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Quaternionic Product of Circles and Cycles and Octonionic Product for Pairs of Circles

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Abstract. This paper concerns with a product of circles induced by the quaternionic product considered in a projective manner. Several properties of this composition law are derived and on this way we arrive at some special numbers as roots or powers of unit. We extend this product to cycles as oriented circles and to pairs of circles by using the algebra of octonions. Three applications of the given products are proposed.

Keywords: Circle, Quaternion, Product, Octonion, Projective Geometry.

2010 Mathematics subject classification: 51N20, 51N14, 11R52, 11R06.

1. Introduction

The aim of this paper is to introduce some products on the set of circles, cycles and spheres. For circles considered in a projective way we use the well-known product of quaternions to define a first product, denoted $\odot_c$, while cycles are defined as oriented circles. By restring $\odot_c$ to the set of circles centered in the origin O we derive a geometric image for a product $R_1 \odot_c R_1$ by taking into account the intersection between a fixed circle $C(O,R)$ and a translated one $C^t(O,R_2)$. In this geometrical setting a new product, denoted $\odot_h$, appears in a natural manner. A detailed study of both these products is the content of section 1. By looking to examples as well as to roots/powers of the unit 1 we obtain some remarkable numbers, some of them algebraic but other of difficult nature.

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Section 2 deals with an octonionic product of pairs of circles inspired by the expression of an octonion as a pair of quaternions. For this new composition law we compute the square of a fixed pair and several products involving the unit circle S^1. We note that the products of the first section are commutative while the considered product of pairs of circles is not.

In the last section we propose three applications of the circle products. The first two of them regards hyperbolic objects, namely the equilateral hyperbola and hyperbolic matrices, but, in general, concerns with multi-valued maps. The last possible application returns to the Euclidean plane geometry and defines a chain of labels for a given polygon with lengths of edges greater than or equal to 1. The usual right triangle with sides $(3, 4, 5)$ is exemplified.

2. Quaternionic product of circles and quaternionic product of cycles

The starting point of this paper is the identification of a given circle C in the Euclidean plane with coordinates (x, y):

$$C : x^2 + y^2 + ax + by + c = 0 \tag{2.1}$$

with a quaternion:

$$q(C) = \bar{k} + a\bar{\imath} + b\bar{\jmath} + c = (c, a, b, 1) \in \mathbb{R}^4. \tag{2.2}$$

The quaternion $q(C)$ is pure imaginary if and only if the origin $O(0, 0)$ belongs to C. Let us point out that the given circle is expressed in a *projective* manner since the coefficient of the quadratic part is chosen as being 1. As is presented in [4, p. 70] the set of circles in a plane is a 3-dimensional projective subspace of the 5-dimensional projective space of conics. Our study will be a mix of elements from Euclidean and projective geometry.

From the real algebra structure of the quaternions it follows a product of circles:

$$C_1 \odot_c C_2 := q^{-1}(q(C_1) \cdot q(C_2)). \tag{2.3}$$

For $C_i, i = 1, 2$ given by (a_i, b_i, c_i) we derive immediately:

$$q(C_1 \odot_c C_2) = (c_1 + c_2)\bar{k} + (b_1 - b_2 + a_1c_2 + a_2c_1)\bar{\imath} + (a_2 - a_1 + b_1c_2 + b_2c_1)\bar{\jmath} + \\ +(c_1c_2 - 1 - a_1a_2 - b_1b_2) \tag{2.4}$$

which gives a non-commutative expression for the coefficients of $\bar{\imath}$ and $\bar{\jmath}$. We remark the degenerated case $c_2 = -c_1$.

Due to the chosen projective setting we restrict our study to circles $C(O, R)$ centered in O and having the radius R; hence their set is a 1-dimensional projective subspace of the projective spaces considered above. For such a circle we have:

$$C(O, R) : (a, b, c) = (0, 0, -R^2) \tag{2.5}$$

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and hence the equation (2.4) yields:

$$q(C(O,R_1)\odot_c C(O,R_2)) = (c_1+c_2)\bar{k}+(c_1c_2-1) = -(R_1^2+R_2^2)\bar{k}+[(R_1R_2)^2-1]. \tag{2.6}$$

