $l = 1$, in (2.1), we obtain

$$\zeta\left(\frac{2-\lambda}{p}\right) = \sum_{k=1}^{n} \frac{1}{k^{\frac{2-\lambda}{p}}} - \frac{pn^{\frac{p+\lambda-2}{p}}}{p+\lambda-2} - \frac{1}{2n^{\frac{2-\lambda}{p}}} + \frac{2-\lambda}{12pn^{1+\frac{2-\lambda}{p}}}\varepsilon_1, \tag{2.4}$$

where $0 < \varepsilon_1 < 1$.

Since $\frac{2}{q} + \frac{\lambda}{p} = 2 + \frac{\lambda-2}{p} = \frac{2p+\lambda-2}{p} > 0 (p + \lambda - 2 > 0)$. Taking $\sigma = \frac{\lambda}{p} + \frac{2}{q}$, $l = 1$, we obtain

$$\zeta\left(\frac{2}{q} + \frac{\lambda}{p}\right) = \sum_{k=1}^{n-1} \frac{1}{k^{\frac{2}{q}+\frac{\lambda}{p}}} + \frac{pn^{-\frac{p+\lambda-2}{p}}}{p+\lambda-2} + \frac{1}{2n^{\frac{2}{q}+\frac{\lambda}{p}}} + \frac{q\lambda+2p}{12pqn^{1+\frac{2}{q}+\frac{\lambda}{p}}}\varepsilon_2, \tag{2.5}$$

where $0 < \varepsilon_2 < 1$.

Since $\frac{2}{p} + \frac{\lambda}{q} = 2 + \frac{\lambda-2}{q} = \frac{2q+\lambda-2}{q} > 0 (q + \lambda - 2 < 0, q < 0)$. Taking $\sigma = \frac{2}{p} + \frac{\lambda}{q}$, $l = 1$, we obtain

$$\zeta\left(\frac{2}{p} + \frac{\lambda}{q}\right) = \sum_{k=1}^{n-1} \frac{1}{k^{\frac{2}{p}+\frac{\lambda}{q}}} + \frac{qn^{-\frac{q+\lambda-2}{q}}}{q+\lambda-2} + \frac{1}{2n^{\frac{2}{p}+\frac{\lambda}{q}}} + \frac{p\lambda+2q}{12pqn^{1+\frac{2}{p}+\frac{\lambda}{q}}}\varepsilon_3, \tag{2.6}$$

[DOI: 10.52547/ijmsi.17.2.87]

where $0 < \varepsilon_3 < 1$.

In addition,

$$\begin{aligned}
\omega(n,\lambda,q) &= \sum_{k=1}^{\infty} \frac{1}{\max\{k^\lambda, n^\lambda\}} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{q}} \\
&= \sum_{k=1}^{n} \frac{1}{\max\{k^\lambda, n^\lambda\}} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{q}} - \frac{1}{n^\lambda} + \sum_{k=n}^{\infty} \frac{1}{\max\{k^\lambda, n^\lambda\}} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{q}} \\
&= \sum_{k=1}^{n} \frac{1}{n^\lambda} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{q}} - \frac{1}{n^\lambda} + \sum_{k=n}^{\infty} \frac{1}{k^\lambda} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{q}} \\
&= \frac{1}{n^{\frac{(q+1)\lambda-2}{q}}} \sum_{k=1}^{n} \frac{1}{k^{\frac{2-\lambda}{q}}} - \frac{1}{n^\lambda} + n^{\frac{2-\lambda}{q}} \sum_{k=n}^{\infty} \frac{1}{k^{\frac{\lambda}{p}+\frac{2}{q}}} \\
&> \frac{1}{n^{\frac{(q+1)\lambda-2}{q}}} - \frac{1}{n^\lambda} + n^{\frac{2-\lambda}{q}} \sum_{k=n}^{\infty} \frac{1}{k^{\frac{\lambda}{p}+\frac{2}{q}}}.
\end{aligned}$$

