Iranian Journal of Mathematical Sciences and Informatics
Vol. 17, No. 1 (2022), pp 207-217
DOI: 10.52547/ijmsi.17.1.207

# Bi-concave Functions Defined by Al-Oboudi Differential Operator

Şahsene Altınkaya

Department of Mathematics, Beykent University, 34500, Istanbul, Turkey

E-mail: sahsenealtinkaya@gmail.com

Abstract. The purpose of the present paper is to introduce a class $D^n_{\Sigma;\delta}C_0(\alpha)$ of bi-concave functions defined by Al-Oboudi differential operator. We find estimates on the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in this class. Several consequences of these results are also pointed out in the form of corollaries.

**Keywords:** Bi-concave functions, Al-Oboudi differential operator, Coefficient estimates.

**2000 Mathematics subject classification:** 30C45.

## 1. Introduction

Let $A$ indicate an analytic function family, which is normalized under the condition of $f(0) = f'(0)-1 = 0$ in the open unit disk $\Delta = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and given by the following Taylor-Maclaurin series:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Further, by $S$ we shall denote the class of all functions in $A$ which are univalent in $\Delta$.

It is well known that every function $f \in S$ has an inverse $f^{-1}$, satisfying $f^{-1}(f(z)) = z,\ (z \in \Delta)$ and $f\left(f^{-1}(w)\right) = w,\ \left(|w| < r_0(f);\ r_0(f) \geq \frac{1}{4}\right),$

Received 15 September 2018; Accepted 27 April 2019

where

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots$$

(for details, see Duren [13]). A function $f \in A$ is said to be bi-univalent in $\Delta$ if both $f$ and $f^{-1}$ are univalent in $\Delta$. Let $\Sigma$ stand for the class of bi-univalent functions defined in the unit disk $\Delta$. For a brief history of functions in the class $\Sigma$, see [25] (see also [10, 11, 14, 17, 20, 26, 27]). More recently, Srivastava *et al.* [25], Altınkaya and Yalcın [3] made an effort to introduce various subclasses of the bi-univalent function class $\Sigma$ and found non-sharp coefficient estimates on the initial coefficients $|a_2|$ and $|a_3|$ (see also [21, 15]). But determination of the bounds for the coefficients

$$|a_n|, \quad n \in \mathbb{N} \setminus \{1, 2\}; \quad \mathbb{N} = \{1, 2, 3, ...\}$$

is still an open problem. In the literature, there are only a few works determining the general coefficient bounds $|a_n|$ for the analytic bi-univalent functions (see, for example [4, 16, 28]).

The study of operators plays an important role in Geometric Function Theory in Complex Analysis and its related fields (see, for example [2, 18, 19]). Recently, the interest in this area has been increasing because it permits detailed investigations of problems with physical applications. For $f \in A$, we consider the following differential operator introduced by Al-Oboudi [1],

$$\begin{aligned} D_\delta^0 f(z) &= f(z), \\ D_\delta^1 f(z) &= (1-\delta) f(z) + \delta f'(z) \quad (\delta \geq 0), \\ &\vdots \\ D_\delta^k f(z) &= D_\delta(D_\delta^{k-1} f(z)) \quad (k \in \mathbb{N}). \end{aligned}$$

Additionally, in view of (1.1), we deduce that

$$D_\delta^k f(z) = z + \sum_{n=2}^{\infty} \left[1 + (n-1)\delta\right]^k a_n z^n \quad (k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$$

with $D_\delta^k f(0) = 0$.

It is of interest to note that $D_1^k$ is the Salagean's differential operator [23].

## 2. Preliminaries

Conformal maps of the unit disk onto convex domains are a classical topic. Recently, Avkhadiev and Wirths [6] discovered that conformal maps onto concave domains (the complements of convex closed sets) have some novel properties.

A function $f : \Delta \to \mathbb{C}$ is said to belong to the family $C_0(\alpha)$ if $f$ satisfies the following conditions:



[ DOI: 10.52547/ijmsi.17.1.207 ]

- $f$ is analytic in $\Delta$ with the standard normalization $f(0) = f'(0)-1 = 0$. In addition it satisfies $f(1) = \infty$.
- $f$ maps $\Delta$ conformally onto a set whose complement with respect to $\mathbb{C}$ is convex.
- The opening angle of $f(\Delta)$ at $\infty$ is less than or equal to $\pi\alpha,\ \alpha \in (1, 2]$.