From the properties of quaternionic product we have that the product above can be also expressed in matrix product manner:

$$(-R_2^2,0,0,1)\cdot\begin{pmatrix} -R_1^2 & 0 & 0 & 1 \\ 0 & -R_1^2 & 1 & 0 \\ 0 & -1 & -R_1^2 & 0 \\ -1 & 0 & 0 & -R_1^2 \end{pmatrix} = ((R_1R_2)^2-1,0,0,-(R_1^2+R_2^2)). \tag{2.7}$$

This circle is a real one for $R_1, R_2 \geq 1$ and we derive the product law:

$$C(O,R_1)\odot_c C(O,R_2) = C(O,R), \quad R := \sqrt{\frac{(R_1R_2)^2-1}{R_1^2+R_2^2}}. \tag{2.8}$$

In conclusion, on the set $M = [1,+\infty]$ we define a *non-internal* law of composition:

$$R_1 \odot_c R_2 := \sqrt{\frac{(R_1R_2)^2-1}{R_1^2+R_2^2}} < \min\{R_1,R_2\} \tag{2.9}$$

and the rest of this section concerns with several of its properties.

Property 1 Is commutative, associative and with neutral element $R=\infty$:

$$R_1 \odot_c R_2 \odot_c R_3 = \sqrt{\frac{(R_1R_2R_3)^2-(R_1^2+R_2^2+R_3^2)}{(R_1R_2)^2+(R_2R_3)^2+(R_3R_1)^2-1}}. \tag{2.10}$$

Property 2 With $R_i = \tan\varphi_i$ we get:

$$\tan\varphi_1 \odot_c \tan\varphi_2 := \sqrt{\frac{-\cos(\varphi_2-\varphi_1)\cos(\varphi_1+\varphi_2)}{\sin^2\varphi_1\cos^2\varphi_2+\sin^2\varphi_2\cos^2\varphi_1}}. \tag{2.11}$$

Property 3 Concerning the unit circle we have:

$$R\odot_c 1 = \sqrt{\frac{R^2-1}{R^2+1}} < 1 \leq R, \quad \lim_{R\to+\infty}(R\odot_c 1) = 1. \tag{2.12}$$

In particular, the unit circle is the square root of the degenerate circle: $S^1 \odot_c S^1 = \{O\}$; in fact $[q(S^1)]^2 = (\bar{k}-1)^2 = -2\bar{k}$. Trigonometrically:

$$\tan\varphi \odot_c 1 = \sqrt{-\cos(2\varphi)}, \quad \varphi \in \left[\frac{\pi}{4},\frac{\pi}{2}\right]. \tag{2.13}$$

Property 4 Concerning the squares we have:

$$R^2_{\odot_c} = \frac{\sqrt{R^4-1}}{\sqrt{2}R} < R, \quad (\tan\varphi)^2_{\odot_c} = \frac{\sqrt{-\cos(2\varphi)}}{\sin(2\varphi)}. \tag{2.14}$$

Hence, the square root of 1 is the number:

$$\sqrt[c]{1} := \sqrt{1+\sqrt{2}} = 1.553773... = 2^{\frac{3}{4}}\cos\frac{\pi}{8}, \quad (\sqrt[c]{1})^4 - 2(\sqrt[c]{1})^2 - 1 = 0 \tag{2.15}$$

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while the square root of $\sqrt[c]{1}$ is the number:

$$\sqrt[2c]{1} := \sqrt{1+\sqrt[4]{2}}\cdot\sqrt[4]{1+\sqrt{2}} = 1.844324..., \quad (\sqrt[2c]{1})^2_{\odot_c} = \sqrt[c]{1}. \tag{2.16}$$

Let us remark that $\sqrt[c]{1}$ is the usual (i.e. real) square root of *the silver ratio* $\Psi := 1+\sqrt{2}$ and an interesting property is given in Corollary 10.2. of [2, p. 26]: $\sqrt[c]{1}$ cannot be expressed as a finite sum of real numbers of type $\pm\sqrt[n_i]{a_i}$, $1 \leq i \leq r$, where $r, n_1, ..., n_r, a_1, ..., a_r \in \mathbb{N}^*$.