By (2.5) and $\frac{2}{q} + \frac{\lambda}{p} = \frac{q\lambda+2p}{pq} > 0$

$$\begin{aligned}
\omega(n,\lambda,q) &> \frac{1}{n^{\frac{(p+1)\lambda-2}{p}}} - \frac{1}{n^\lambda} + n^{\frac{2-\lambda}{q}} \left[\frac{pn^{-\frac{p+\lambda-2}{p}}}{p+\lambda-2} + \frac{1}{2n^{\frac{2}{q}+\frac{\lambda}{p}}} + \frac{q\lambda+2p}{12pqn^{1+\frac{2}{q}+\frac{\lambda}{p}}}\varepsilon_2 \right] \\
&> \frac{1}{n^{\frac{(p+1)\lambda-2}{p}}} - \frac{1}{n^\lambda} + n^{\frac{2-\lambda}{q}} \left[\frac{pn^{-\frac{p+\lambda-2}{p}}}{p+\lambda-2} + \frac{1}{2n^{\frac{2}{q}+\frac{\lambda}{p}}} \right] \\
&= \frac{1}{n^{\frac{(p+1)\lambda-2}{p}}} - \frac{1}{n^\lambda} + \frac{qn^{\,1-\lambda}}{q+\lambda-2} + \frac{1}{2n^\lambda} \\
&= \frac{1}{n^{\frac{(p+1)\lambda-2}{p}}} - \frac{1}{2n^\lambda} + \frac{qn^{\,1-\lambda}}{q+\lambda-2} \\
&> \frac{qn^{\,1-\lambda}}{q+\lambda-2}.
\end{aligned}$$

Using the last result and the inequality for $\omega(n,\lambda,q)$ above, we obtain (2.2).

$$\begin{aligned}
\omega(n,\lambda,p) &= \sum_{k=1}^{\infty} \frac{1}{\max\{k^\lambda, n^\lambda\}} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}} \\
&= \sum_{k=1}^{n} \frac{1}{\max\{k^\lambda, n^\lambda\}} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}} - \frac{1}{n^\lambda} + \sum_{k=n}^{\infty} \frac{1}{\max\{k^\lambda, n^\lambda\}} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}} \\
&= \sum_{k=1}^{n} \frac{1}{n^\lambda} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}} - \frac{1}{n^\lambda} + \sum_{k=n}^{\infty} \frac{1}{k^\lambda} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}} \\
&= \frac{1}{n^{\frac{(p+1)\lambda-2}{p}}} \sum_{k=1}^{n} \frac{1}{k^{\frac{2-\lambda}{p}}} - \frac{1}{n^\lambda} + n^{\frac{2-\lambda}{p}} \sum_{k=n}^{\infty} \frac{1}{k^{\frac{2}{p}+\frac{\lambda}{q}}}.
\end{aligned}$$

By (2.4) and (2.6)

$$
\begin{aligned}
\omega(n,\lambda,p) &< \frac{1}{n^{\frac{(p+1)\lambda-2}{p}}}\left[\zeta\left(\frac{2-\lambda}{p}\right)+\frac{pn^{\frac{p+\lambda-2}{p}}}{p+\lambda-2}+\frac{1}{2n^{\frac{2-\lambda}{p}}}\right]-\frac{1}{n^{\lambda}} \\
&\quad +n^{\frac{2-\lambda}{p}}\left[\frac{qn^{-\frac{q+\lambda-2}{q}}}{q+\lambda-2}+\frac{1}{2n^{\frac{2}{p}+\frac{\lambda}{q}}}+\frac{p\lambda+2q}{12p\,qn^{1+\frac{2}{p}+\frac{\lambda}{q}}}\right] \\
&= \frac{1}{n^{\frac{(p+1)\lambda-2}{p}}}\zeta\left(\frac{2-\lambda}{p}\right)+\frac{pn^{\,1-\lambda}}{p+\lambda-2}+\frac{1}{2n^{\lambda}}-\frac{1}{n^{\lambda}}+\frac{qn^{\,1-\lambda}}{q+\lambda-2} \\
&\quad +\frac{1}{2n^{\lambda}}+\frac{p\lambda+2q}{12p\,qn^{1+\lambda}} \\
&= \frac{1}{n^{\frac{(p+1)\lambda-2}{p}}}\zeta\left(\frac{2-\lambda}{p}\right)+\frac{pq\,\lambda\;n^{\,1-\lambda}}{(p+\lambda-2)(q+\lambda-2)}+\frac{p\lambda+2q}{12pqn^{1+\lambda}} \\
&= n^{1-\lambda}\left\{\frac{pq\,\lambda}{(p+\lambda-2)(q+\lambda-2)}-\frac{1}{n^{\frac{p+\lambda-2}{p}}}\left[-\zeta\left(\frac{2-\lambda}{p}\right)\right.\right. \\
&\quad \left.\left.-\frac{p\lambda+2q}{12p\,qn^{\frac{p-\lambda+2}{p}}}\right]\right\}.
\end{aligned}
$$