The class $C_0(\alpha)$ is referred to as the class of concave univalent functions and for a detailed discussion about concave functions, we refer to Avkhadiev et al. [7], Cruz and Pommerenke [12] and references there in.

In particular, the inequality

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < 0 \qquad (z \in \Delta)$$

is used - sometimes also as a definition - for concave functions $f \in C_0$ (see e.g. [22] and others).

Bhowmik et al. [9] showed that an analytic function $f$ maps $\Delta$ onto a concave domain of angle $\pi\alpha$, if and only if $\Re(P_f(z)) > 0$, where

$$P_f(z) = \frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1+z}{1-z} - 1 - z\frac{f''(z)}{f'(z)}\right].$$

There has been a number of investigations on basic subclasses of concave univalent functions (see, for example [5], [8] and [24]).

Let us recall now the following definition required in sequel.

**Definition 2.1.** Let the functions $h, p : \Delta \to \mathbb{C}$ be so constrained that

$$\min\left\{\Re\left(h\left(z\right)\right), \Re\left(p\left(z\right)\right)\right\} > 0$$

and

$$h\left(0\right) = p\left(0\right) = 1.$$

Motivated by each of the above definitions, we now define a new subclass of bi-concave analytic functions involving Al-Oboudi differential operator $D_\delta^k$.

**Definition 2.2.** A function $f \in \Sigma$ given by (1.1) is said to be in the class

$$D_{\Sigma;\delta}^k C_0(\alpha) \qquad (k \in \mathbb{N}_0,\ \delta \geq 0,\ \alpha \in (1, 2],\ z, w \in \Delta)$$

if the following conditions are satisfied:

$$\frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1+z}{1-z} - 1 - z\frac{\left[D_{\Sigma;\delta}^k f(z)\right]''}{\left[D_{\Sigma;\delta}^k f(z)\right]'}\right] \in h(\Delta) \tag{2.1}$$

and

$$\frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1-w}{1+w} - 1 - w\frac{\left[D_{\Sigma;\delta}^k g(w)\right]''}{\left[D_{\Sigma;\delta}^k g(w)\right]'}\right] \in p(\Delta), \tag{2.2}$$

where $g = f^{-1}$.



[ DOI: 10.52547/ijmsi.17.1.207 ]

*Remark* 2.3. There are several choices of $k$ and $\delta$ which would provide interesting subclasses of the class $D^k_{\Sigma;\delta}C_0(\alpha)$. For example,

(i) For $k=0$, it can be directly verified that the functions $h$ and $p$ satisfy the hypotheses of Definition 2.1. Now if $f \in C_{\Sigma;0}(\alpha)$ then

$$f \in \Sigma, \quad \frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1+z}{1-z} - 1 - z\frac{f''(z)}{f'(z)}\right] \in h(\Delta) \quad (z \in \Delta)$$

and

$$\frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1+w}{1-w} - 1 - w\frac{g''(w)}{g'(w)}\right] \in p(\Delta) \quad (w \in \Delta),$$

where $g = f^{-1}$.

(ii) For $\delta = 1$, it can be directly verified that the functions $h$ and $p$ satisfy the hypotheses of Definition 2.1. Now if $f \in D^k_{\Sigma}C_0(\alpha)$ then

$$f \in \Sigma, \quad \frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1+z}{1-z} - 1 - z\frac{\left[D^k_\Sigma f(z)\right]''}{\left[D^k_\Sigma f(z)\right]'}\right] \in h(\Delta) \quad (k \in \mathbb{N}_0,\ z \in \Delta)$$

and

$$\frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1-w}{1+w} - 1 - w\frac{\left[D^k_\Sigma g(w)\right]''}{\left[D^k_\Sigma g(w)\right]'}\right] \in p(\Delta) \quad (k \in \mathbb{N}_0, w \in \Delta),$$

where $g = f^{-1}$.

## 3. Main Results and Their Consequences

We begin by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $D^k_{\Sigma;\delta}C_0(\alpha)$.

