Iranian Journal of Mathematical Sciences and Informatics
Vol. 17, No. 1 (2022), pp 165-176
DOI: 10.52547/ijmsi.17.1.165

Translation Surfaces of the Third Fundamental Form in Lorentz-Minkowski Space

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Abstract. In this paper we study translation surfaces with the non-degenerate third fundamental form in Lorentz- Minkowski space $\mathbb{L}^3$. As a result, we classify translation surfaces satisfying an equation in terms of the position vector field and the Laplace operator with respect to the third fundamental form III on the surface.

Keywords: Surfaces of coordinate finite type, Translation surfaces, Laplace operator.

2000 Mathematics subject classification: 53A05, 53A07, 53C40.

1. Introduction

Let $\mathbb{L}^3$ be the 3-dimensional Lorentz-Minkowski space, that is, the space $\mathbb{R}^3$ endowed with the metric

$$g_L(X,X) = dx^2 + dy^2 - dz^2,$$

where $X = (x, y, z) \in \mathbb{R}^3$.

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Received 29 August 2018; Accepted 16 July 2019

For two vectors $V = (v_1, v_2, v_3)$ and $W = (w_1, w_2, w_3)$ in $\mathbb{L}^3$ the Lorentz cross product of X and Y is defined by

$$V \wedge W = (v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, -v_1 w_2 + v_2 w_1).$$

Assume $r : M^2 \to \mathbb{L}^3$ is an isometric immersion of M^2 into $\mathbb{L}^3$. Then the position vector of M^2 in $\mathbb{L}^3$ satisfies [7]

$$\Delta r = -2H, \tag{1.1}$$

where H is the mean curvature vector of M^2 in $\mathbb{L}^3$. It follows from (1.1) that M^2 is minimal in $\mathbb{L}^3$ if and only if the immersion r is harmonic.

The notion of finite type immersion of submanifolds of a Euclidean space has been widely used in classifying and characterizing well known Riemannian submanifolds [6]. B.-Y. Chen posed the problem of classifying the finite type submanifolds in the 3-dimensional Euclidean space $\mathbb{E}^3$. These can be regarded as a generalization of minimal submanifolds.

The notion of finite type immersion has played an important role in classifying and characterizing various submanifolds in Euclidean space.

Since then the theory of submanifolds of finite type has been studied by many geometers.

In [9] F. Dillen, W. Goemans, I. Van de Woestyne have derived a classification of translation surfaces in $\mathbb{E}^3$ and $\mathbb{E}^3_1$, satisfying the Weingarten condition.

A well known result due to Takahashi [18] states that minimal surfaces and spheres are the only surfaces in $\mathbb{E}^3$ satisfying the condition$\Delta r = \lambda r,\ \lambda \in \mathbb{R}$.

In [11] Ferrandez, Garay and Lucas proved that the surfaces of $\mathbb{E}^3$ satisfying

$$\Delta H = AH,\ A \in Mat(3,3),$$

are either minimal, or an open piece of sphere or of a right circular cylinder.

In [10] F. Dillen, J. Pas and L. Verstraelen proved that the only surfaces in $\mathbb{E}^3$ satisfying

$$\Delta r = Ar + B,\ A \in Mat(3,3),\ B \in Mat(3,1),$$

are the minimal surfaces, the spheres and the circular cylinders.

The authors [2] classified the factorable surfaces in the three-dimensional Euclidean and Lorentzian spaces, whose component functions are eigenfunctions of their Laplace operator.

M. E. Aydin studied constant curvatures of translation surfaces in the three dimensional simply Isotropic space [1]. Also, Y. Yuan and H. L. Liu dealth with translation surfaces of some new types in 3-Minkowski space [21]. B. Bukcu, M. K. Karacan and D. W. Yoon, classified translation surfaces of Type 2 in the three dimensional simply isotropic space $\mathbb{I}^1_3$ satisfying $\Delta^J x_i = \lambda_i x_i,\ J = I$ $, II\ , III$, where $\lambda_i \in \mathbb{R}$. Δ^J denotes the Laplace operator with respect to the fundamental forms $I\ , II$ and III [5].

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In $\mathbb{I}_3^1$, when the generating curves lie in perpendicular planes, three types of translation surfaces exist [5, 15].

Type 1: The surface M^2 is parametrized by

$$r(u,v) = (u, v, f(u) + g(v)),$$

and the translated curves are $(u, 0, f(u))$, $(0, v, g(v))$.

Type 2: The surface M^2 is parametrized by

$$r(u,v) = (u, f(u) + g(v), v),$$

and the translated curves are $(u, f(u), 0)$, $(0, g(v), v)$.