In (2.4), taking $n=1$, by $2-p<\lambda\leq 2$, we obtain

$$
\begin{aligned}
\zeta\left(\frac{2-\lambda}{p}\right) &= 1-\frac{p}{p+\lambda-2}-\frac{1}{2}+\frac{(2-\lambda)\varepsilon_1}{12p} \\
&< \frac{1}{2}-\frac{p}{p+\lambda-2}+\frac{2-\lambda}{12p} \\
&= -\frac{(\lambda-2-3p)(\lambda-2-2p)}{12p(p+\lambda-2)} \\
&< 0.
\end{aligned}
$$

So for $n\geq 1$, $n\in N$, $2-p<\lambda\leq 2$, we have

$$
\begin{aligned}
&-\zeta\left(\frac{2-\lambda}{p}\right)+\frac{2-\lambda}{12pn^{1+\frac{2-\lambda}{p}}} \\
&\leq -\zeta\left(\frac{2-\lambda}{p}\right)+\frac{2-\lambda}{12p} \\
&= \frac{(\lambda-2-3p)(\lambda-2-2p)}{12p(p+\lambda-2)}+\frac{2-\lambda}{12p}
\end{aligned}
$$

[DOI: 10.52547/ijmsi.17.2.87]

$$
\begin{aligned}
&= \frac{(\lambda-2-3p)(\lambda-2-2p)+(2-\lambda)(p+\lambda-2)}{12p(p+\lambda-2)}\\
&= \frac{(\lambda-2)^2+(-5p-p-\lambda+2)(\lambda-2)+6p^2}{12p(p+\lambda-2)}\\
&= \frac{6p(2-\lambda)+6p^2}{12p(p+\lambda-2)}\\
&= \frac{(2-\lambda)+p}{2(p+\lambda-2)}\\
&\le \frac{2+p}{2(p+\lambda-2)}.
\end{aligned}
$$

Using the last result and the inequality for $\omega(n,\lambda,p)$ above, we obtain (2.3). □

3. Main Results

Theorem 3.1. *If* $0<p<1$, $\frac{1}{p}+\frac{1}{q}=1$, $2-p<\lambda\le 2$, $a_n\ge 0$, $b_n\ge 0$, *for* $n\ge 1, n\in N$ *and* $0<\sum\limits_{n=1}^{\infty}a_n^p<\infty$, $0<\sum\limits_{n=1}^{\infty}b_n^q<\infty$, *then*

$$
\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^\lambda,n^\lambda\}} > \left\{\sum_{n=1}^{\infty}\frac{q}{q+\lambda-2}n^{1-\lambda}a_n^p\right\}^{\frac{1}{p}} \times\left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda)-\frac{p+2}{2(p+\lambda-2)n^{\frac{p+\lambda-2}{p}}}\right]n^{1-\lambda}b_n^q\right\}^{\frac{1}{q}}, \quad (3.1)
$$

where $\kappa(\lambda)=\frac{pq\lambda}{(p+\lambda-2)(q+\lambda-2)}>0$.

Proof. By the reverse Hölder 's inequality [2], we have

$$
\begin{aligned}
\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^\lambda,n^\lambda\}} &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[\frac{a_m}{\max\{m^\lambda,n^\lambda\}^{\frac{1}{p}}}\left(\frac{m}{n}\right)^{\frac{2-\lambda}{pq}}\right]\\
&\quad\times\left[\frac{b_n}{\max\{m^\lambda,n^\lambda\}^{\frac{1}{q}}}\left(\frac{n}{m}\right)^{\frac{2-\lambda}{pq}}\right]\\
&\ge \left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[\frac{a_m^p}{\max\{m^\lambda,n^\lambda\}}\left(\frac{m}{n}\right)^{\frac{2-\lambda}{q}}\right]\right\}^{\frac{1}{p}}\\
&\quad\times\left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[\frac{b_n^q}{\max\{m^\lambda,n^\lambda\}}\left(\frac{n}{m}\right)^{\frac{2-\lambda}{p}}\right]\right\}^{\frac{1}{q}}\\
&= \left\{\sum_{m=1}^{\infty}\omega(m,\lambda,q)a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\omega(n,\lambda,p)b_n^q\right\}^{\frac{1}{q}}.
\end{aligned}
$$