**Theorem 3.1.** *Let $f$ given by (1.1) be in the class $D^k_{\Sigma;\delta}C_0(\alpha)$. Then*

$$|a_2| \leq \min\left\{\sqrt{\frac{(\alpha+1)^2}{4(1+\delta)^{2k}} + \frac{(\alpha-1)^2\left(\left|h'(0)\right|^2+\left|p'(0)\right|^2\right)}{32(1+\delta)^{2k}} + \frac{(\alpha^2-1)\left(\left|h'(0)\right|+\left|p'(0)\right|\right)}{8(1+\delta)^{2k}}},\right.$$
$$\left.\sqrt{\frac{(\alpha-1)\left(\left|h''(0)\right|+\left|p''(0)\right|\right)}{16|2(1+\delta)^{2k}-3(1+2\delta)^k|} + \frac{(\alpha+1)}{2|2(1+\delta)^{2k}-3(1+2\delta)^k|}}\right\} \tag{3.1}$$

*and*

$$|a_3| \leq \min$$
$$\left\{\frac{8(\alpha+1)^2+(\alpha-1)^2\left(\left|h'(0)\right|^2+\left|p'(0)\right|^2\right)}{32(1+\delta)^{2k}} + \frac{(\alpha^2-1)\left(\left|h'(0)\right|+\left|p'(0)\right|\right)}{8(1+\delta)^{2k}} + \frac{(\alpha-1)\left(\left|h''(0)\right|+\left|p''(0)\right|\right)}{48(1+2\delta)^k},\right.$$
$$\left.\frac{\left|3(\alpha-1)(1+2\delta)^k-(\alpha-1)(1+\delta)^{2k}\right|\left|h''(0)\right|+(\alpha-1)(1+\delta)^{2k}\left|p''(0)\right|}{24(1+\delta)^{2k}|2(1+\delta)^{2k}-3(1+2\delta)^k|} + \frac{\alpha+1}{2|2(1+\delta)^{2k}-3(1+2\delta)^k|}\right\}. \tag{3.2}$$



[ DOI: 10.52547/ijmsi.17.1.207 ]

*Proof.* Let $f \in D^k_{\Sigma;\delta}C_0(\alpha)$ and $g$ be the analytic extension of $f^{-1}$ to $\Delta$. It follows from (2.1) and (2.2) that

$$\frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1+z}{1-z}-1-z\frac{\left[D^k_{\Sigma;\delta}f(z)\right]''}{\left[D^k_{\Sigma;\delta}f(z)\right]'}\right]=h\left(z\right) \tag{3.3}$$

and

$$\frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1-w}{1+w}-1-w\frac{\left[D^k_{\Sigma;\delta}g(w)\right]''}{\left[D^k_{\Sigma;\delta}g(w)\right]'}\right]=p\left(w\right), \tag{3.4}$$

where $h$ and $p$ satisfy the conditions of Definition 2.1. Furthermore, the functions $h\left(z\right)$ and $p\left(w\right)$ have the following Taylor-Maclaurin series expansions:

$$h\left(z\right)=1+h_1z+h_2z^2+\cdots$$

and

$$p\left(w\right)=1+p_1w+p_2w^2+\cdots,$$

respectively. Now, equating the coefficients in (3.3) and (3.4), we get

$$\frac{2\left[(\alpha+1)-2(1+\delta)^k a_2\right]}{\alpha-1}=h_{1,} \tag{3.5}$$

$$\frac{2\left[(\alpha+1)+4(1+\delta)^{2k}a_2^2-6(1+2\delta)^k a_3\right]}{\alpha-1}=h_2, \tag{3.6}$$

$$-\frac{2\left[(\alpha+1)-2(1+\delta)^k a_2\right]}{\alpha-1}=p_{1,} \tag{3.7}$$

$$\frac{2\left[(\alpha+1)+4(1+\delta)^{2k}a_2^2-6(1+2\delta)^k (2a_2^2-a_3)\right]}{\alpha-1}=p_2. \tag{3.8}$$

From (3.5) and (3.7), we find that

$$h_1=-p_1. \tag{3.9}$$

Also, from (3.5), we can write

$$a_2=\frac{\alpha+1}{2(1+\delta)^k}-\frac{h_1(\alpha-1)}{4(1+\delta)^k}. \tag{3.10}$$

Next, by using (3.5), (3.7), (3.9) and (3.10), we get

$$a_2^2=\frac{(\alpha+1)^2}{4(1+\delta)^{2k}}+\frac{(\alpha-1)^2\left(h_1^2+p_1^2\right)}{32(1+\delta)^{2k}}-\frac{\left(\alpha^2-1\right)\left(h_1-p_1\right)}{8(1+\delta)^{2k}}. \tag{3.11}$$