Type 3: The surface M^2 is parametrized by

$$r(u,v) = \frac{1}{2}(f(u) + g(v), u - v + \pi, u + v),$$

and the translated curves are $\frac{1}{2}(f(u), u + \frac{\pi}{2}, u - \frac{\pi}{2})$, $(g(v), \frac{\pi}{2} - v, \frac{\pi}{2} + v)$.

In [17] G. Kaimakamis, B.J. Papantoniou and K. Petoumenos classified and proved that such surfaces of revolution in the 3-dimensional Lorentz-Minkowski space $\mathbb{E}_1^3$ satisfying

$$\Delta^{III}\overrightarrow{r} = A\overrightarrow{r}$$

are either minimal or Lorentz hyperbolic cylinders or pseudospheres of real or imaginary radius. S. Stamatakis and H. Al-Zoubi in [16] classified the surfaces of revolution with non zero Gaussian curvature in $\mathbb{E}^3$ under the condition

$$\Delta^{III} r = Ar, \ A \in Mat(3, \mathbb{R}).$$

In [8] M. Choi and D. W. Yoon studied the helicoidal surfaces with the non-degenerate third fundamental form in Minkowski 3-space.

Recently, the authors [3] studied the translation surfaces in the 3-dimensional Euclidean and Lorentz-Minkowski space under the condition $\Delta^{III} r_i = \mu_i r_i$, $\mu_i \in \mathbb{R}$, where Δ^{III} denotes the Laplacian of the surface with respect to the third fundamental form III.

In this paper we study translation surfaces with the non-degenerate third fundamental form in Lorentz- Minkowski space $\mathbb{L}^3$. As a result, we classify translation surfaces satisfying an equation in terms of the position vector field and the Laplace operator with respect to the third fundamental form III on the surface.

2. Preliminaries

A submanifold M^2 of a 3-dimensional Euclidean space $\mathbb{E}^3$ is said to be of finite type if each component of its position vector field r can be written as a finite sum of eigenfunctions of the Laplacian Δ of M^2, that is, if

$$r = r_0 + \sum_{i=1}^{k} r_i,$$

[DOI: 10.52547/ijmsi.17.1.165]

where r_i are $\mathbb{E}^3$ -valued eigenfunctions of the Laplacian of (M^2, r) [6]:

$$\Delta r_i = \lambda_i r_i,\ \lambda_i \in \mathbb{R},\ i = 1, 2, .., k.$$

If λ_i are different, then M^2 is said to be of k-type.

The coefficients of the first fundamental form and the second fundamental form are

$$\begin{aligned} E &= g(r_u, r_u),\ F = g(r_u, r_v),\ G = g(r_v, r_v); \\ L &= g(r_{uu}, \mathbf{N}),\ M = g(r_{uv}, \mathbf{N}),\ N = g(r_{vv}, \mathbf{N}), \end{aligned}$$

where $r_u = \frac{\partial r}{\partial u}$, $r_v = \frac{\partial r}{\partial v}$ and $\mathbf{N}$ is the unit normal vector to M^2.

In the classical literature, one write the third fundamental form as

$$III = e_{11} du^2 + 2e_{12} du dv + e_{22} dv^2.$$

The Laplace-Beltrami operator of a smooth function $\varphi : M^2 \to \mathbb{R}, (u, v) \mapsto \varphi(u, v)$ with respect to the first fundamental form of the surface M^2 is the operator Δ, defined by

$$\Delta\varphi = \frac{-1}{\sqrt{|EG - F^2|}} \left[\frac{\partial}{\partial u} \left(\frac{G\varphi_u - F\varphi_v}{\sqrt{|EG - F^2|}} \right) + \frac{\partial}{\partial v} \left(\frac{E\varphi_v - F\varphi_u}{\sqrt{|EG - F^2|}} \right) \right].$$

The second Beltrami differential operator with respect to the third fundamental form III is defined by

$$\Delta^{III} = \frac{-1}{\sqrt{|e|}} \big(\frac{\partial}{\partial x^i} (\sqrt{|e|} e^{ij} \frac{\partial}{\partial x^j}) \big), \tag{2.1}$$

where $e = \det(e_{ij})$ and e^{ij} denote the components of the inverse tensor of e_{ij}.