Also, the first relation (2.14) means that $\odot_c$ is a "shrinking" composition and we point out that Ψ is a quadratic Pisot-Vijayaraghavan number considered as solution of:

$$x^2 - 2x - 1 = 0. \tag{2.17}$$

The conjugate of Ψ with respect to this algebraic equation is:

$$-\Psi^{-1} = 1-\sqrt{2} = -0.44.... \tag{2.18}$$

Let us point out that from the point of view of endomorphisms on smooth manifolds the silver mean is treated in [5, p. 16] and a fourth order square root of unit is called *almost electromagnetic structure* in [8, p. 721]. The continuous fraction of these remarkable numbers are easy to compute with Mathematica:

$$\sqrt[c]{1} = [1; 1, 1, 4, 6, 1, 2, 2, 2, 1, 1, 6, ...], \ \sqrt[2c]{1} = [1; 1, 5, 2, 2, 1, 3, 2, 2, 4, ...]. \tag{2.19}$$

Property 5 We have an interesting relationship between the right hand side of the first equation (2.14) and a research of Fagnano on lemniscate from 1718. More precisely, the substitution $R = \frac{1}{u}$ in the obtained ratio gives:

$$\frac{\sqrt{R^4-1}}{\sqrt{2}R} = \frac{\sqrt{1-u^4}}{\sqrt{2}u} \tag{2.20}$$

and we reformulate the Fagnano result after [9, p. 15]:

Proposition 2.1. *If:*

$$\frac{\sqrt{1-u^4}}{\sqrt{2}u} = \frac{z}{\sqrt{1-\sqrt{1-z^4}}} \tag{2.21}$$

then:

$$\frac{du}{\sqrt{1-u^4}} = \frac{dz}{2\sqrt{1-z^4}}. \tag{2.22}$$

The usual inverses of $\sqrt[c]{1}$ and $\sqrt[2c]{1}$ are:

$$\frac{1}{\sqrt[c]{1}} = \sqrt{\sqrt{2}-1} = 0.643594... = 2^{\frac{3}{4}}\sin\frac{\pi}{8}, \quad \frac{1}{\sqrt[2c]{1}} = 0.5442299.... \tag{2.23}$$

In conclusion, we have the non-internal composition law on the set $M^{-1} = (0, 1]$:

$$u_1 \odot_c u_2 := \sqrt{\frac{1-(u_1u_2)^2}{u_1^2+u_2^2}}. \tag{2.24}$$

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For example:

$$\left(\frac{1}{2}\right)^2_{\odot_c} = \sqrt{\frac{3}{2}} > 1, \quad \sin t \odot_c \cos t = \sqrt{1 - \frac{\sin^2(2t)}{4}} \tag{2.25}$$

$$\sin t \odot_c 1 = \frac{\cos t}{\sqrt{1+\sin^2 t}}, \quad \cos t \odot_c 1 = \frac{\sin t}{\sqrt{1+\cos^2 t}},$$

$$\tan\varphi \odot_c 1 = \sqrt{\cos(2\varphi)}, \quad \varphi \in \left[0, \frac{\pi}{2}\right]. \tag{2.26}$$

We think the last relation explains the appearance of lemniscate in our study.

Property 6 Let $a < b < c$ be a Pythagorean triple: $a^2 + b^2 = c^2$. We consider $R_1 = \frac{c}{a}$ and $R_2 = \frac{c}{b}$:

$$\begin{cases} \frac{c}{a} \odot_c \frac{c}{b} = \frac{\sqrt{c^4-(ab)^2}}{c} = \frac{\sqrt{a^4+a^2b^2+b^4}}{c} \\ \frac{c}{a} \odot_c 1 = \frac{b}{\sqrt{a^2+c^2}}, \quad \frac{c}{b} \odot_c 1 = \frac{a}{\sqrt{b^2+c^2}}. \end{cases} \tag{2.27}$$

Property 7 From (2.10) we obtain also the third powers:

$$R^3_{\odot_c} = R\sqrt{\frac{R^4-3}{3R^4-1}}, \quad R \in [\sqrt[4]{3} = 1.316074, +\infty) \tag{2.28}$$

$$(R^2_{\odot_c}) \odot_c 1 = \sqrt{\frac{R^4-2R^2-1}{R^4+2R^2-1}}. \tag{2.29}$$

Property 8 In the following we propose a geometric interpretation of the considered product of $C_i(O, R_i)$. Suppose $R_1 \le R_2$ and translate C_2 to have the new center $A(d, 0)$. Suppose that C_1 and the translated C_2^t intersect in the points $P_{1,2}(x_0, \pm y_0)$ and we search d such that $y_0 = R_1 \odot_c R_2$. The points P are given by the system:

$$x^2 + y^2 = R_1^2, \quad (x-d)^2 + y^2 = R_2^2. \tag{2.30}$$

Solving for x it results ([1]):

$$x_0 = \frac{d^2 + R_1^2 - R_2^2}{2d} \tag{2.31}$$

and returning to the first equation we arrive at the following equation in d:

$$R_1^2 - \frac{(R_1R_2)^2 - 1}{R_1^2 + R_2^2} = \left(\frac{d^2 + R_1^2 - R_2^2}{2d}\right)^2. \tag{2.32}$$