By adding (3.6) to (3.8), we get

$$a_2^2=\frac{(\alpha-1)\left(h_2+p_2\right)}{8\left[2(1+\delta)^{2k}-3(1+2\delta)^k\right]}-\frac{\alpha+1}{2\left[2(1+\delta)^{2k}-3(1+2\delta)^k\right]}. \tag{3.12}$$

Therefore, we find from the equations (3.11) and (3.12) that

$$|a_2|^2\le\frac{(\alpha+1)^2}{4(1+\delta)^{2k}}+\frac{(\alpha-1)^2\left(\left|h'(0)\right|^2+\left|p'(0)\right|^2\right)}{32(1+\delta)^{2k}}+\frac{\left(\alpha^2-1\right)\left(\left|h'(0)\right|+\left|p'(0)\right|\right)}{8(1+\delta)^{2k}}$$



[ DOI: 10.52547/ijmsi.17.1.207 ]

and

$$|a_2|^2 \le \frac{(\alpha-1)\left(|h''(0)|+|p''(0)|\right)}{16\left|2(1+\delta)^{2k}-3(1+2\delta)^k\right|}+\frac{(\alpha+1)}{2\left|2(1+\delta)^{2k}-3(1+2\delta)^k\right|}.$$

Similarly, subtracting (3.8) from (3.6), we have

$$a_3 = a_2^2 - \frac{(\alpha-1)(h_2-p_2)}{24(1+2\delta)^k}. \tag{3.13}$$

Then, upon substituting the value of in view of $a_2^2$ from (3.11) and (3.12) into (3.13), it follows that

$$a_3 = \frac{(\alpha+1)^2}{4(1+\delta)^{2k}}+\frac{(\alpha-1)^2\left(h_1^2+p_1^2\right)}{32(1+\delta)^{2k}}-\frac{\left(\alpha^2-1\right)(h_1-p_1)}{8(1+\delta)^{2k}}-\frac{(\alpha-1)(h_2-p_2)}{24(1+2\delta)^k}$$

and

$$a_3 = \frac{(\alpha-1)(h_2+p_2)}{8\left[2(1+\delta)^{2k}-3(1+2\delta)^k\right]}-\frac{\alpha+1}{2\left[2(1+\delta)^{2k}-3(1+2\delta)^k\right]}-\frac{(\alpha-1)(h_2-p_2)}{24(1+2\delta)^k}.$$

Consequently, we have

$$|a_3| \le \frac{8(\alpha+1)^2+(\alpha-1)^2\left(\left|h'(0)\right|^2+\left|p'(0)\right|^2\right)}{32(1+\delta)^{2k}}+\frac{\left(\alpha^2-1\right)\left(\left|h'(0)\right|+\left|p'(0)\right|\right)}{8(1+\delta)^{2k}}+\frac{(\alpha-1)\left(\left|h''(0)\right|+\left|p''(0)\right|\right)}{48(1+2\delta)^k}$$

and

$$|a_3| \le \frac{\left|3(\alpha-1)(1+2\delta)^k-(\alpha-1)(1+\delta)^{2k}\right|\left|h''(0)\right|+(\alpha-1)(1+\delta)^{2k}\left|p''(0)\right|}{24(1+\delta)^{2k}\left|2(1+\delta)^{2k}-3(1+2\delta)^k\right|}+\frac{\alpha+1}{2\left|2(1+\delta)^{2k}-3(1+2\delta)^k\right|}.$$

This completes the proof of the theorem. □

It is easily seen that, by specializing the functions $h$ and $p$ involved in the Theorem, several coefficient estimates can be obtained as special cases.

**Corollary 3.2.** *If we set*

$$h(z) = \left(\frac{1+z}{1-z}\right)^\gamma = 1+2\gamma z+2\gamma^2 z^2+\ldots \qquad (0<\gamma\le 1),$$

$$p(z) = \left(\frac{1-z}{1+z}\right)^\gamma = 1-2\gamma z+2\gamma^2 z^2+\ldots \qquad (0<\gamma\le 1),$$

*then inequalities (3.1) and (3.2) become*

$$|a_2| \le \min\left\{\sqrt{\frac{(\alpha+1)^2+(\alpha-1)^2\gamma^2+2\left(\alpha^2-1\right)\gamma}{4(1+\delta)^{2k}}}, \sqrt{\frac{(\alpha+1)+(\alpha-1)\gamma^2}{2\left|2(1+\delta)^{2k}-3(1+2\delta)^k\right|}}\right\}$$