If $r = (r_1, r_2, r_3)$ is a function of class C^2 then we set

$$\Delta^{III} r = (\Delta^{III} r_1, \Delta^{III} r_2, \Delta^{III} r_3).$$

The mean curvature H and the Gauss curvature K_G are, respectively, defined by

$$H = \frac{EN + GL - 2FM}{2(EG - F^2)}$$

and

$$K_G = \frac{LN - M^2}{EG - F^2}.$$

3. Translation surfaces of some new types in $\mathbb{L}^3$

The study of the translation surfaces in $\mathbb{L}^3$, H. Liu [13] investigated the translation surfaces with constant mean curvature or constant Gauss curvature. In [19] I. V. de Woestijne studied minimal translation surfaces.

[DOI: 10.52547/ijmsi.17.1.165]

Definition 3.1. A surface M^2 in $\mathbb{L}^3$ is called a translation surface if it is given by the graph

$$z(u,\ v) = f(u) + g(v),$$

where f and g are smooth functions on some interval of $\mathbb{R}$.

A minimal translation surface in a 3-dimensional Euclidean space $\mathbb{E}^3$ must be a plane or a Scherk surface which is the graph of the function

$$z(u,\ v) = \frac{1}{\alpha}\log|\cos(\alpha u)| - \frac{1}{\alpha}\log|\cos(\alpha v)|,$$

where α is a non-zero constant.

In 3-dimensional Minkowski space $\mathbb{L}^3$, according to the spacelike direction, timelike direction and lightlike direction, the translation surfaces can be considered belonging to

six types [21].

Type 1. Along spacelike direction and spacelike direction;

Type 2. Along spacelike direction and timelike direction;

Type 3. Along lightlike direction and lightlike direction;

Type 4. Along lightlike direction and spacelike direction;

Type 5. Along timelike direction and lightlike direction;

Type 6. Along timelike direction and timelike direction.

Let M^2 be a 2-dimensional surface, of the 3-dimensional Minkowski space $\mathbb{L}^3$. Using the standard coordinate system of $\mathbb{L}^3$ we denote the parametric representation of the surface $r(u,v)$ by

$$r(u,v) = (r_1(u,v),\ r_2(u,v),\ r_3(u,v)).$$

Translation surfaces S_a of types 5 and 6 can be written as [21]

$$S_a:\quad r(u,v) = (r_1(u,v),\ r_2(u,v),\ r_3(u,v)) = (f(u+av) + g(v),\ u,\ v)\ . \tag{3.1}$$

(i) When $|a| = 1$, the surfaces S_a is translation surface of type 5.

(ii) When $|a| > 1$, the surfaces S_a is translation surface of type 6.

The coefficients of the first fundamental form of the surface S_a are:

$$E = 1 + f_u^2,\ F = f_u(af_v + g_v),\ G = (af_v + g_v)^2 - 1. \tag{3.2}$$

We define smooth function W as

$$W = \sqrt{\varepsilon(EG - F^2)} = \sqrt{\varepsilon((af_v + g_v)^2 - f_u^2 - 1)},$$

where $\varepsilon = \pm 1$.

The unit normal vector of M^2 is given by

$$\mathbf{N} = \frac{1}{W}(1, -f_u, af_v + g_v).$$

Then the coefficients of the second fundamental form of the surface S_a are:

$$L = \frac{f_{uu}}{W},\ M = \frac{af_{uv}}{W},\ N = \frac{a^2 f_{vv} + g_{vv}}{W}. \tag{3.3}$$

[DOI: 10.52547/ijmsi.17.1.165]

From these we find that the mean curvature H and the curvature K_G of (3.1) are given by

$$H = \frac{(1+f_u^2)(a^2 f_{vv}+g_{vv}) - 2af_u f_{uv}(af_v+g_v) + f_{uu}((af_v+g_v)^2-1)}{2W^3} \tag{3.4}$$

and

$$K_G = \frac{f_{uu}(a^2 f_{vv}+g_{vv}) - a^2 f_{uv}^2}{W^4}. \tag{3.5}$$

By a transformation

$$\begin{cases} x = u + av \\ y = v, \end{cases} \tag{3.6}$$

and $\frac{\partial(x,y)}{\partial(u,v)} \neq 0$.

$$S_a: \quad r(x,y) = (f(x)+g(y),\ x-ay,\ y)\ . \tag{3.7}$$

From (3.7) we have

$$E = 1 + f_x^2,\ F = -a + f_x g_y,\ G = -1 + a^2 + g_y^2.$$

The unit normal vector of M^2 is given by

$$\mathbf{N} = \frac{1}{W}(1, -f_x, af_x + g_y).$$

The coefficients of the second fundamental form of the surface S_a are:

$$L = \frac{f_{xx}}{W},\ M = 0,\ N = \frac{g_{yy}}{W}.$$

From (3.4) and (3.5) we get

$$H = \frac{(1+f_x^2)g_{yy} + (-1+a^2+g_y^2)f_{xx}}{2W^3} \tag{3.8}$$

and

$$K_G = \frac{f_{xx}g_{yy}}{W^4}, \tag{3.9}$$

where $W = \sqrt{\varepsilon((af_x+g_y)^2 - f_x^2 - 1)}$.

