Bi-concave Functions Defined by Al-Oboudi Differential Operator

Şahsene Altınkaya

Department of Mathematics, Beykent University, 34500, Istanbul, Turkey
E-mail: sahsenealtinkaya@gmail.com

Abstract. The purpose of the present paper is to introduce a class $D^p_{\lambda,\beta}C_0(\alpha)$ of bi-concave functions defined by Al-Oboudi differential operator. We find estimates on the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in this class. Several consequences of these results are also pointed out in the form of corollaries.

Keywords: Bi-concave functions, Al-Oboudi differential operator, Coefficient estimates.


1. Introduction

Let $A$ indicate an analytic function family, which is normalized under the condition of $f(0) = f'(0) - 1 = 0$ in the open unit disk $\Delta = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and given by the following Taylor-Maclaurin series:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Further, by $S$ we shall denote the class of all functions in $A$ which are univalent in $\Delta$.

It is well known that every function $f \in S$ has an inverse $f^{-1}$, satisfying $f^{-1}(f(z)) = z$, $(z \in \Delta)$ and $f(f^{-1}(w)) = w$, $(|w| < r_0(f) ; r_0(f) \geq \frac{1}{4})$.
where
\[ f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots \]
(for details, see Duren [13]). A function \( f \in A \) is said to be bi-univalent in \( \Delta \) if both \( f \) and \( f^{-1} \) are univalent in \( \Delta \). Let \( \Sigma \) stand for the class of bi-univalent functions defined in the unit disk \( \Delta \). For a brief history of functions in the class \( \Sigma \), see [25] (see also [10, 11, 14, 17, 20, 26, 27]). More recently, Srivastava et al. [25], Altınkaya and Yalcın [3] made an effort to introduce various subclasses of the bi-univalent function class \( \Sigma \) and found non-sharp coefficient estimates on the initial coefficients \( |a_2| \) and \( |a_3| \) (see also [21, 15]). But determination of the bounds for the coefficients \( |a_n|, \ n \in \mathbb{N}\setminus\{1, 2\}; \ \mathbb{N} = \{1, 2, 3, \ldots\} \) is still an open problem. In the literature, there are only a few works determining the general coefficient bounds \( |a_n| \) for the analytic bi-univalent functions (see, for example [4, 16, 28]).

The study of operators plays an important role in Geometric Function Theory in Complex Analysis and its related fields (see, for example [2, 18, 19]). Recently, the interest in this area has been increasing because it permits detailed investigations of problems with physical applications. For \( f \in A \), we consider the following differential operator introduced by Al-Oboudi [1],

\[ D^0_\delta f(z) = f(z), \]
\[ D^1_\delta f(z) = (1 - \delta)f(z) + \delta f'(z) \quad (\delta \geq 0), \]
\[ \vdots \]
\[ D^k_\delta f(z) = D_\delta(D^{k-1}_\delta f(z)) \quad (k \in \mathbb{N}). \]

Additionally, in view of (1.1), we deduce that
\[ D^k_\delta f(z) = z + \sum_{n=2}^{\infty} \left[ 1 + (n - 1)\delta \right]^k a_n z^n \quad (k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}) \]
with \( D^k_\delta f(0) = 0 \).

It is of interest to note that \( D^k_\delta \) is the Salagean’s differential operator [23].

2. Preliminaries

Conformal maps of the unit disk onto convex domains are a classical topic. Recently, Avkhadiev and Wirths [6] discovered that conformal maps onto concave domains (the complements of convex closed sets) have some novel properties.

A function \( f : \Delta \to \mathbb{C} \) is said to belong to the family \( C_0(\alpha) \) if \( f \) satisfies the following conditions:
• $f$ is analytic in $\Delta$ with the standard normalization $f(0) = f'(0) - 1 = 0$. In addition it satisfies $f(1) = \infty$.
• $f$ maps $\Delta$ conformally onto a set whose complement with respect to $\mathbb{C}$ is convex.
• The opening angle of $f(\Delta)$ at $\infty$ is less than or equal to $\pi\alpha$, $\alpha \in (1, 2]$.

The class $C_0(\alpha)$ is referred to as the class of concave univalent functions and for a detailed discussion about concave functions, we refer to Avkhadiev et al. [7], Cruz and Pommerenke [12] and references there in.

In particular, the inequality
\[
\Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) < 0 \quad (z \in \Delta)
\]
is used - sometimes also as a definition - for concave functions $f \in C_0$ (see e.g. [22] and others).