*and*

$$|a_3| \le \min\left\{\frac{(\alpha+1)^2+(\alpha-1)^2\gamma^2+2\left(\alpha^2-1\right)\gamma}{4(1+\delta)^{2k}}+\frac{(\alpha-1)\gamma^2}{6(1+2\delta)^k},\right.$$
$$\left.\frac{\left|3(\alpha-1)(1+2\delta)^k-(\alpha-1)(1+\delta)^{2k}\right|\gamma^2+(\alpha-1)(1+\delta)^{2k}\gamma^2}{6(1+\delta)^{2k}\left|2(1+\delta)^{2k}-3(1+2\delta)^k\right|}+\frac{\alpha+1}{2\left|2(1+\delta)^{2k}-3(1+2\delta)^k\right|}\right\}.$$


[ DOI: 10.52547/ijmsi.17.1.207 ]

**Corollary 3.3.** *If we let*

$$h(z)=\frac{1+(1-2\beta)z}{1-z}=1+2(1-\beta)z+2(1-\beta)z^2+\cdots\qquad(0\le\beta<1),$$

$$p(z)=\frac{1-(1-2\beta)z}{1+z}=1-2(1-\beta)z+2(1-\beta)z^2+\cdots\qquad(0\le\beta<1),$$

*then inequalities (3.1) and (3.2) become*

$$|a_2|\le\min\left\{\sqrt{\frac{(\alpha+1)^2+(\alpha-1)^2(1-\beta)^2+2(\alpha^2-1)(1-\beta)}{4(1+\delta)^{2k}}},\sqrt{\frac{(\alpha+1)+(\alpha-1)(1-\beta)}{2|2(1+\delta)^{2k}-3(1+2\delta)^k|}}\right\}$$

*and*

$$\begin{aligned}|a_3|\quad\le\quad\min\Bigg\{&\frac{(\alpha+1)^2+(\alpha-1)^2(1-\beta)^2+2\left(\alpha^2-1\right)(1-\beta)}{4(1+\delta)^{2k}}+\frac{(\alpha-1)(1-\beta)}{6(1+2\delta)^k},\\&\frac{\left|3(\alpha-1)(1+2\delta)^k-(\alpha-1)(1+\delta)^{2k}\right|(1-\beta)+(\alpha-1)(1+\delta)^{2k}(1-\beta)}{6(1+\delta)^{2k}|2(1+\delta)^{2k}-3(1+2\delta)^k|}+\frac{\alpha+1}{2|2(1+\delta)^{2k}-3(1+2\delta)^k|}\Bigg\}.\end{aligned}$$

**Theorem 3.4.** *Let $f$ given by (1.1) be in the class $C_{\Sigma;0}(\alpha)$. Then*

$$\begin{aligned}|a_2|\quad\le\quad\min\Bigg\{&\sqrt{\frac{(\alpha+1)^2}{4}+\frac{(\alpha-1)^2\left(|h'(0)|^2+|p'(0)|^2\right)}{32}+\frac{\left(\alpha^2-1\right)\left(|h'(0)|+|p'(0)|\right)}{8}},\\&\sqrt{\frac{(\alpha-1)\left(|h''(0)|+|p''(0)|\right)}{16}+\frac{(\alpha+1)}{2}}\Bigg\}\end{aligned}\tag{3.14}$$

*and*

$$\begin{aligned}|a_3|\quad\le\quad\min\Bigg\{&\frac{8(\alpha+1)^2+(\alpha-1)^2\left(|h'(0)|^2+|p'(0)|^2\right)}{32}+\frac{\left(\alpha^2-1\right)\left(|h'(0)|+|p'(0)|\right)}{8}+\frac{(\alpha-1)\left(|h''(0)|+|p''(0)|\right)}{48},\\&\frac{\left|3(\alpha-1)(1+2\delta)^n-(\alpha-1)(1+\delta)^{2n}\right|\left|h''(0)\right|+(\alpha-1)(1+\delta)^{2n}\left|p''(0)\right|}{24}+\frac{\alpha+1}{2}\Bigg\}.\end{aligned}\tag{3.15}$$