[DOI: 10.52547/ijmsi.17.2.87]

Since $0 < p < 1$ and $q < 0$. By (2.2), (2.3), we obtain (3.1). Theorem 3.1 is proved. □

Theorem 3.2. *If* $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $2 - p < \lambda \le 2$, $a_n \ge 0$, $b_n \ge 0$, *for* $n \ge 1, n \in N$ *and* $0 < \sum_{n=1}^{\infty} n^{1-\lambda} b_n^q < \infty$, *then*

$$\sum_{n=1}^{\infty}\left(\frac{q}{q+\lambda-2}n^{1-\lambda}\right)^{1-q}\left(\sum_{m=1}^{\infty}\frac{b_m}{\max\{m^\lambda, n^\lambda\}}\right)^q > \sum_{n=1}^{\infty}\left[\kappa(\lambda) - \frac{p+2}{2(p+\lambda-2)n^{\frac{p+\lambda-2}{p}}}\right] n^{1-\lambda} b_n^q. \tag{3.2}$$

where $\kappa(\lambda) = \frac{p\,q\lambda}{(p+\lambda-2)(q+\lambda-2)} > 0$.

Inequalities (3.2) *and* (3.1) *are equivalent.*

Proof. Let

$$a_n = \left(\frac{q}{q+\lambda-2}n^{1-\lambda}\right)^{1-q}\left[\sum_{m=1}^{\infty}\frac{b_m}{\max\{m^\lambda, n^\lambda\}}\right]^{q-1}, \qquad n \in N.$$

By (3.1), we have

$$\begin{aligned}\left\{\sum_{n=1}^{\infty}\frac{q}{q+\lambda-2}n^{1-\lambda}a_n^p\right\}^q &= \left\{\sum_{n=1}^{\infty}\left(\frac{q}{q+\lambda-2}n^{1-\lambda}\right)^{1-q}\right.\\ &\quad\left.\times\left(\sum_{m=1}^{\infty}\frac{b_m}{\max\{m^\lambda, n^\lambda\}}\right)^q\right\}^q\\ &= \left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_n b_m}{\max\{m^\lambda, n^\lambda\}}\right\}^q\\ &> \left\{\sum_{n=1}^{\infty}\frac{q}{q+\lambda-2}n^{1-\lambda}a_n^p\right\}^{q-1}\left\{\sum_{n=1}^{\infty}[\kappa(\lambda)\right.\\ &\quad\left.-\frac{p+2}{2(p+\lambda-2)n^{\frac{p+\lambda-2}{p}}}\right] n^{1-\lambda}b_n^q\Bigg\}.\end{aligned}$$

Then we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{q}{q+\lambda-2}n^{1-\lambda}a_n^p &= \sum_{n=1}^{\infty}\left(\frac{q}{q+\lambda-2}n^{1-\lambda}\right)^{1-q}\left(\sum_{m=1}^{\infty}\frac{b_m}{\max\{m^\lambda, n^\lambda\}}\right)^q\\ &> \sum_{n=1}^{\infty}\left[\kappa(\lambda) - \frac{p+2}{2(p+\lambda-2)n^{\frac{p+\lambda-2}{p}}}\right] n^{1-\lambda}b_n^q.\end{aligned}$$

[DOI: 10.52547/ijmsi.17.2.87]