Its solution greater than R_1 is:

$$d = \frac{\sqrt{R_1^4+1} + \sqrt{R_2^4+1}}{\sqrt{R_1^2+R_2^2}} \tag{2.33}$$

and the corresponding x_0:

$$x_0 = \frac{\sqrt{R_1^4+1}}{\sqrt{R_1^2+R_2^2}} < R_1. \tag{2.34}$$

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The symmetric expression of d above yields the possibility to define a new commutative product, denoted $\odot_h$ from "horizontal":

$$R_1 \odot_h R_2 := \frac{\sqrt{R_1^4+1}+\sqrt{R_2^4+1}}{\sqrt{R_1^2+R_2^2}}. \tag{2.35}$$

For example, the square is:

$$R^2_{\odot_h} = \frac{\sqrt{2(R^4+1)}}{R} > R, \quad 1^2_{\odot_h} = 2. \tag{2.36}$$

This means that there is no root for 1 and:

$$1^3_{\odot_h} = \frac{\sqrt{17}+\sqrt{2}}{\sqrt{5}} = 2.493273..., \quad 1^4_{\odot_h} = \frac{\sqrt{34}}{2} = 2.915475. \tag{2.37}$$

The above ratio is solution of the algebraic equation:

$$5x^4 - 38x^2 + 45 = 0. \tag{2.38}$$

Also, allowing in (2.33) strictly positive numbers not necessary greater then 1 we have other examples:

$$\sin t \odot_h \cos t = \sqrt{\sin^4 t + 1} + \sqrt{\cos^4 t + 1} > 2,$$
$$\frac{a}{c} \odot_h \frac{b}{c} = \frac{\sqrt{a^4+c^4}+\sqrt{b^4+c^4}}{c^2} > 2. \tag{2.39}$$

Recall that the quaternion (2.1) has an Euclidean norm:

$$\|q(C)\|^2 = 1 + a^2 + b^2 + c^2 = 1 + R^4 \tag{2.40}$$

and then the given products are:

$$C_1 \odot_h C_2 = C(O,R_1) \odot_h C(O,R_2) = \frac{\|q(C_1)\| + \|q(C_2\|}{\sqrt{\sqrt{\|q(C_1)\|^2-1}+\sqrt{\|q(C_2)\|^2-1}}},$$
$$C^2_{\odot_h} = \frac{2\|q(C)\|}{\sqrt{2\sqrt{\|q(C)\|^2-1}}} \tag{2.41}$$
$$C_1 \odot_c C_2 = \sqrt{\frac{\sqrt{(\|q(C_1\|^2-1)(\|q(C_2\|^2-1)}-1}{\sqrt{\|q(C_1)\|^2-1}+\sqrt{\|q(C_2)\|^2-1}}},$$
$$C^2_{\odot_c} = \sqrt{\frac{\|q(C)\|^2-2}{2\sqrt{\|q(C)\|^2-1}}}. \tag{2.42}$$

Property 9 Since $R_1^2 + R_2^2 \geq 2R_1R_2$ we have the inequalities:

$$R_1 \odot_c R_2 \leq \sqrt{\frac{1}{2}\left(R_1R_2 - \frac{1}{R_1R_2}\right)} < \sqrt{\frac{R_1R_2}{2}},$$
$$R^2_{\odot_c} < \frac{R}{\sqrt{2}} < R < R^2_{\odot_h} < \sqrt{2}\left(R + \frac{1}{R}\right). \tag{2.43}$$