Bhowmik et al. [9] showed that an analytic function $f$ maps $\Delta$ onto a concave domain of angle $\pi\alpha$, if and only if $\Re (P_f(z)) > 0$, where
\[
P_f(z) = \frac{2}{\alpha - 1} \left[ \frac{\alpha + 1}{2} \frac{1 + z}{1 - z} - 1 - z \frac{f''(z)}{f'(z)} \right].
\]

There has been a number of investigations on basic subclasses of concave univalent functions (see, for example [5], [8] and [24]).

Let us recall now the following definition required in sequel.

**Definition 2.1.** Let the functions $h, p : \Delta \to \mathbb{C}$ be so constrained that
\[
\min \{ \Re (h(z)), \Re (p(z)) \} > 0
\]
and
\[
h(0) = p(0) = 1.
\]

Motivated by each of the above definitions, we now define a new subclass of bi-concave analytic functions involving Al-Oboudi differential operator $D^k_\Sigma$.

**Definition 2.2.** A function $f \in \Sigma$ given by (1.1) is said to be in the class
\[
D^k_{\Sigma,\delta} C_0(\alpha) \quad (k \in \mathbb{N}_0, \; \delta \geq 0, \; \alpha \in (1, 2], \; z, w \in \Delta)
\]
if the following conditions are satisfied:
\[
\frac{2}{\alpha - 1} \left[ \frac{\alpha + 1}{2} \frac{1 + z}{1 - z} - 1 - z \frac{D^k_{\Sigma,\delta} f(z)}{D^k_{\Sigma,\delta} f(z)} \right] \in h(\Delta) \quad (2.1)
\]
and
\[
\frac{2}{\alpha - 1} \left[ \frac{\alpha + 1}{2} \frac{1 + w}{1 + w} - 1 - w \frac{D^k_{\Sigma,\delta} g(w)}{D^k_{\Sigma,\delta} g(w)} \right] \in p(\Delta), \quad (2.2)
\]
where $g = f^{-1}$.
Remark 2.3. There are several choices of $k$ and $\delta$ which would provide interesting subclasses of the class $D^k_{\Sigma,\delta}C_0(\alpha)$. For example,

(i) For $k = 0$, it can be directly verified that the functions $h$ and $p$ satisfy the hypotheses of Definition 2.1. Now if $f \in C_{\Sigma,\delta}C_0(\alpha)$ then
\[
 f \in \Sigma, \quad \frac{2}{\alpha - 1} \left[ \frac{\alpha + 1 + z}{2} - 1 - z \frac{f''(z)}{f'(z)} \right] \in h(\Delta) \quad (z \in \Delta)
\]
and
\[
 \frac{2}{\alpha - 1} \left[ \frac{\alpha + 1 + w}{2} - 1 - w \frac{g''(w)}{g'(w)} \right] \in p(\Delta) \quad (w \in \Delta),
\]
where $g = f^{-1}$.

(ii) For $\delta = 1$, it can be directly verified that the functions $h$ and $p$ satisfy the hypotheses of Definition 2.1. Now if $f \in D^k_{\Sigma,\delta}C_0(\alpha)$ then
\[
 f \in \Sigma, \quad \frac{2}{\alpha - 1} \left[ \frac{\alpha + 1 + z}{2} - 1 - z \frac{D^k_{\Sigma^0}f(z)}{[D^k_{\Sigma^0}f(z)]'} \right] \in h(\Delta) \quad (k \in \mathbb{N}_0, \ z \in \Delta)
\]
and
\[
 \frac{2}{\alpha - 1} \left[ \frac{\alpha + 1 + w}{2} - 1 - w \frac{D^k_{\Sigma^0}g(w)}{[D^k_{\Sigma^0}g(w)]'} \right] \in p(\Delta) \quad (k \in \mathbb{N}_0, w \in \Delta),
\]
where $g = f^{-1}$.

3. Main Results and Their Consequences

We begin by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $D^k_{\Sigma,\delta}C_0(\alpha)$.