**Corollary 3.5.** *If we set*

$$h(z)=\left(\frac{1+z}{1-z}\right)^\gamma=1+2\gamma z+2\gamma^2z^2+\ldots\qquad(0<\gamma\le1),$$

$$p(z)=\left(\frac{1-z}{1+z}\right)^\gamma=1-2\gamma z+2\gamma^2z^2+\ldots\qquad(0<\gamma\le1),$$

*then inequalities (3.14) and (3.15) become*

$$|a_2|\le\min\left\{\sqrt{\frac{(\alpha+1)^2+(\alpha-1)^2\gamma^2+2\left(\alpha^2-1\right)\gamma}{4}},\sqrt{\frac{(\alpha+1)+(\alpha-1)\gamma^2}{2}}\right\}$$

*and*

$$|a_3| \leq \min\left\{\frac{(\alpha+1)^2+(\alpha-1)^2\gamma^2+2\left(\alpha^2-1\right)\gamma}{4}+\frac{(\alpha-1)\gamma^2}{6},\right.$$
$$\left.\frac{\gamma^2(\alpha-1)+(\alpha+1)}{2}\right\}.$$

**Corollary 3.6.** *If we let*

$$h(z)=\frac{1+(1-2\beta)z}{1-z}=1+2(1-\beta)z+2(1-\beta)z^2+\cdots \qquad (0\leq\beta<1),$$

$$p(z)=\frac{1-(1-2\beta)z}{1+z}=1-2(1-\beta)z+2(1-\beta)z^2+\cdots \qquad (0\leq\beta<1),$$

*then inequalities (3.14) and (3.15) become*

$$|a_2|\leq\min\left\{\sqrt{\frac{(\alpha+1)^2+(\alpha-1)^2(1-\beta)^2+2(\alpha^2-1)(1-\beta)}{4}},\sqrt{\frac{(\alpha+1)+(\alpha-1)(1-\beta)}{2}}\right\}$$

*and*

$$|a_3| \leq \min\left\{\frac{(\alpha+1)^2+(\alpha-1)^2(1-\beta)^2+2\left(\alpha^2-1\right)(1-\beta)}{4}+\frac{(\alpha-1)(1-\beta)}{6},\right.$$
$$\left.\frac{(1-\beta)(\alpha-1)+(\alpha+1)}{2}\right\}.$$

**Theorem 3.7.** *Letf given by (1.1) be in the class $D^n_\Sigma C_0(\alpha)$. Then*

$$|a_2| \leq \min\left\{\sqrt{\frac{(\alpha+1)^2}{2^{2k+2}}+\frac{(\alpha-1)^2\left(\left|h'(0)\right|^2+\left|p'(0)\right|^2\right)}{2^{2k+5}}+\frac{\left(\alpha^2-1\right)\left(\left|h'(0)\right|+\left|p'(0)\right|\right)}{2^{2k+3}}},\right.$$
$$\left.\sqrt{\frac{(\alpha-1)\left(|h''(0)|+|p''(0)|\right)}{16(3^{k+1}-2^{2k+1})}+\frac{(\alpha+1)}{2(3^{k+1}-2^{2k+1})}}\right\} \qquad (3.16)$$

*and*

$$|a_3| \leq \min\left\{\frac{8(\alpha+1)^2+(\alpha-1)^2\left(\left|h'(0)\right|^2+\left|p'(0)\right|^2\right)}{2^{2k+5}}+\frac{\left(\alpha^2-1\right)\left(\left|h'(0)\right|+\left|p'(0)\right|\right)}{2^{2k+3}}+\frac{(\alpha-1)\left(\left|h''(0)\right|+\left|p''(0)\right|\right)}{16.3^{k+1}},\right. \qquad (3.17)$$
$$\left.\frac{(\alpha-1)\left(3^{k+1}-2^{2k}\right)\left|h''(0)\right|+(\alpha-1)2^{2k}\left|p''(0)\right|}{3.2^{2k+3}(3^{k+1}-2^{2k+1})}+\frac{\alpha+1}{2(3^{k+1}-2^{2k+1})}\right\}.$$

**Corollary 3.8.** *If we set*

$$h(z)=\left(\frac{1+z}{1-z}\right)^\gamma=1+2\gamma z+2\gamma^2z^2+\ldots \qquad (0<\gamma\leq1),$$

$$p(z)=\left(\frac{1-z}{1+z}\right)^\gamma=1-2\gamma z+2\gamma^2z^2+\ldots \qquad (0<\gamma\leq1),$$