The components of the third fundamental form of the surface M^2 is given by

$$\begin{aligned} e_{11} &= g_L(\frac{\partial \mathbf{N}}{\partial x}, \frac{\partial \mathbf{N}}{\partial x}) = \frac{\varepsilon f_{xx}^2 G}{W^4}, \quad e_{12} = g_L(\frac{\partial \mathbf{N}}{\partial x}, \frac{\partial \mathbf{N}}{\partial y}) = -\frac{\varepsilon f_{xx} g_{yy} F}{W^4}, \quad (3.10) \\ e_{22} &= g_L(\frac{\partial \mathbf{N}}{\partial y}, \frac{\partial \mathbf{N}}{\partial y}) = \frac{\varepsilon g_{yy}^2 E}{W^4}, \end{aligned}$$

hence

$$\sqrt{|e|} = K_G W,$$

where $\varepsilon = \pm 1$.

[DOI: 10.52547/ijmsi.17.1.165]

4. Translation surfaces of type 5 in $\mathbb{E}^3_1$ satisfying $\Delta r_i = \lambda_i r_i$

In this part we explore the classification of the translation surfaces M^2 of some new types in $\mathbb{E}^3_1$ satisfying the condition $\Delta r_i = \lambda_i r_i$. The Laplacian Δ of M^2 can be expressed as follows:

$$\Delta\varphi = \frac{-\varepsilon}{W^3}\left[W\varepsilon(G\varphi_{xx} + E\varphi_{yy} - 2F\varphi_{xy}) + Q(x,y)\varphi_x + P(x,y)\varphi_y\right], \qquad (4.1)$$

where

$$Q(x,y) = H_1(f_x - a(af_x + g_y)), \quad P(x,y) = -H_1(af_x + g_y), \quad H_1 = EN + GL - 2FM.$$

We have the following

Theorem 4.1. *Let M^2 be an affine translation surface given by (3.1) in $\mathbb{E}^3_1$. Then M^2 satisfies the equation $\Delta r_i = \lambda_i r_i$ $(i = 1, 2, 3)$ if and only if one of the following statement is true:*

1) M^2 has zero mean curvature everywhere.

2) M^2 is parametrized as

$$r(u,v) = \left(\frac{\varepsilon\lambda_3 v\sqrt{-\lambda_3 v^2 - c}}{\lambda_1\sqrt{-\lambda_3 v^2 - c + 1}},\ u,\ v\right),\ \lambda_3 y^2 + c < 0.$$

5. Translation surfaces in $\mathbb{L}^3$ satisfying $\Delta^{III} r_i = \lambda_i r_i$

In this part we explore the classification of the translation surfaces M^2 of some new types in $\mathbb{L}^3$ satisfying the condition

$$\Delta^{III} r_i = \lambda_i r_i. \qquad (5.1)$$

By a straightforward computation, the Laplacian Δ^{III} of the third fundamental form III on M with the help of (3.10) and (2.1) turns out to be

$$\begin{aligned}\Delta^{III} \quad = \quad & -W^2\left(\left(\frac{1+f_x^2}{f_{xx}^2}\right)\frac{\partial^2}{\partial x^2} + \left(\frac{-1+a^2+g_y^2}{g_{yy}^2}\right)\frac{\partial^2}{\partial y^2} + 2\left(\frac{-a+f_xg_y}{f_{xx}g_{yy}}\right)\frac{\partial^2}{\partial x\partial y}\right.\\ & \left. + \frac{1}{f_{xx}}\left(\frac{1+f_x^2}{f_{xx}}\right)_x\frac{\partial}{\partial x} + \frac{1}{g_{yy}}\left(\frac{-1+a^2+g_y^2}{g_{yy}}\right)_y\frac{\partial}{\partial y}\right). \qquad (5.2)\end{aligned}$$