**Theorem 3.1.** Let $f$ given by (1.1) be in the class $D^k_{\Sigma,\delta}C_0(\alpha)$. Then

\[
 |a_2| \leq \min \left\{ \sqrt{\frac{(\alpha+1)^2 + (\alpha-1)^2}{4(1+\delta)^{2\alpha}} + \frac{(\alpha^2-1)(h'(0)+|p'(0)|)}{8(1+\delta)^{2\alpha}}}, \right. \right.
\]
\[
 \left. \left. \frac{(\alpha-1)(h''(0)+|p''(0)|)}{16(1+\delta)^{2\alpha}-3(1+2\delta)^{2\alpha}} + \frac{(\alpha+1)}{2(1+\delta)^{2\alpha}-3(1+2\delta)^{2\alpha}} \right\} \quad (3.1) \right.
\]

and

\[
 |a_3| \leq \min \left\{ \frac{8(\alpha+1)^2 + (\alpha-1)^2}{32(1+\delta)^{2\alpha}} + \frac{(\alpha^2-1)(h'(0)+|p'(0)|)}{8(1+\delta)^{2\alpha}} + \frac{(\alpha-1)(h''(0)+|p''(0)|)}{48(1+\delta)^{2\alpha}}, \right. \right.
\]
\[
 \left. \left. \frac{3(\alpha-1)(1+2\delta)^{2\alpha}-(\alpha-1)(1+\delta)^{2\alpha}}{24(1+\delta)^{2\alpha}-3(1+2\delta)^{2\alpha}} |h''(0)+|p''(0)|| + \frac{(\alpha+1)}{2(1+\delta)^{2\alpha}-3(1+2\delta)^{2\alpha}} \right\} \right. \quad (3.2) \right.
\]
Proof. Let \( f \in D_{\Sigma;\delta}^k C_0(\alpha) \) and \( g \) be the analytic extension of \( f^{-1} \) to \( \Delta \). It follows from (2.1) and (2.2) that

\[
\frac{2}{\alpha - 1} \left[ \frac{\alpha + 1}{2} + z - 1 - z \frac{D_{\Sigma;\delta}^k f(z)}{[D_{\Sigma;\delta}^k f(z)]''} \right] = h(z) \tag{3.3}
\]

and

\[
\frac{2}{\alpha - 1} \left[ \frac{\alpha + 1}{2} - w - 1 - w \frac{D_{\Sigma;\delta}^k g(w)}{[D_{\Sigma;\delta}^k g(w)]''} \right] = p(w), \tag{3.4}
\]

where \( h \) and \( p \) satisfy the conditions of Definition 2.1. Furthermore, the functions \( h(z) \) and \( p(w) \) have the following Taylor-Maclaurin series expansions:

\[
h(z) = 1 + h_1 z + h_2 z^2 + \cdots
\]

and

\[
p(w) = 1 + p_1 w + p_2 w^2 + \cdots,
\]

respectively. Now, equating the coefficients in (3.3) and (3.4), we get

\[
\frac{2}{\alpha - 1} \left[ (\alpha + 1) - 2(1 + \delta)^k a_2 \right] = h_1, \tag{3.5}
\]

\[
\frac{2}{\alpha - 1} \left[ (\alpha + 1) + 4(1 + \delta)^2 a_2^3 - 6(1 + 2\delta)^k a_3 \right] = h_2, \tag{3.6}
\]

\[
\frac{2}{\alpha - 1} \left[ (\alpha + 1) - 2(1 + \delta)^k a_2 \right] = p_1, \tag{3.7}
\]

\[
\frac{2}{\alpha - 1} \left[ (\alpha + 1) + 4(1 + \delta)^2 a_2^3 - 6(1 + 2\delta)^k(2a_2^3 - a_3) \right] = p_2. \tag{3.8}
\]

From (3.5) and (3.7), we find that

\[
h_1 = -p_1. \tag{3.9}
\]

Also, from (3.5), we can write

\[
a_2 = \frac{\alpha + 1}{2(1 + \delta)^k} \frac{h_1 (\alpha - 1)}{4(1 + \delta)^k}. \tag{3.10}
\]

Next, by using (3.5), (3.7), (3.9) and (3.10), we get

\[
a_2^2 = \frac{(\alpha + 1)^2}{4(1 + \delta)^{2k}} + \frac{(\alpha - 1)^2 (h_2^2 + p_2^2)}{32(1 + \delta)^{2k}} - \frac{(\alpha^2 - 1)(h_1 - p_1)}{8(1 + \delta)^{2k}}. \tag{3.11}
\]

By adding (3.6) to (3.8), we get

\[
a_2^2 = \frac{(\alpha - 1)(h_2 + p_2)}{8[2(1 + \delta)^{2k} - 3(1 + 2\delta)^k]} - \frac{\alpha + 1}{2[2(1 + \delta)^{2k} - 3(1 + 2\delta)^k]} \tag{3.12}
\]