[ DOI: 10.52547/ijmsi.17.1.207 ]

*then inequalities (3.16) and (3.17) become*

$$|a_2| \le \min\left\{\sqrt{\frac{(\alpha+1)^2+(\alpha-1)^2\gamma^2+2\left(\alpha^2-1\right)\gamma}{2^{2k+2}}}, \sqrt{\frac{(\alpha+1)+(\alpha-1)\gamma^2}{2(3^{k+1}-2^{2k+1})}}\right\}$$

*and*

$$|a_3| \le \min\left\{\frac{(\alpha+1)^2+(\alpha-1)^2\gamma^2+2\left(\alpha^2-1\right)\gamma}{2^{2k+2}} + \frac{(\alpha-1)\gamma^2}{2.3^{k+1}},\right.$$
$$\left.\frac{(\alpha-1)\left(3^{k+1}-2^{2k}\right)\gamma^2+(\alpha-1)2^{2k}\gamma^2}{3.2^{2k+1}(3^{k+1}-2^{2k+1})} + \frac{\alpha+1}{2(3^{k+1}-2^{2k+1})}\right\}.$$

**Corollary 3.9.** *If we let*

$$h(z) = \frac{1+(1-2\beta)z}{1-z} = 1 + 2(1-\beta)z + 2(1-\beta)z^2 + \cdots \qquad (0 \le \beta < 1),$$

$$p(z) = \frac{1-(1-2\beta)z}{1+z} = 1 - 2(1-\beta)z + 2(1-\beta)z^2 + \cdots \qquad (0 \le \beta < 1),$$

*then inequalities (3.16) and (3.17) become*

$$|a_2| \le \min\left\{\sqrt{\frac{(\alpha+1)^2+(\alpha-1)^2(1-\beta)^2+2(\alpha^2-1)(1-\beta)}{2^{2k+2}}}, \sqrt{\frac{(\alpha+1)+(\alpha-1)(1-\beta)}{2(3^{k+1}-2^{2k+1})}}\right\}$$

*and*

$$|a_3| \le \min\left\{\frac{(\alpha+1)^2+(\alpha-1)^2(1-\beta)^2+2\left(\alpha^2-1\right)(1-\beta)}{2^{2k+2}} + \frac{(\alpha-1)(1-\beta)}{2.3^{k+1}},\right.$$
$$\left.\frac{(\alpha-1)\left(3^{k+1}-2^{2k}\right)(1-\beta)+(\alpha-1)2^{2k}(1-\beta)}{3.2^{2k+1}(3^{k+1}-2^{2k+1})} + \frac{\alpha+1}{2(3^{k+1}-2^{2k+1})}\right\}.$$

## Acknowledgments

The authors would like to thank the referee for useful and helpful comments and suggestions.

## References

1. F. M. Al-Oboudi, On Univalent Functions Defined by a Generalized Slgean Operator, *Int. J. Math. Math. Sci.*, **2004**, (2004), 1429-1436.
2. A. Akgl, On Second-order Differential Subordinations for a Class of Analytic Functions Defined by Convolution, *J. Nonlinear Sci. Appl.*, **10**, (2017), 954-963.
3. Ş. Altınkaya, S. Yalçın, Initial Coefficient Bounds for a General Class of Bi-univalent Functions, *International Journal of Analysis*, Article ID 867871, **2014**, (2014), 1-4.
4. Ş. Altınkaya, S. Yalçın, Coefficient Bounds for a Subclass of Bi-univalent Functions, *TWMS Journal of Pure and Applied Mathematics*, **6**, (2015) 180-185.
5. Ş. Altınkaya, S. Yalçın, General Properties of Multivalent Concave Functions Involving Linear Operator of Carlson-Shaffer Type, *Competes rendus de l'Acadmie bulgare des Sciences*, **69**(12), (2016), 1533-1540.
6. F. G. Avkhadiev, K. J. Wirths, Convex Holes Produce Lower Bounds for Coefficients, *Complex Variables, Theory and Application*, **47**, (2002), 556–563.