By using (5.1) and (5.2) we have the following equations

$$\frac{1+f_x^2}{f_{xx}} + \frac{a^2-1+g_y^2}{g_{yy}} + \frac{f_x}{f_{xx}}\left(\frac{1+f_x^2}{f_{xx}}\right)_x + \frac{g_y}{g_{yy}}\left(\frac{a^2-1+g_y^2}{g_{yy}}\right)_y = -\frac{\lambda_1(f+g)}{W^2} \qquad (5.3)$$

$$\frac{1}{f_{xx}}\left(\frac{1+f_x^2}{f_{xx}}\right)_x - \frac{a}{g_{yy}}\left(\frac{a^2-1+g_y^2}{g_{yy}}\right)_y = -\frac{\lambda_2(x-ay)}{W^2} \qquad (5.4)$$

$$\frac{1}{g_{yy}}\left(\frac{a^2-1+g_y^2}{g_{yy}}\right)_y = -\frac{\lambda_3 y}{W^2}. \qquad (5.5)$$

Differentiating (5.5) with respect to x we obtain

$$f_{xx}\left(\frac{1}{g_{yy}}\left(\frac{a^2-1+g_y^2}{g_{yy}}\right)_y\right)(a^2f_x+ag_y-f_x)=0. \tag{5.6}$$

If $a^2f_x+ag_y-f_x=0$, then $f_{xx}=0$ and $g_{yy}=0$, it is a contradiction.
Then there exists $d_1\in\mathbb{R}^*$ such that

$$\frac{a^2-1+g_y^2}{g_{yy}}=d_1. \tag{5.7}$$

We get also, by the equation (5.5), $\lambda_3=0$.
Then, (5.3) and (5.4) can be written as the forms:

$$\frac{1+f_x^2}{f_{xx}}+\frac{f_x}{f_{xx}}\left(\frac{1+f_x^2}{f_{xx}}\right)_x+d_1=-\frac{\lambda_1(f+g)}{W^2} \tag{5.8}$$

$$\frac{1}{f_{xx}}\left(\frac{1+f_x^2}{f_{xx}}\right)_x=-\frac{\lambda_2(x-ay)}{W^2}. \tag{5.9}$$

Differentiating (5.9) with respect to y, we have

$$\frac{2\varepsilon}{f_{xx}}\left(\frac{1+f_x^2}{f_{xx}}\right)_x g_{yy}(af_x+g_y)=a\lambda_2.$$

Using (5.7), we have

$$\frac{2\varepsilon}{f_{xx}}\left(\frac{1+f_x^2}{f_{xx}}\right)_x(af_x+g_y)(-1+a^2+g_y^2)=d_1a\lambda_2.$$

Taking the derivative with respect to y, we have

$$\frac{2\varepsilon}{f_{xx}}\left(\frac{1+f_x^2}{f_{xx}}\right)_x g_{yy}(-1+a^2+3g_y^2+2af_xg_y)=0. \tag{5.10}$$

We discuss by cases:
Case 1. If $-1+a^2+3g_y^2+2af_xg_y=0$, then $f_{xx}=0$, a contradiction.
Case 2. If $\left(\frac{1+f_x^2}{f_{xx}}\right)_x=0$. Then (5.9) implies $\lambda_2=0$.
Then there exists $d_2\in\mathbb{R}^*$ such that

$$\frac{1+f_x^2}{f_{xx}}=d_2. \tag{5.11}$$

Substituting in (5.8), we have

$$-W^2(d_1+d_2)=\lambda_1(f+g). \tag{5.12}$$

Let $d_1+d_2\neq 0$. On differentiating (5.12), with respect to y, we find

$$af_x=-g_y-\frac{\lambda_1 g_y}{2(d_1+d_2)\varepsilon g_{yy}},$$

[DOI: 10.52547/ijmsi.17.1.165]

and so $f_{xx} = 0$, a contradiction. So, it is $d_1 + d_2 = 0$. Consequently, from (5.12), it is $\lambda_1 = 0$. Putting $d_1 = -d_2 = k$ and integrating (5.11) we find

$$f(x) = k \ln \left| c_0 \cos(-\frac{1}{k}x + c_1) \right|, \quad c_0 \in \mathbb{R}^*. \tag{5.13}$$

5.1. **Translation surfaces of type** 5 **in** $\mathbb{L}^3$**.** In this paragraph we explore the classification of the translation surfaces M^2 of type 5 satisfying (5.1).