Therefore, we find from the equations (3.11) and (3.12) that

\[
|a_2|^2 \leq \frac{(\alpha + 1)^2}{4(1 + \delta)^{2k}} + \frac{(\alpha - 1)^2 (|h'(0)|^2 + |p'(0)|^2)}{32(1 + \delta)^{2k}} + \frac{(\alpha^2 - 1)(|h'(0)| + |p'(0)|)}{8(1 + \delta)^{2k}}.
\]
and
\[ |a_2|^2 \leq \frac{(\alpha - 1) |h''(0)| + |p''(0)|}{16 |h'(0)|^2 + 3 |p'(0)|} + \frac{(\alpha + 1)}{2 |h'(0)|^2 + 3 |p'(0)|}. \]

Similarly, subtracting (3.8) from (3.6), we have
\[ a_3 = a_2^2 - \frac{(\alpha - 1) (h_2 - p_2)}{2(1 + \delta)^k}. \] (3.13)

Then, upon substituting the value of in view of \(a_2^2\) from (3.11) and (3.12) into (3.13), it follows that
\[ a_3 = \frac{(\alpha + 1)^2}{4(1 + \delta)^{2k}} + \frac{(\alpha - 1)^2 (h_1^2 + p_1^2)}{32(1 + \delta)^{2k}} - \frac{(\alpha - 1) (h_1 - p_1)}{8(1 + \delta)^{2k}} - \frac{(\alpha - 1) (h_2 - p_2)}{24(1 + \delta)^{2k}} \]
and
\[ a_3 = \frac{(\alpha - 1) (h_2 + p_2)}{8 |2(1 + \delta)^{2k} - 3 |1 + 2\delta)^{2k}|} - \frac{(\alpha - 1) (h_2 - p_2)}{2 |2(1 + \delta)^{2k} - 3 |1 + 2\delta)^{2k}|} \]

Consequently, we have
\[ |a_3| \leq \frac{8(\alpha + 1)^2 + (\alpha - 1)^2 (h''(0))^2 + |p''(0)|^2}{32(1 + \delta)^{2k}} + \frac{(\alpha - 1) |h''(0)| + |p''(0)|}{48(1 + \delta)^{2k}} \]
and
\[ |a_3| \leq \frac{3(\alpha - 1) (1 + 2\delta)^{2k} - (\alpha - 1) (1 + \delta)^{2k} |h''(0)| + (\alpha - 1) (1 + \delta)^{2k} |p''(0)|}{24(1 + \delta)^{2k} |2(1 + \delta)^{2k} - 3 |1 + 2\delta)^{2k}|} + \frac{\alpha + 1}{2 |2(1 + \delta)^{2k} - 3 |1 + 2\delta)^{2k}|} \].

This completes the proof of the theorem. \(\square\)

It is easily seen that, by specializing the functions \(h\) and \(p\) involved in the Theorem, several coefficient estimates can be obtained as special cases.

**Corollary 3.2.** If we set
\[ h(z) = \left( \frac{1 + z}{1 - z} \right)^\gamma = 1 + 2 \gamma z + 2 \gamma^2 z^2 + \ldots \quad (0 < \gamma \leq 1), \]
\[ p(z) = \left( \frac{1 - z}{1 + z} \right)^\gamma = 1 - 2 \gamma z + 2 \gamma^2 z^2 + \ldots \quad (0 < \gamma \leq 1), \]
then inequalities (3.1) and (3.2) become
\[ |a_2| \leq \min \left\{ \sqrt{\frac{(\alpha + 1)^2 + (\alpha - 1)^2 \gamma^2 + 2(\alpha^2 - 1)\gamma}{4(1 + \delta)^{2k}}} \right\} \]
and
\[ |a_3| \leq \min \left\{ \sqrt{\frac{(\alpha + 1)^2 + (\alpha - 1)^2 \gamma^2 + 2(\alpha^2 - 1)\gamma}{4(1 + \delta)^{2k}}} + \frac{(\alpha - 1) \gamma^2}{6(1 + \delta)^{2k}} \right\} \].
Corollary 3.3. If we let
\[ h(z) = \frac{1 + (1 - 2\beta) z}{1 - z} = 1 + 2(1 - \beta) z + 2(1 - \beta) z^2 + \cdots \quad (0 \leq \beta < 1), \]
\[ p(z) = \frac{1 - (1 - 2\beta) z}{1 + z} = 1 - 2(1 - \beta) z + 2(1 - \beta) z^2 + \cdots \quad (0 \leq \beta < 1), \]
then inequalities (3.14) and (3.15) become
\[ |a_2| \leq \min \left\{ \sqrt{\frac{(\alpha + 1)^2 + (\alpha - 1)^2(1 - \beta)^2 + 2(\alpha^2 - 1)(1 - \beta)}{4(1 + \delta)^{2k}}}, \sqrt{\frac{(\alpha + 1) + (\alpha - 1)(1 - \beta)}{2(1 + \delta)^{2k} - 3(1 + 2k)^2}} \right\}, \]
and
\[ |a_3| \leq \min \left\{ \frac{(\alpha + 1)^2 + (\alpha - 1)^2(1 - \beta)^2 + 2(\alpha^2 - 1)(1 - \beta)}{4(1 + \delta)^{2k}} + \frac{(\alpha - 1)(1 - \beta)(1 + 2k)}{6(1 + \delta)^{2k}}, \right\} \]