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Property 10 Returning to the circles C_1 and C_2^t discussed in the property 8 we have the power of the point A with respect to C_1:

$$P(A, C_1) = d^2 - R_1^2. \tag{2.44}$$

Also in order to get a quaternionic product of non-concentric circles we compute with (2.4):

$$q(C_1 \odot_c C_2^t) = (d^2 - R_1^2 - R_2^2)\bar{k} + 2dR_1^2\bar{i} - 2d\bar{j} - [R_1^2(d^2 - R_2^2) + 1] \tag{2.45}$$

and the coefficient of $\bar{k}$ is:

$$d - (R_1^2 + R_2^2) = \frac{2[1 + \sqrt{(R_1^4 + 1)(R_2^4 + 1)} - (R_1R_2)^2]}{R_1^2 + R_2^2} = P(A, C_1) + P(A, C_2^t) \tag{2.46}$$

while the real part of the quaternion (1.45) is $P(O, C_1) \cdot P(O, C_2^t) - 1$. The equation (2.6) is the limit $d \to 0$ of (2.45).

Property 11 We extend the previous products from circles to cycles i.e. *oriented circles*. Hence a cycle $\mathcal{C}$ is a pair $\mathcal{C} := (C, \varepsilon := \pm 1)$ with $\varepsilon = +1$ if the sense of the circle C is trigonometric and $\varepsilon = -1$ if the sense of C is clockwise. Then we introduce:

$$\mathcal{C}_1 \odot_p \mathcal{C}_2 := (C_1 \odot_p C_2, \varepsilon_1 \cdot \varepsilon_2), \quad p \in \{c, h\}. \tag{2.47}$$

Also, the products $\odot_{c,h}$ can be considered for general n-dimensional spheres $S^n(O, R) \subset \mathbb{R}^{n+1}$ with $n \geq 2$ by using the expressions (2.8) respectively (2.35). For example, the well-known Hopf fibration is the Riemannian submersion $S^3(1) \to S^2(\frac{1}{2})$ ([3, p. 1205]) and hence we compute:

$$1 \odot_c 2 = \sqrt{\frac{3}{5}}, \quad 1 \odot_h \frac{1}{2} = \frac{4\sqrt{2} + \sqrt{17}}{2\sqrt{5}}. \tag{2.48}$$

We finish this section with the remark that we can avoid the degeneration $(S^1)^2_{\odot_c} = \{O\}$ by considering the para-complex algebra $\mathbb{R}[X]/(x^2 - 1)$ instead of the complex algebra. Since in this new algebra the square of $\bar{k}$ is $+1$ we arrive at a new product $\odot_{pc}$ on $\mathbb{R}^*_+ = (0, +\infty)$:

$$x \odot_{pc} y = \sqrt{\frac{(xy)^2 + 1}{x^2 + y^2}}. \tag{2.49}$$

The triple $(\mathbb{R}^*_+, \odot_{pc}, 1)$ is a commutative monoid with:

$$x \odot_{pc} y \odot_{pc} z = \sqrt{\frac{(xyz)^2 + x^2 + y^2 + z^2}{(xy)^2 + (yz)^2 + (zx)^2 + 1}}. \tag{2.50}$$

Several interesting relationships between commutative monoids, Pythagorean triples and products on conics are studied on [7].

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3. Octonionic product for pairs of circles

Recall that an octonion $o \in \mathbb{O}$ can be thought as a pair of quaternions $o := (q_1, q_2)$ and their non-associative product is:

$$o_1 \cdot o_2 = (p_1, p_2) \cdot (q_1, q_2) := (p_1 q_1 - \bar{q_2} p_2, q_2 p_1 + p_2 \bar{q_1}) \tag{3.1}$$

with bar the usual conjugation of quaternions. It follows that a pair of circles $\mathcal{P} = (C_1, C_2)$ can be considered as an octonion $o(\mathcal{P}) := (q(C_1), q(C_2))$ and we define the product:

$$\mathcal{P}_1 \odot_o \mathcal{P}_2 = o(\mathcal{P}_1) \cdot o(\mathcal{P}_2). \tag{3.2}$$

If $C_i = C_i(O, R_i)$, $1 \leq i \leq 4$ then a long but straightforward computation yields:

$$(R_1, R_2) \odot_o (R_3, R_4) := \left(\sqrt{\frac{(R_1 R_3)^2 - (R_2 R_4)^2 - 2}{R_1^2 + R_3^2 + R_2^2 - R_4^2}}, \sqrt{\frac{(R_1 R_4)^2 + (R_2 R_3)^2}{R_1^2 + R_3^2 + R_4^2 - R_2^2}} \right) \tag{3.3}$$

with the conditions:

$$(R_1 R_3)^2 \geq (R_2 R_4)^2 + 2, \quad R_1^2 + R_3^2 > \max\{R_2^2 - R_4^2, R_4^2 - R_2^2\}. \tag{3.4}$$

Remark 3.1. i) The quaternionic product is not commutative but the products $\odot_{c,h}$ are commutative. The octonionic product $\odot_o$ is also non-commutative.
ii) Having the model of the first section we can introduce an octonionic product on pairs of cycles (with $(\varepsilon_1 \varepsilon_3, \varepsilon_2 \varepsilon_4)$ on the second level) as well as for pairs of n-spheres.