[ DOI: 10.52547/ijmsi.17.1.207 ]

7. F. G. Avkhadiev , C. Pommerenke, K. J. Wirths, Sharp Inequalities for the Coefficients of Concave Schlicht Functions, *Comment. Math. Helv.*, **81**, (2006), 801-807.
8. H. Bayram, Ş. Altınkaya, General Properties of Concave Functions Defined by the Generalized Srivastava-Attiya Operator, *Journal of Computational Analysis and Applications*, **23**, (2017), 408-416.
9. B. Bhowmik, S. Ponnusamy, K. J. Wirths, Characterization and the Pre-Schwarzian Norm Estimate for Concave Univalent Functions, *Monatsh Math.*, **161**, (2010), 59-75.
10. D. A. Brannan, T. S. Taha, On Some Classes of Bi-univalent Functions, *Studia Universitatis Babeş-Bolyai Mathematica*, **31**, (1986), 70-77.
11. D. A. Brannan, J. G. Clunie, *Aspects of Contemporary Complex Analysis*, (Proceedings of the NATO Advanced Study Institute Held at University of Durham: July 1-20, 1979), New York: Academic Press, 1980.
12. L. Cruz, C. Pommerenke, On Concave Univalent Functions, *Complex Var. Elliptic Equ.*, **52**, (2007), 153-159.
13. P. L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, Springer, New York, USA, **259**, 1983.
14. B. A. Frasin, M. K. Aouf, New Subclasses of Bi-univalent Functions, *Appl. Math. Lett.*, **24**, (2011), 1569–1573.
15. S. Hajiparvaneh, A. Zireh, Coefficient Estimates for Subclass of Analytic and Bi-univalent Functions Defined by Differential Operator, *Tbilisi Mathematical Journal*, **10**, (2017), 91-102.
16. S. G. Hamidi, J. M. Jahangiri, Faber Polynomial Coefficient Estimates for Analytic Bi-close-to-convex Functions, *C. R. Acad. Sci. Paris, Ser. I*, **352**, (2014), 17-20.
17. M. Lewin, On a Coefficient Problem for Bi-univalent Functions, *Proceeding of the American Mathematical Society*, **18**, (1967), 63-68.
18. A. A. Lupaş, On a Certain Subclass of Analytic Functions Defined by a Generalized Slgean Operator and Ruscheweyh Derivative, *Carpathian Journal of Mathematics*, **28**, (2012), 183-190.
19. G. Murugunsundaramoorthy, K. Vijaya, N. Magesh, On Applications of Generalized Integral Operator to a Subclass of Analytic Functions, *Acta Universitatis Apulensis*, **25**, (2011), 165-176.
20. E. Netanyahu, The Minimal Distance of the Image Boundary from the Origin and the Second Coefficient of a Univalent Function in $|z| < 1$, *Archive for Rational Mechanics and Analysis*, **32**, (1969), 100-112.
21. A. B. Patil, U.H. Naik, On New Subclasses of Bi-univalent Functions Associated with Al-Oboudi Differential Operator, *International Journal of Pure and Applied Mathematics*, **110**, (2016), 143-151.
22. J. Pfaltzgra, B. Pinchuk, A Variational Method for Classes of Meromorphic Functions, *J. Analyse Math.*, **24**, (1971) , 101-150.
23. G. S. Slgean, *Subclasses of Univalent Functions*, Complex Analysis-Fifth Romanian-Finnish Seminar, Part 1 (Bucharest, 1981), Lecture Notes in Math. 1013, Springer, Berlin, 1983, 362–372.
24. F. M. Sakar, H. Özlem Güney, Coefficient Estimates for Bi-concave Functions, *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.*, **68**, (2019), 53-60.
25. H. M. Srivastava, A.K. Mishra, P. Gochhayat, Certain Subclasses of Analytic and Bi-univalent Functions, *Appl. Math. Lett.*, **23**, (2010), 1188-1192.
26. Q. H. Xu, Y.-C. Gui, H. M. Srivastava, Coefficient Estimates for a Certain Subclass of Analytic and Bi-univalent Functions, *Appl. Math. Lett.*, **25**, (2012), 990–994.
27. T. Yavuz, Coefficient Estimates for a New Subclass of Bi-univalent Functions Defined by Convolution, *Creat. Math. Inform.*, **27**, (2018) , 89–94.

28. A. Zireh, E. A. Adegani, S. Bulut, Faber Polynomial Coefficient Estimates for a Comprehensive Subclass of Analytic Bi-univalent Functions Defined by Subordination, *Bull. Belg. Math. Soc. Simon Stevin*, **23**, (2016), 487-504.



[ DOI: 10.52547/ijmsi.17.1.207 ]