Then (5.7) can be rewritten as

$$\frac{g_{yy}}{g_y^2} = \frac{1}{k}.$$

A simple integration implies that there exist $(c_2, c_3) \in \mathbb{R}^* \times \mathbb{R}$ such that

$$g(y) = -k \ln |c_2(y + c_3)|, \quad c_2 \in \mathbb{R}^*.$$

So

$$\begin{cases} f(x) = k \ln \left| c_0 \cos(-\frac{1}{k}x + c_1) \right|, & c_0 \in \mathbb{R}^* \\ g(y) = -k \ln |c_2(y + c_3)|, & c_2 \in \mathbb{R}^*. \end{cases}$$

Substituting these functions in (3.7), we obtain

$$S_a : \quad r(x, y) = (k \ln \left| \frac{c_0 \cos(-\frac{1}{k}x + c_1)}{c_2(y + c_3)} \right|, \ x - ay, \ y), \qquad c_0, c_2 \in \mathbb{R}^* \ .$$

Theorem 5.1. *Let M^2 be a translation surface of type* 5 *given by (3.7) in* $\mathbb{L}^3$*. Then $\Delta^{III} r_i = \lambda_i r_i$, $(i = 1, 2, 3)$ if and only if M^2 is the surface of Scherk*

$$r(x, y) = (k \ln \left| \frac{c_0 \cos(-\frac{1}{k}x + c_1)}{c_2(y + c_3)} \right|, \ x - ay, \ y), \qquad c_0, c_2 \in \mathbb{R}^*, \quad c_1, c_3 \in \mathbb{R}.$$

5.2. **Translation surfaces of type** 6 **in** $\mathbb{L}^3$**.** In this paragraph we explore the classification of the translation surfaces M^2 of type 6 satisfying (5.1).

Then (5.7) can be rewritten as

$$\frac{g_{yy}}{a^2 - 1 + g_y^2} = \frac{1}{k}.$$

A simple integration implies that there exist $(c_2, c_3) \in \mathbb{R}^* \times \mathbb{R}$ such that

$$g(y) = -k \ln \left| c_2 \cos \frac{\sqrt{a^2 - 1}}{k} (y + c_3) \right|.$$

So

$$\begin{cases} f(x) = k \ln \left| c_0 \cos(-\frac{1}{k}x + c_1) \right|, & (c_0, c_1) \in \mathbb{R}^* \times \mathbb{R} \\ g(y) = -k \ln \left| c_2 \cos \frac{\sqrt{a^2-1}}{k} (y + c_3) \right|, & (c_2, c_3) \in \mathbb{R}^* \times \mathbb{R}. \end{cases}$$

Substituting these functions in (3.7), we obtain

$$S_a : \quad r(x, y) = (k \ln \left| \frac{c_0 \cos(-\frac{1}{k}x + c_1)}{c_2 \cos \frac{\sqrt{a^2-1}}{k} (y + c_3)} \right|, \ x - ay, \ y) \ .$$

[DOI: 10.52547/ijmsi.17.1.165]

Theorem 5.2. *Let M^2 be a translation surface of type* 6 *given by (3.7) in* $\mathbb{L}^3$. *Then* $\Delta^{III} r_i = \lambda_i r_i$, $(i = 1, 2, 3)$ *if and only if* M^2 *is the surface of Scherk*

$$r(x,y) = (k \ln \left| \frac{c_0 \cos(-\frac{1}{k}x + c_1)}{c_2 \cos \frac{\sqrt{a^2-1}}{k}(y + c_3)} \right|, \ x - ay, \ y), \quad c_0, c_2 \in \mathbb{R}^*, \quad c_1, c_3 \in \mathbb{R}.$$

6. The Extrinsic Geometry of the Second Fundamental Form

6.1. Translation surfaces that satisfy $2H = \alpha K_G$.

Definition 6.1 ([5])**.** A surface of the three dimensional space $\mathbb{L}^3$ is said to be $III-$ harmonic if it satisfies the condition $\Delta^{III} r = 0$.

Theorem 6.2 ([21])**.** *Let S_a be a translation surface of type* 5 *or* 6 *in* $\mathbb{L}^3$ *whose Gauss curvature K_G and mean curvature H satisfy* $bH + cK_G = 0$ $(bc \neq 0)$. *Then it is congruent to a plane or an open part of it.*

In this section we study translation surfaces that satisfy the relation

$$2H = \alpha K_G, \ \alpha \in \mathbb{R}. \tag{6.1}$$

Theorem 6.3. *If* $\frac{2H}{K_G} = \alpha \in \mathbb{R}$, *then*

$$\Delta^{III} r(x,y) = \frac{2H}{K_G} N.$$

Proof. Equation (6.1) writes as

$$\frac{1 + f_x^2}{f_{xx}} + \frac{a^2 - 1 + g_y^2}{g_{yy}} = \frac{\alpha}{W}.$$