Theorem 3.4. Let \( f \) given by (1.1) be in the class \( C_{\Sigma,0}(\alpha) \). Then
\[ |a_2| \leq \min \left\{ \sqrt{\frac{(\alpha + 1)^2}{4} + \frac{(\alpha - 1)^2}{32} \left( |h'(0)|^2 + |p'(0)|^2 \right)} + \frac{(\alpha^2 - 1)(|h'(0)| + |p'(0)|)}{8}, \right\} \quad (3.14) \]
and
\[ |a_3| \leq \min \left\{ \frac{8(\alpha + 1)^2 + (\alpha - 1)^2(1 + \delta)^{2n}}{32} \left( |h'(0)|^2 + |p'(0)|^2 \right) + \frac{(\alpha^2 - 1)(|h'(0)| + |p'(0)|)}{8} + \frac{(\alpha - 1)(|h''(0)| + |p''(0)|)}{48}, \right\} \quad (3.15) \]

Corollary 3.5. If we set
\[ h(z) = \left( \frac{1 + z}{1 - z} \right)^{\gamma} = 1 + 2\gamma z + 2\gamma^2 z^2 + \cdots \quad (0 < \gamma \leq 1), \]
\[ p(z) = \left( \frac{1 - z}{1 + z} \right)^{\gamma} = 1 - 2\gamma z + 2\gamma^2 z^2 + \cdots \quad (0 < \gamma \leq 1), \]
then inequalities (3.14) and (3.15) become
\[ |a_2| \leq \min \left\{ \sqrt{\frac{(\alpha + 1)^2 + (\alpha - 1)^2(1 - \beta)^2 + 2(\alpha^2 - 1)(1 - \beta)}{4(1 + \delta)^{2k}}}, \sqrt{\frac{(\alpha + 1) + (\alpha - 1)(1 - \beta)}{2(1 + \delta)^{2k} - 3(1 + 2k)^2}} \right\}. \]
and
\[ |a_3| \leq \min \left\{ \frac{(\alpha+1)^2 + (\alpha-1)^2 \gamma^2 + 2(\alpha^2-1)\gamma}{4} + \frac{(\alpha-1)\gamma^2}{6}, \frac{\gamma^2(\alpha-1) + (\alpha+1)^2}{2} \right\}. \]

**Corollary 3.6.** If we let
\[ h(z) = \frac{1 + (1 - 2\beta)z}{1 - z} = 1 + 2(1 - \beta)z + 2(1 - \beta)z^2 + \cdots \quad (0 \leq \beta < 1), \]
\[ p(z) = \frac{1 - (1 - 2\beta)z}{1 + z} = 1 - 2(1 - \beta)z + 2(1 - \beta)z^2 + \cdots \quad (0 \leq \beta < 1), \]
then inequalities (3.14) and (3.15) become
\[ |a_2| \leq \min \left\{ \sqrt{\frac{(\alpha+1)^2 + (\alpha-1)^2(1-\beta)^2 + 2(\alpha^2-1)(1-\beta)}{4}}, \sqrt{\frac{(\alpha+1) + (\alpha-1)(1-\beta)}{2}} \right\} \]
and
\[ |a_3| \leq \min \left\{ \frac{(\alpha+1)^2 + (\alpha-1)^2(1-\beta)^2 + 2(\alpha^2-1)(1-\beta)}{4} + \frac{(\alpha-1)(1-\beta)}{6}, \frac{(1-\beta)(\alpha-1) + (\alpha+1)}{2} \right\}. \]