Example 3.2. i) Considering the unit circle on the first pair it results:

$$(S^1, S^1) \odot_o (C_3 \sim R_3, C_4 \sim R_4) = \left(\sqrt{\frac{R_3^2 - R_4^2 - 2}{R_3^2 - R_4^2 + 2}}, 1 \right), \quad R_3^2 \geq R_4^2 + 2. \tag{3.5}$$

ii) Considering the unit circle in the second pair we have:

$$(C_1 \sim R_1, C_2 \sim R_2) \odot_o (S^1, S^1) = \left(\sqrt{\frac{R_1^2 - R_2^2 - 2}{R_1^2 + R_2^2}}, \sqrt{\frac{R_1^2 + R_2^2}{R_1^2 - R_2^2 + 2}} \right),$$
$$R_1^2 \geq R_2^2 + 2. \tag{3.6}$$

iii) The squares are given by:

$$(C_1 \sim R_1, C_2 \sim R_2)^2_{\odot_o} = \left(\frac{\sqrt{R_1^4 - R_2^4 - 2}}{\sqrt{2} R_1}, R_2 \right), \quad R_1^4 \geq R_2^4 + 2. \tag{3.7}$$

For example $(\sqrt{2}, 1)^2_{\odot_o} = (\frac{1}{2}, 1)$ and $(2, 1)^2_{\odot_o} = (\frac{\sqrt{13}}{2\sqrt{2}}, 1)$. The condition of (3.7) gives the non-existence of square for pairs of equal circles.
iii) If the unit circle is distributed in both factors we have:

$$(C_1, S^1) \odot_o (C_3, S^1) = \left(\sqrt{\frac{(R_1 R_3)^2 - 3}{R_1^2 + R_3^2}}, 1 \right), \quad R_1 R_3 \geq \sqrt{3} \tag{3.8}$$

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while the product $(S^1, C_2) \odot_o (S^1, C_4)$ is impossible since the first condition (2.4) does not holds. The last products with S^1 are:

$$\begin{cases} (C_1, S^1) \odot_o (S^1, C_4) = \left(\sqrt{\frac{R_1^2 - R_4^2 - 2}{R_1^2 - R_4^2 + 2}}, \sqrt{\frac{(R_1 R_4)^2 + 1}{R_1^2 + R_4^2}} \right), & R_1^2 \geq R_4^2 + 2, \\ (S^1, C_2) \odot_o (C_3, S^1) = \left(\sqrt{\frac{R_3^2 - R_2^2 - 2}{R_2^2 + R_3^2}}, \sqrt{\frac{(R_2 R_3)^2 + 1}{R_3^2 - R_2^2 + 2}} \right), & R_3^2 \geq R_2^2 + 2. \end{cases} \tag{3.9}$$

From (3.8) it results the squares:

$$(C, S^1)^2_{\odot_o} = \left(\frac{\sqrt{R^4 - 3}}{\sqrt{2} R}, 1 \right), \quad R \geq \sqrt[4]{3}. \tag{3.10}$$

4. Applications

In this section we consider three applications of the given products.

Application 1 We define a 2-valued composition law on the main sheet of the equilateral hyperbola:

$$H_e : xy = 1, \quad x, y \in (0, +\infty). \tag{4.1}$$

For a point $P \in H_e$ let:

$$\max P := \max\{x(P), y(P)\} \geq 1. \tag{4.2}$$

We define a product on H_e:

$$P_1 \odot_h P_2 = \{A, B \in H_e; \max A = x(A) = \max(P_1) \odot_h \max(P_2),$$
$$\max B = y(B) = \max(P_1) \odot_h \max(P_2)\}. \tag{4.3}$$

For example:

$$(1,1)^2_{\odot_h} = \{A(2, \frac{1}{2}), B(\frac{1}{2}, 2)\}. \tag{4.4}$$

The point $E(1,1) \in H_e$ does not belongs to the image of this composition law. □