On the other-hand, by the reverse Hölder 's inequality [2], we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_n b_m}{\max\{m^\lambda, n^\lambda\}} = \sum_{n=1}^{\infty}\left[\left(\frac{q}{q+\lambda-2}n^{1-\lambda}\right)^{-\frac{1}{p}}\sum_{m=1}^{\infty}\frac{b_m}{\max\{m^\lambda, n^\lambda\}}\right] \times\left[\left(\frac{q}{q+\lambda-2}n^{1-\lambda}\right)^{\frac{1}{p}}a_n\right]$$
$$\geq \left[\sum_{n=1}^{\infty}\left(\frac{q}{q+\lambda-2}n^{1-\lambda}\right)^{1-q}\left(\sum_{m=1}^{\infty}\frac{b_m}{\max\{m^\lambda, n^\lambda\}}\right)^q\right]^{\frac{1}{q}} \times\sum_{n=1}^{\infty}\left[\frac{q}{q+\lambda-2}n^{1-\lambda}a_n^p\right]^{\frac{1}{p}}.$$

From (3.2), it follows that

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_n b_m}{\max\{m^\lambda, n^\lambda\}} > \sum_{n=1}^{\infty}\left[\frac{q}{q+\lambda-2}n^{1-\lambda}a_n^p\right]^{\frac{1}{p}} \times\left[\sum_{n=1}^{\infty}\left(\kappa(\lambda)-\frac{p+2}{2(p+\lambda-2)n^{\frac{p+\lambda-2}{p}}}\right)n^{1-\lambda}b_n^q\right]^{\frac{1}{q}}.$$

Then, (3.2) and (3.1) are equivalent. Theorem 3.2 is proved. □

In inequality (3.1), taking $\lambda = 2$, we obtain:

Corollary 3.3. *If* $0 < p < 1$, $\frac{1}{p}+\frac{1}{q}=1$,$a_n \geq 0$, $b_n \geq 0$, *for* $n \geq 1, n \in N$ *and* $0 < \sum_{n=1}^{\infty} a_n^p < \infty$, $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, *then*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^2, n^2\}} > \left\{\sum_{n=1}^{\infty}\frac{1}{n}a_n^p\right\}^{\frac{1}{p}} \times\left\{\sum_{n=1}^{\infty}\left[1-\frac{p+2}{2pn}\right]\frac{1}{n}b_n^q\right\}^{\frac{1}{q}}. \quad (3.3)$$

Acknowledgments

The author would like to thank the referee made several helpful comments. This research is funded by Research Foundation of Chongqing University of Science and Technology, the project No. is ck2017zkyb019.

References

1. G. Hardy, J. E. Littlewood, G. Polya, *Inequalities,* Cambridge Univ. Press, 1934.
2. J. Kuang, *Applied Inequalities,* Sandong Science and Technology Press, Jinan, 2004.
3. J. Kuang, L. Debnath, On New Generalizations of Hilbert's Inequality and Their Applications, *J. Math. Anal. Appl.*, **245**, (2000), 248-265.

[DOI: 10.52547/ijmsi.17.2.87]

4. K. Knopp, *Theory and Application of Infinite Series,* Blackie & Son Limited, London, 1928.
5. M. Krnic, J. Pecaric, General Hilbert's and Hardy's Inequalities, *Math. Inequal. Appl.*, **8**(1), (2005), 29-51.
6. G. Xi, On Generalizations and Reinforcements of a Hilbert's Type Inequality, *Rocky Mountain J. Mathematics*, **42**(1), (2012), 329-337.
7. G. Xi, A Generalization of the Hilbert's Type Inequality, *Mathematical Inequalities & Applications*, **18**(4), (2015), 1501-1510.
8. G. Xi, Y. Xi, L. Wang, A Reverse Hilbert's Type Inequality, *Applied Mechanics and Materials*, **701-702**, (2015), 06-109.
9. B. Yang, A New Hilbert-Type Inequality, *Bull. Belg. Math. Soc.*, **13**, (2006), 479-487.
10. B. Yang, On Best Extensions of Hardy-Hilbert's Inequality with Two Parameters, *J. Inequal. Pure Appl. Math.*, **6**(3), (2005), Article 81.
11. B. Yang, On a Strengthened Version of the More Accurate Hardy-Hilbert's Inequality, *Acta. Math. Sinica*, **42**(6), (1999), 1103-1110.
12. B. Yang, Y. Zhu, Inequality on the Hurwitz Zeta-Function Restricted to the axis of Positive Reals, *Acta Scientiarum Naturalium Universitatis Sunyatseni*, **36**(2), (1997), 30-35.

[DOI: 10.52547/ijmsi.17.2.87]