**Theorem 3.7.** Let \( f \) given by (1.1) be in the class \( D_\alpha C_0(\alpha) \). Then
\[ |a_2| \leq \min \left\{ \sqrt{\frac{(\alpha+1)^2}{2^{2k+2}} + \frac{(\alpha-1)^2(\frac{|h'(0)|^2 + |p'(0)|^2}{2^{2k+2}}) + \frac{(\alpha^2-1)(\frac{|h'(0)| + |p'(0)|}{2^{2k+2}})}{2^{2k+2}}}{\sqrt{\frac{(\alpha-1)(|h''(0)| + |p''(0)|)}{16(3^{k+1} - 2^{2k+1})} + \frac{(\alpha + 1)}{2(3^{k+1} - 2^{2k+1})}}}, \right\} \tag{3.16} \]
and
\[ |a_3| \leq \min \left\{ \frac{8(\alpha+1)^2 + (\alpha-1)^2\left(\frac{|h'(0)|^2 + |p'(0)|^2}{2^{2k+2}}\right)}{2^{2k+2}} + \frac{(\alpha^2-1)(\frac{|h'(0)| + |p'(0)|}{2^{2k+2}}) + (\alpha-1)(\frac{|h''(0)| + |p''(0)|}{16(3^{k+1} - 2^{2k+1})})}{2(3^{k+1} - 2^{2k+1})}, \right\} \tag{3.17} \]

**Corollary 3.8.** If we set
\[ h(z) = \left(\frac{1 + z}{1 - z}\right)^\gamma = 1 + 2\gamma z + 2\gamma^2 z^2 + ... \quad (0 < \gamma \leq 1), \]
\[ p(z) = \left(\frac{1 - z}{1 + z}\right)^\gamma = 1 - 2\gamma z + 2\gamma^2 z^2 + ... \quad (0 < \gamma \leq 1), \]
then inequalities (3.16) and (3.17) become
\[ |a_2| \leq \min \left\{ \sqrt{\frac{(\alpha+1)^2+(\alpha-1)^2+2(\alpha^2-1)}{2k+2}}, \sqrt{\frac{(\alpha+1)+(\alpha-1)\gamma}{2(3k+1-2k^2+1)}} \right\} \]
and
\[ |a_3| \leq \min \left\{ \frac{(\alpha+1)^2+(\alpha-1)^2+2(\alpha^2-1)\gamma}{2k+2} + \frac{(\alpha-1)^2}{2,3k+1}, \frac{(\alpha-1)(3^{k+1}-2^k)\gamma^2+(\alpha-1)2^k\gamma^2}{3,2k+1+(3^{k+1}-2k^2+1)} + \frac{\alpha+1}{2(3^{k+1}-2k^2+1)} \right\}. \]

**Corollary 3.9.** If we let
\[ h(z) = \frac{1 + (1 - 2\beta)z}{1 - z} = 1 + 2 (1 - \beta) z + 2 (1 - \beta) z^2 + \cdots \quad (0 \leq \beta < 1), \]
\[ p(z) = \frac{1 - (1 - 2\beta)z}{1 + z} = 1 - 2 (1 - \beta) z + 2 (1 - \beta) z^2 + \cdots \quad (0 \leq \beta < 1), \]
then inequalities (3.16) and (3.17) become
\[ |a_2| \leq \min \left\{ \sqrt{\frac{(\alpha+1)^2+(\alpha-1)^2(1-\beta)^2+2(\alpha^2-1)(1-\beta)}{2k+2}}, \sqrt{\frac{(\alpha+1)+(\alpha-1)(1-\beta)}{2(3k+1-2k^2+1)}} \right\} \]
and
\[ |a_3| \leq \min \left\{ \frac{(\alpha+1)^2+(\alpha-1)^2(1-\beta)^2+2(\alpha^2-1)(1-\beta)}{2k+2} + \frac{(\alpha-1)(1-\beta)}{2,3k+1}, \frac{(\alpha-1)(3^{k+1}-2^k)(1-\beta)+(\alpha-1)2^k(1-\beta)}{3,2k+1+(3^{k+1}-2k^2+1)} + \frac{\alpha+1}{2(3^{k+1}-2k^2+1)} \right\}. \]

**ACKNOWLEDGMENTS**

The authors would like to thank the referee for useful and helpful comments and suggestions.

**REFERENCES**