Application 2 Another multi-valued product can be introduced on the set of hyperbolic matrices following the approach of Section 3.3 from [6, p. 300]. A matrix $\gamma \in SL_2(\mathbb{R})$ is called *hyperbolic* if its eigenvalues are real and distinct; let us denotes $SL_2^H(\mathbb{R})$ their set. Since the characteristic polynomial of arbitrary γ is:

$$f_\gamma(x) = x^2 - tr(\gamma)x + 1 \tag{4.5}$$

it follows that $\gamma \in SL_2^H(\mathbb{R})$ if and only if $|tr(\gamma)| > 2$ and then its eigenvalues are reciprocal numbers. Let $e(\gamma)$ be the eigenvalue which is larger than 1 and define the norm of γ as:

$$N(\gamma) := e(\gamma)^2. \tag{4.6}$$

We introduce a product on $SL_2^H(\mathbb{R})$:

$$\gamma_1 \odot_h \gamma_2 = \{\gamma \in SL_2^H(\mathbb{R}); e(\gamma) = e(\gamma_1) \odot_h e(\gamma_2)\}. \tag{4.7}$$

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From (2.35) the norm of an arbitrary $\gamma \in \gamma_1 \odot_h \gamma_2$ is:

$$N(\gamma) = \frac{N(\gamma_1)^2 + N(\gamma_2)^2 + 2 + 2\sqrt{(N(\gamma_1)^2+1)(N(\gamma_2)^2+1)}}{N(\gamma_1) + N(\gamma_2)}. \tag{4.8}$$

For example fix $\gamma \in SL_2^H(\mathbb{R})$ of diagonal form:

$$\gamma = \gamma(R) = diag(R, \frac{1}{R}), \quad R > 1. \tag{4.9}$$

The relation (2.36) yields the norm of an arbitrary $\gamma \in \gamma(R)^2_{\odot_h}$:

$$N(\gamma) = \frac{2(R^4+1)}{R^2} > 2R^2 = 2N(\gamma(R)). \tag{4.10}$$

□

Remark 4.1. Before the next application we introduce here a matrix intermezzo in relationship with the matrix product (2.7). We associate a 2×2 matrix to the circle $C(O,R)$ through:

$$m(C(O,R)) := \begin{pmatrix} -R^2 & 1 \\ -1 & -R^2 \end{pmatrix} = -R^2 I_2 + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -R^2 I_2 + m(\bar{k}) \tag{4.11}$$

and then, as is expected:

$$m(C(O, R_1 \odot_c R_2)) = m(C(O,R_1)) \cdot m(C(O,R_2)). \tag{4.12}$$

The elements of this correspondence are:

$$tr(m(C)) = -2R^2, \quad \det(m(C)) = R^4 + 1, \quad f(m(C)) = (x+R^2)^2 + 1 \tag{4.13}$$

and we can derive expressions for $\odot_{c,h}$ similar to $(2.41-2)$ but in terms of trace and/or determinant of the corresponding matrices.

Application 3 In this application we associate a p-label to each vertex of a polygon $\mathcal{P} = P_1...P_n$ with $p \in \{c,h\}$. Suppose that any length $l_i = \|P_iP_{i+1}\|$ belongs to $M = [1, +\infty)$ and then the p-number of the vertex P_1 is defined as:

$$p_i := l_{i-1} \odot_p l_i. \tag{4.14}$$

For example, let the right triangle ΔABC with legs $\|AB\| = 3$ and $\|AC\| = 4$. Then the c-chain of ΔABC is:

$$c(\Delta ABC) := (c_A, c_B, c_C) = \left(\frac{\sqrt{143}}{5}, \sqrt{\frac{112}{17}}, \sqrt{\frac{399}{41}} \right). \tag{4.15}$$

Also, a regular polygon with sides of length 1 has a vanishing c-chain and a constant h-chain $(2, ..., 2)$. □

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Remark 4.2. We finish this work with another geometrical interpretation for the initial product $\odot_c$. Suppose that $a = R_1$ and $b = R_2$ are the legs of the right triangle ΔABC having the hypotenuse c. Then:

$$R_1 \odot_c R_2 = \frac{\sqrt{4S^2(ABC) - 1}}{c} \tag{4.16}$$

where $S(ABC)$ is the area of ΔABC.

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