Differentiating with respect to x and y, we have

$$\frac{1}{f_{xx}} \left(\frac{1 + f_x^2}{f_{xx}} \right)_x = \frac{-\alpha\varepsilon(a g_y + (a^2 - 1) f_x)}{W^3} \tag{6.2}$$

$$\frac{1}{g_{yy}} \left(\frac{a^2 - 1 + g_y^2}{g_{yy}} \right)_y = \frac{-\alpha\varepsilon(a f_x + g_y)}{W^3}. \tag{6.3}$$

From (5.2), (6.2) and (6.3) we have

$$\begin{cases} \Delta^{III} r_1(x,y) = \frac{\alpha\varepsilon}{W} \\ \Delta^{III} r_2(x,y) = \frac{-\alpha\varepsilon f_x}{W} \\ \Delta^{III} r_3(x,y) = \frac{\alpha\varepsilon(a f_x + g_y)}{W}. \end{cases} \tag{6.4}$$

Then

$$\begin{aligned} \Delta^{III} r(x,y) &= (\Delta^{III} r_1(x,y), \ \Delta^{III} r_2(x,y), \ \Delta^{III} r_3(x,y)) \\ &= \frac{\alpha\varepsilon}{W}(1, \ -f_x, \ (a f_x + g_y)) \\ &= \alpha\varepsilon N. \end{aligned}$$

□

[DOI: 10.52547/ijmsi.17.1.165]

Corollary 6.4. *Let M^2 be a translation surface of type 5 or 6 given by (3.7) in $\mathbb{L}^3$. Then M^2 is $III-$ harmonic if and only if M^2 has zero mean curvature.*

6.2. **The Mean Curvature of the Second Fundamental Form.** The second fundamental form is an important notion in the classical differential geometry of surfaces.

The second mean curvature H_{II} of non-degenerate second fundamental form in Lorentz- Minkowski space $\mathbb{L}^3$ is defined by [20]. The mean curvature of the second fundamental form H_{II} is introduced as a measure for the rate of change of the $II-$ area under a normal deformation [12].

The mean curvature of the second fundamental form is defined by

$$H_{II} = H - \frac{1}{2}\Delta^{II}(\ln\sqrt{|K_G|}), \tag{6.5}$$

where Δ^{II} denotes the Laplacian operator of non-degenerate second fundamental form, that is,

$$\Delta^{II} = \frac{1}{\sqrt{|\det\ II|}}\sum_{i,j}\frac{\partial}{\partial u^i}(\sqrt{|\det\ II|}L^{ij}\frac{\partial}{\partial u^j}),$$

where $(L^{ij}) = (L_{ij})^{-1}$, where L_{ij} are the coefficients of second fundamental forms II, and $\{u^i\}$ is rectangular coordinate system in $\mathbb{E}^3$.

Definition 6.5. (1) A non developable surface is called $II-flat$ if the second Gaussian curvature vanishes identically.

(2) A non developable surface is called $II-minimal$ if the second mean curvature vanishes identically.

Proposition 6.6. *H_{II} can be equivalently expressed as*

$$H_{II} = H - \widetilde{H}, \tag{6.6}$$

where

$$\widetilde{H} = \left(\frac{M}{W^2}\left(\frac{1}{\sqrt{K}}\right)_y - \frac{N}{W^2}\left(\frac{1}{\sqrt{K}}\right)_x\right)_x + \left(\frac{M}{W^2}\left(\frac{1}{\sqrt{K}}\right)_x - \frac{L}{W^2}\left(\frac{1}{\sqrt{K}}\right)_y\right)_y.$$

There does not exist a polynomial affine translation surface with constant mean curvature in $\mathbb{E}^3$ [4].

Proposition 6.7. *The second mean H_{II} curvature on M^2 is*

$$\begin{aligned} H_{II} \;\; = \;\; & -\frac{1}{8W^3}\left[W^4\left(\frac{2f''f^{(4)}-3f'''^2}{f''^3}+\frac{2g''g^{(4)}-3g'''^2}{g''^3}\right)\right. \\ & +4\varepsilon W^2\left(\frac{(-ag'+(1-a^2)f')f'''}{f''}-\frac{(af'+g')g'''}{g''}\right) \\ & +4f''(-1+a^2+g'^2+2a^2g'^2+2(1-a^2)^2f'^2+4a(1+a^2)f'g') \\ & \left.+4g''(-1+f'^2+2a^2f'^2+2g'^2)\right]. \end{aligned} \tag{6.7}$$

[DOI: 10.52547/ijmsi.17.1.165]

Acknowledgments

The authors would like to express their sincere gratitude to the referee for the valuable suggestions which improve the paper.

References

1. M. E. Aydin, A Generalization of Translation Surfaces with Constant Curvature in the Isotropic Space, *J. Geom.*, **107**, (2016), 603-615.
2. M. Bekkar, B. Senoussi, Factorable Surfaces in the Three-dimensional Euclidean and Lorentzian Spaces Satisfying $\Delta r_i = \lambda_i r_i$, *J. Geom.*, **103**, (2012), 17 - 29.
3. M. Bekkar, B. Senoussi, Translation Surfaces in the 3-dimensional Space Satisfying $\Delta^{III} r_i = \mu_i r_i$, *J. Geom.*, **103**, (2012), 367-374.
4. H. G. Bozok, M. Ergüt, Polynomial Affine Translation Surfaces in Euclidean 3-Space, *Bol. Soc. Paran. Mat.*, **373**, (2019), 195-202.
5. B. Bukcu, M. K. Karacan, D. W. Yoon, Translation Surfaces of Type 2 in the Three Dimensional Simply Isotropic Space $\mathbb{I}_3^1$., *Bull. Korean Math. Soc.*, **54**, (2017), 953 - 965.
6. B.-Y. Chen, *Total Mean Curvature and Submanifolds of Finite Type*, World Scientific, Singapore, (second edition) 2015.
7. B.-Y. Chen, Finite Type Submanifolds in Pseudo-Euclidean Spaces and Applications, *Kodai Math. J.*, **6**, (1985), 358-374.
8. M. Choi, D. W. Yoon, Helicoidal Surfaces of the Third Fundamental Form in Minkowski 3-space, *Bull. Korean Math. Soc.*, **52**, (2015), 1569-1578.
9. F. Dillen, W. Goemans, I. Van de Woestyne, Translation Surfaces of Weingarten Type in 3-space, *Bull. Transilv. Univ. Brasov.*, **15**, (2008) 109-122.
10. F. Dillen, J. Pas, I. Verstraelen, On Surfaces of Finite Type in Euclidean 3-space, *Kodai Math. J.*, **13**, (1990), 10-21.
11. A. Ferrandez, O. J. Garay, P. Lucas, *On a Certain Class of Conformally Flat Euclidean Hypersurfaces*, Proc. of the Conf, in Global Analysis and Global Differential Geometry, Berlin. 1990.
12. S. Haesen, S. Verpoort, L. Verstraelen, The Mean Curvature of the Second Fundamental Form, *Houston J. of Math.*, **34**(3), (2008), 703-719.
13. H. Liu, Translation Surfaces with Constant Mean Curvature in 3-dimensional Spaces, *J. Geom.*, **64**, (1999), 141-149.
14. B. Senoussi, M. Bekkar, Helicoidal Surfaces in the 3-dimensional Lorentz - Minkowski Space $\mathbb{E}_1^3$ *satisfying* $\Delta^{III} r = Ar$, *Tsukuba J. Math.*, **37**, (2013), 339 - 353.
15. Z. M. Sipus, Translation Surfaces of Constant Curvatures in a Simply Isotropic Space, *Period Math. Hungar.*, **68**, (2014), 160 - 175.
16. S. Stamatakis, H. Al-Zoubi, Surfaces of Revolution Satisfying $\Delta^{III} x = Ax$, *J. Geom. Graph*, **14**, (2010), 181-186.
17. G. Kaimakamis, B. J. Papantoniou, K. Petoumenos, Surfaces of Revolution in the 3-dimensional Lorentz-Minkowski Space $\mathbb{E}_1^3$ Satisfying $\Delta^{III} \overrightarrow{r} = A\overrightarrow{r}$, *Bull. Greek. Math. Soc.*, **50**, (2005), 76 - 90.
18. T. Takahashi, Minimal Immersions of Riemannian Manifolds, *J. Math. Soc. Japan*, **18**, (1966), 380-385.
19. I. V. de Woestijne, Minimal Surfaces of 3-dimensional Minkowski Space, in Geometry and Topology of Submanifolds II, *World Scientific Publ.*, (1990), 344-369.
20. S. Verpoort, *The Geometry of the Second Fundamental Form*: Curvature Properties and Variational Aspects, PhD. Thesis, Katholieke Universiteit Leuven, Belgium, 2008.
21. Y. Yuan and H. Liu, Some New Traslation Surfaces in 3 *-Minkowski space*, *Journal of Mathematical Research and Exposition*, **31**, (2011), 1123-1128.